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Signed Domatic Number of Directed Circulant Graphs

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Abstract: A function $f : V \to \{-1, 1\}$ is a signed dominating function (SDF) of a directed graph D ([4]) if for every vertex $v \in V$,

$$f(N^-[v]) = \sum_{u \in N^-[v]} f(u) \geq 1.$$

In this paper, we introduce the concept of signed efficient dominating function (SEDF) for directed graphs. A SDF of a directed graph D is said to be SEDF if for every vertex $v \in V$, $f(N^-[v]) = 1$ when $|N^-[v]|$ is odd and $f(N^-[v]) = 2$ when $|N^-[v]|$ is even. We study the signed domatic number $d_S(D)$ of directed graphs. Actually, we give a lower bound for signed domination number $\gamma_S(G)$ and an upper bound for $d_S(G)$. Also we characterize some classes of directed circulant graphs for which $d_S(D) = \delta^-(D) + 1$. Further, we find a necessary and sufficient condition for the existence of SEDF in circulant graphs in terms of covering projection.

Key Words: Signed graphs, signed domination, signed efficient domination, covering projection.

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§1. Introduction

Let D be a simple finite digraph with vertex set $V(D) = V$ and arc set $E(D) = E$. For every vertex $v \in V$, in-neighbors of v and out-neighbors of v are defined by $N^-[v] = N_D^-[v] = \{u \in V : (u, v) \in E\}$ and $N^+[v] = N_D^+[v] = \{u \in V : (v, u) \in E\}$ respectively. For a vertex $v \in V$, $d_D^+(v) = d^+(v) = |N^+(v)|$ and $d_D^-(v) = d^-(v) = |N^-(v)|$ respectively denote the outdegree and indegree of the vertex v. The minimum and maximum indegree of D are denoted by $\delta^-(D)$ and $\Delta^-(D)$ respectively. Similarly the minimum and maximum outdegree of D are denoted by $\delta^+(D)$ and $\Delta^+(D)$ respectively.

In [2], J.E. Dunbar et al. introduced the concept of signed domination number of an undirected graph. In 2005, Bohdan Zelinka [1] extended the concept of signed domination in directed graphs.

A function $f : V \to \{-1, 1\}$ is a signed dominating function (SDF) of a directed graph D

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([4]) if for every vertex $v \in V$,

$$f(N^-[v]) = \sum_{u\in N^-[v]} f(u) \geq 1.$$

The signed domination number, denoted by $\gamma_S(D)$, is the minimum weight of a signed dominating function of D [4]. In this paper, we introduce the concept of signed efficient dominating function (SEDF) for directed graphs. A SDF of a directed graph D is said to be SEDF if for every vertex $v \in V$, $f(N^-[v]) = 1$ when $|N^-[v]|$ is odd and $f(N^-[v]) = 2$ when $|N^-[v]|$ is even.

A set $\{f_1, f_2, \cdots, f_d\}$ of signed dominating functions on a graph (directed graph) G with the property that

$$\sum_{i=1}^{d} f_i(x) \leq 1$$

for each vertex $x \in V(G)$, is called a signed dominating family on G. The maximum number of functions in a signed dominating family on G is the signed domatic number of G, denoted by $d_S(G)$.

The signed domatic number of undirected and simple graphs was introduced by Volkmann and Zelinka [6]. They determined the signed domatic number of complete graphs and complete bipartite graphs. Further, they obtained some bounds for domatic number. They proved the following results.

Theorem 1.1([6]) *Let G be a graph of order $n(G)$ with signed domination number $\gamma_S(G)$ and signed domatic number $d_S(G)$. Then $\gamma_S(G).d_S(G) \leq n(G)$.*

Theorem 1.1([6]) *Let G be a graph with minimum degree $\delta(G)$, then $1 \leq d_S(G) \leq \delta(G) + 1$.*

In this paper, we study some of the properties of signed domination number and signed domatic number of directed graphs. Also, we study the signed domination number and signed domatic number of directed circulant graphs $Cir(n, A)$. Further, we obtain a necessary and sufficient condition for the existence of SEDF in $Cir(n, A)$ in terms of covering projection.

§2. Signed Domatic Number of Directed Graphs

In this section, we study the signed domatic number of directed graphs. Actually, we give a lower bound for $\gamma_S(G)$ and an upper bound for $d_S(G)$.

Theorem 2.1 *Let D be a directed graph of order n with signed domination number $\gamma_S(D)$ and signed domatic number $d_S(D)$. Then $\gamma_S(D)d_S(D) \leq n$.*

Proof Let $d = d_S(D)$ and $\{f_1, f_2, \cdots, f_d\}$ be a corresponding signed dominating family on D. Then

$$\begin{aligned} d\gamma_S(D) &= \sum_{i=1}^{d}\gamma_S(D) \leq \sum_{i=1}^{d}\sum_{v\in V(D)} f_i(v) \\ &= \sum_{v\in V(D)}\sum_{i=1}^{d} f_i(v) \leq \sum_{v\in V(D)} 1 = n. \end{aligned}$$ □

In [4], H. Karami et al. proved the following result.

Theorem 2.2([4]) *Let D be a digraph of order n in which $d^+(x) = d^-(x) = k$ for each $x \in V$, where k is a nonnegative integer. Then $\gamma_S(D) \geq \frac{n}{k+1}$.*

In the view of Theorems 2.1 and 2.2, we have the following corollary.

Corollary 2.3 *Let D be a digraph of order n in which $d^+(x) = d^-(x) = k$ for each $x \in V$, where k is a nonnegative integer. Then $d_S(D) \leq k + 1$.*

The next result is a more general form of the above corollary.

Theorem 2.4 *Let D be a directed graph with minimum in degree $\delta^-(D)$, then $1 \leq d_S(D) \leq \delta^-(D) + 1$.*

Proof Note that the function $f : V(D) \to \{+1, -1\}$, defined by $f(v) = +1$ for all $v \in V(D)$, is a SDF and $\{f\}$ is a signed domatic family on D. Hence $d_S(D) \geq 1$. Let $d = d_S(D)$ and $\{f_1, f_2, \cdots, f_d\}$ be a corresponding signed dominating family of D. Let $v \in V$ be a vertex of minimum degree $\delta^-(D)$.
Then,

$$\begin{aligned} d &= \sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d}\sum_{x\in N^-[v]} f_i(x) \\ &= \sum_{x\in N^-[v]}\sum_{i=1}^{d} f_i(x) \leq \sum_{x\in N^-[v]} 1 = \delta^-(D) + 1. \end{aligned}$$ □

Theorem 2.5([6]) *The signed domination number is an odd integer.*

Remark 2.6 The signed domination number of a directed graph may not be an odd integer. For example, for the directed circulant graph $Cir(10, \{1, 2, 3, 4\})$, the signed domination number is 2.

Theorem 2.7 *Let D be a directed graph such that $d^+(x) = d^-(x) = 2g$ for each $x \in V$ and let $u \in V(D)$. If $d = d_S(D) = 2g + 1$ and $\{f_1, f_2, \cdots f_d\}$ is a signed domatic family of D, then*

$$\sum_{i=1}^{d} f_i(u) = 1 \quad \text{and} \quad \sum_{x\in N^-[u]} f_i(x) = 1$$

for each $u \in V(D)$ *and each* $1 \le i \le 2g+1$*.*

Proof Since $\sum_{i=1}^{d} f_i(u) \le 1$, this sum has at least g summands which have the value -1. Since $\sum_{x \in N^-[u]} f_i(x) \ge 1$ for each $1 \le i \le 2g+1$, this sum has at least $g+1$ summands which have the value 1.

Also the sum

$$\sum_{x \in N^-[u]} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in N^-[u]} f_i(x)$$

has at least dg summands of value -1 and at least $d(g+1)$ summands of value 1. Since the sum

$$\sum_{x \in N^-[u]} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in N^-[u]} f_i(x)$$

contains exactly $d(2g+1)$ summands, it is easy to observe that $\sum_{i=1}^{d} f_i(u)$ have exactly g summands of value -1 and $\sum_{x \in N^-[u]} f_i(x)$ has exactly $g+1$ summands of value 1 for each $1 \le i \le r+1$. Hence we must have

$$\sum_{i=1}^{d} f_i(u) = 1 \quad \text{and} \quad \sum_{x \in N^-[u]} f_i(x) = 1$$

for each $u \in V(D)$ and for each $1 \le i \le 2g+1$. □

§3. Signed Domatic Number and SEDF in Directed Circulant Graphs

Let Γ be a finite group and e be the identity element of Γ. A generating set of Γ is a subset A such that every element of A can be written as a product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. Then the corresponding *Cayley graph* is a graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, y)_a | x, y \in V(G), y = xa \text{ for some } a \in A\}$, denoted by $Cay(\Gamma, A)$. It may be noted that G is connected regular graph degree of degree $|A|$. A Cayley graph constructed out of a finite cyclic group $(\mathbb{Z}_n, \oplus_n)$ is called a circulant graph and it is denoted by $Cir(n, A)$, where A is a generating set of $\mathbb{Z}_n$. When we leave the condition that $a \in A$ implies $a^{-1} \in A$, then we get directed circulant graphs. In a directed circulant graph $Cir(n, A)$, for every vertex v, $|N^-[v]| = |N^+[v]| = |A| + 1$.

Throughout this section, $n(\ge 3)$ is a positive integer, $\Gamma = (\mathbb{Z}_n, \oplus_n)$, where $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ and $D = Cir(n, A)$, where $A = \{1, 2, \cdots, r\}$ and $1 \le r \le n-1$. From here, the operation $\oplus_n$ stands for modulo n addition in $\mathbb{Z}_n$. In this section, we characterize the the circulant graphs for which $d_S(D) = \delta^-(D) + 1$. Also we find a necessary and sufficient condition for the existence of SEDF in $Cir(n, A)$ in terms of covering projection.

Theorem 3.1 *Let* $n \ge 3$ *and* $1 \le r \le n-1$ *(r is even) be integers and* $D = Cir(n, \{1, 2, \cdots, r\})$

be a directed circulant graph. Then $d_S(D) = r+1$ *if, and only if,* $r+1$ *divides* n.

Proof Assume that $d_S(D) = r+1$ and $\{f_1, f_2, \ldots f_{r+1}\}$ is a signed domatic family on D. Since $d^+(v) = d^-(v) = r$, for all $v \in V(D)$, by Theorems 2.1 and 2.2, we have $\gamma_S(D) = \frac{n}{r+1}$.

Suppose n is not a multiple of $r+1$. Then $n = k(r+1) + i$ for some $1 \leq i \leq r$. Let $t = gcd(i, r+1)$. Then there exist relatively prime integers p and q such that $r+1 = qt$ and $i = pt$. Let a and b be the smallest integers such that $a(r+1) = bn$. Then $gcd(a,b) = 1$; otherwise a and b will not be the smallest.

Now $aqt = a(r+1) = b(k(r+1)+i) = b(kqt+pt) = bt(kq+p)$. That is $aq = b(kp+q)$. Note that $gcd(a,b) = gcd(p,q) = 1$. Hence $a = kp+q$ and $b = q$. Thus the subgroup $< r+1 >$ of the finite cyclic group $\mathbb{Z}_n$, generated by $r+1$, must have $kp+q$ elements. But $t = \frac{r+1}{q} = \frac{n}{kp+q}$. Thus the subgroup $< t >$ of $\mathbb{Z}_n$, generated by the element t, also have $kp+q$ elements and hence $< t > = < r+1 >$. Since $d_S(D) = r+1$ and $\{f_1, f_2, \ldots f_{r+1}\}$ is a signed domatic family of D, by Theorem 2.7, we have

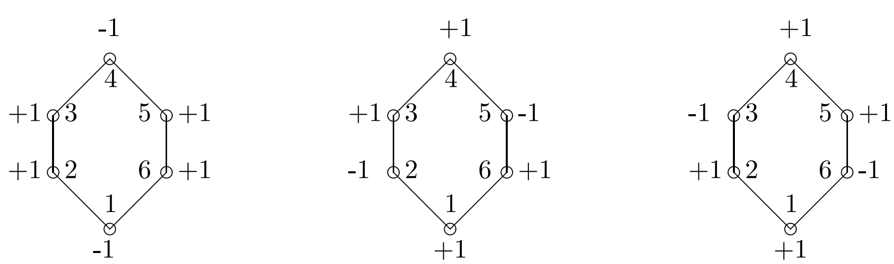
$$\sum_{i=1}^{d} f_i(u) = 1 \quad \text{and} \quad \sum_{x \in N^-[u]} f_i(x) = 1$$

for each $u \in V(D)$ and each $1 \leq i \leq r+1$.

From the above fact and since $|N^-[v]| = r+1$ for all $v \in V(D)$, it is follows that if $f(a) = +1$, then $f(a \oplus_n (r+1)) = +1$ and if $f(a) = -1$, then $f(a \oplus_n (r+1)) = -1$. Thus all the elements of the subgroup $< t >$ have the same sign and hence all the elements in each of the co-set of $< t >$ have the same sign. By Lagranges theorem on subgroups, $\mathbb{Z}_n$ can be written as the union of co-sets of $< t > = < r+1 >$. This means that $\gamma_S(D)$ must be a multiple of the number of elements of $< t >$, that is a multiple of $\frac{n}{t}$ (since n is a multiple of t). Since $t < r+1$, it follows that $\frac{n}{r+1} < \frac{n}{t} \leq \gamma_S(D)$, a contradiction to $\gamma_S(D) = \frac{n}{r+1}$.

Conversely suppose $r+1$ divides n. By theorem 2.4, $d_S(D) \leq r+1$. For each $1 \leq i \leq r+1$, define $f_i(i) = f_i(i \oplus_{r+1} 1) = \ldots = f_i(i \oplus_{r+1} (g-1)) = -1$ and $f_i(i \oplus_{r+1} g) = \ldots = f_i(i \oplus_{r+1} 2g) = +1$, where $g = \frac{r}{2}$, and for the remaining vertices, $f_i(v) = f_i(v \; mod(r+1))$ for $v \in \{r+2, r+3, \ldots, n\}$.

Notice that $\{f_1, f_2, \cdots, f_{r+1}\}$ are SDFs on D with the property that $\sum_{i=1}^{r+1} f_i(x) \leq 1$ for each vertex $x \in V(D)$. Hence $d_S(D) \geq r+1$. □



The SDF f_1 The SDF f_2 The SDF f_3

Fig.1

Example 3.2 Let $n = 6$ and $r = 2$. Then n is a multiple of $r+1$, and $r+1 = 3$ SDFs f_1, f_2 and f_3 (as discussed in the above theorem) of $D = Cir(6, \{1,2\})$ are as given in Fig.1 following, where $V(D) = \{1,2,3,4,5,6\}$.

Theorem 3.3 *Let $n \geq 3$ be an integer and $1 \leq r \leq n-1$ be an integer. Let $D = Cir(n, \{1,2,\cdots,r\})$ be a directed circulant graph. If n is a multiple of $r+1$, then $\gamma_S(D) = \dfrac{n}{r+1}$.*

Proof Assume that n is a multiple of $r+1$. By Theorem 2.2, we have $\gamma_S(D) \geq \frac{n}{r+1}$. It remains to show that there exists a SDF f with the property that $f(D) = \frac{n}{r+1}$.

Define a function f on $V(D)$ by $f(1) = f(2) = \ldots = f(g) = -1$ and $f(g+1) = f(g+2) = \cdots = f(2g+1) = +1$, where $g = \frac{r}{2}$; and for the remaining vertices, $f(v) = f(v\ mod(r+1))$ for $v \in \{r+2, r+3, \cdots, n\}$.

It is clear that f is a SDF and

$$f(D) = (g+1)\left(\frac{n}{r+1}\right) - (g)\left(\frac{n}{r+1}\right) = \frac{n}{r+1}. \qquad \square$$

A graph $\tilde{G}$ is called a covering graph of G with covering projection $f : \tilde{G} \to G$ if there is a surjection $f : V(\tilde{G}) \to V(G)$ such that $f\mid_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ with $\tilde{v} \in f^{-1}(v)$ ([5]).

In 2001, J.Lee has studied the domination parameters through covering projections ([5]). In this paper, we introduce the concept of covering projection for directed graphs and we study the SDF through covering projections.

A directed graph D is called a covering graph of another directed graph H with covering projection $f : D \to H$ if there is a surjection $f : V(D) \to V(H)$ such that $f\mid_{N^+(u)} : N^+(u) \to N^+(v)$ and $f\mid_{N^-(u)} : N^-(u) \to N^-(v)$ are bijections for any vertex $v \in V(H)$ with $u \in f^{-1}(v)$.

Lemma 3.4 *Let $f : D \to H$ be a covering projection from a directed graph D on to another directed graph H. If H has a SEDF, then so is D.*

Proof Let $f : D \to H$ be a covering projection from a directed graph D on to another directed graph H. Assume that H has a SEDF $h : V(H) \to \{1,-1\}$.

Define a function $g : V(D) \to \{1,-1\}$ defined by $g(u) = h(f(u))$ for all $u \in V(D)$. Since h is a function form $V(H)$ to $\{1,-1\}$ and $f : V(D) \to V(H)$, g is well defined. We prove that for the graph D, g is a SEDF.

Firstly, we prove $g(N^-[u]) = 1$ when $u \in V(D)$ and $|N^-[u]|$ is odd. In fact, let $u \in V(D)$ and assume that $|N^-[u]|$ is odd. Since f is a covering projection, $|N^-(u)|$ and $|N^-(f(u))|$ are equal. Also $f|_{N^-(u)} : N^-(u) \to N^-(f(u))$ is a bijection. Also for each vertex $x \in N^-[u]$, we have $g(x) = h(f(x))$. Since $h(N^-[f(u)]) = 1$, we have $g(N^-[u]) = 1$. Similarly, we can prove that $g(N^-[u]) = 2$ when $u \in V(G)$ and $|N^-[u]|$ is even. Hence g is a SEDF on D. $\square$

Theorem 3.5 *Let $D = Cir(n, \{1,2,\cdots,r\})$, $r = 2g$ and $\gamma_S(D) = \dfrac{n}{r+1}$. Then D has a SEDF if and only if, there exists a covering projection from D onto the graph $H = Cir(r+1, \{1,2,\cdots,r\})$.*

Proof Suppose D has a SEDF f. Then $\sum_{x\in N^-[u]} f(x) = 1$ for all $u \in v(D)$. Thus we can have $f(a \oplus_n r+1) = \pm 1$ when ever $f(a) = \pm 1$. Thus the elements of the subgroup $< r+1 >$, generated by $r+1$ have the same sign.

Suppose n is not a multiple of $r+1$, then $n = i(r+1)+j$ for some $1 \leq j \leq r$. Let $t = gcd(r+1, j)$. Then by Theorem 3.1, we have $\gamma_S(D) > \frac{n}{r+1}$, a contradiction. Hence n must be a multiple of $r+1$.

In this case, define $F : D \to H = Cir(r+1, \{1, 2, \cdots, r\})$, defined by $F(x) = x\ (mod\ r+1)$. Note that, $|N^-[x]| = |N^+[x]| = |N^-[y]| = |N^+[y]| = r+1$ for all $x \in V(D)$ and $y \in V(H)$. We prove that the function F is a covering projection.

Let $x \in V(D)$. Then $F(x) = x(mod\ (r+1)) = i$ for some $i \in V(H)$ with $1 \leq i \leq r+1$. Note that by the definition of D and H, $N^+(x) = \{x \oplus_n 1, x \oplus_n 2, \ldots, x \oplus_n r\}$ and $N^+(i) = \{i \oplus_{r+1} 1, i \oplus_{r+1} 2, \ldots, i \oplus_{r+1} r\}$.

Also, for each $1 \leq g \leq r+1$, we have $F(x \oplus_n g) = (x \oplus_n g)(mod\ (r+1)) = (x \oplus_{r+1} g)(mod\ (r+1))$ (since n is a multiple of $r+1$).

Thus $F(x \oplus_n g) = (i \oplus_{r+1} g)(mod\ (r+1))$ (since $x(mod(r+1)) = i$). Thus $F|_{N^+(x)} : N^+(x) \to N^+(F(x))$ is a bijection. Similarly, we can prove that $F|_{N^-(x)} : N^-(x) \to N^-(F(x))$ is also a bijection and hence F is a covering projection from D onto H.

Conversely, suppose there exists a covering projection F from D onto the graph $H = Cir(r+1, \{1, 2, \cdots, r\})$. Define $h : V(H) \to \{+1, -1\}$ defined by $h(x) = -1$ when $1 \leq x \leq g$ and $h(x) = +1$ when $g+1 \leq x \leq 2g+1$. Then h is a SEDF of H and hence by Lemma 3.4, G has a SEDF. □

Theorem 3.6 *Let $D = Cir(n, \{1, 2, \cdots, r\})$, r be an odd integer and $\gamma_S(D) = \dfrac{n}{r+1}$. Then D has a SEDF if and only if, there exists a covering projection from D onto the graph $H = Cir(r+1, \{1, 2, \cdots, r\})$.*

Proof Suppose D has a SEDF f. Let $H = Cir(r+1, \{1, 2, \cdots, r\})$. Note that, $|N^-[x]| = |N^+[x]| = |N^-[y]| = |N^+[y]| = r+1 = 2g$ (say), an even integer, for all $x \in V(D)$ and $y \in V(H)$. Thus

$$\sum_{x\in N^-[u]} f(x) = 2$$

for all $u \in v(D)$. Thus we can have $f(a \oplus_n r+1) = \pm 1$ when ever $f(a) = \pm 1$. Thus the elements of the subgroup $< r+1 >$, generated by $r+1$ have the same sign.

Suppose n is not a multiple of $r+1$, As in the proof of Theorem 3.5, we can get a contradiction. Also the function F defined in Theorem 3.5 is a covering projection from D onto H.

Conversely suppose there exists a covering projection F from D onto the graph $H = Cir(r+1, \{1, 2, \cdots, r\})$. Define $h : V(H) \to \{+1, -1\}$ defined by $h(x) = -1$ when $1 \leq x \leq g-1$ and $h(x) = +1$ when $g \leq x \leq 2g$. Note that h is a SEDF of H and

$$\sum_{x\in N^-[u]} f(x) = 2$$

for all $u \in v(G)$. Thus by Lemma 3.4, G has a SEDF. $\square$

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