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On Generalized Quasi-Kenmotsu Manifolds

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Abstract: We present a brief analysis on some properties of generalized quasi-Sasakian manifolds, discuss some important properties, particularly, regard the integrability conditions of this kind of manifolds in this paper.

Key Words: Riemannian manifold, semi-Riemannian manifold, quasi-Sasakian structure, integrability.

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§1. Introduction

An interesting topic in the differential geometry is the theory of submanifolds in space endowed with additional structures ([5], [6]). Cr-submanifolds of Kaehler manifolds were studied by A.Bejancu, B.Y.Chen, N.Papaghiuc etc. have studied semi-invariant submnaifolds in Sasakian manifolds ([1], [9]). The notion of Kenmotsu manifolds was defined by K.Kenmotsu in 1972 ([10]). N.Papaghiuc have studied semi-invariant submanifolds in a Kenmotsu manifold ([11]). He also studied the geometry of leaves on a semi-invariant $\xi^{\perp}$-submanifolds in a Kenmotsu manifolds ([12]).

§2. Preliminaries

Definition 2.1 *An $(2n+1)$-dimensional semi-Riemannian manifold $(\tilde{M},\tilde{g})$ is said to be an indefinite almost contact manifold if it admits an indefinite almost contact structure (ϕ,ξ,η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field and η is a 1-form, satisfying*

$$\phi^2X = -X+\eta(X)\xi, \quad \eta\circ\phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\tilde{g}(\phi X,\phi Y) = \tilde{g}(X,Y) - \epsilon\eta(X)\eta(Y), \tag{2.2}$$

$$\tilde{g}(X,\xi) = \epsilon\eta(X), \tag{2.3}$$

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$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y) \tag{2.4}$$

for all vector fields X, Y *on* $\tilde{M}$ *and where* $\epsilon = \tilde{g}(\xi, \xi) = \pm 1$ *and* $\tilde{\nabla}$ *is the Levi-Civita (L-C) connection for a semi-Riemannian metric* $\tilde{g}$. *Let* $F(\tilde{M})$ *be the algebra of the smooth functions on* $\tilde{M}$.

Definition 2.2 *An almost contact manifold* $\tilde{M}(\phi, \xi, \eta)$ *is said to be normal if*

$$N_\phi(X, Y) + 2d\eta(X, Y)\xi = 0$$

where

$$N_\phi(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \qquad X, Y \in \Gamma(T\tilde{M})$$

is the nijenhuis tensor field corresponding to the tensor fields ϕ. *The fundamental* 2*-form* Φ *on* $\tilde{M}$ *is defined by*

$$\Phi(X, Y) = \tilde{g}(X, \phi Y).$$

In [7]-[8], the authors studied hypersurfaces of an almost contact metric manifold $\tilde{M}$. In this paper we define hypersurfaces of an almost contact metric manifold $\tilde{M}$ whose structure tensor field satisfy the following relation

$$(\tilde{\nabla}_X \phi)Y = \tilde{g}(\tilde{\nabla}_{\phi^2 X}\xi, Y)\xi - \eta(Y)\tilde{\nabla}_{\phi^2 X}\xi, \tag{2.5}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the metric tensor $\tilde{g}$. We name this manifold $\tilde{M}$ equipped with an almost contact metric structure satisfying from (2.5) as generalized Quasi-Kenmotsu manifold, in short G.Q.K.

We define a $(1,1)$ tensor field F by

$$FX = \tilde{\nabla}_X \xi. \tag{2.6}$$

Let us now state the following proposition:

Proposition 2.1 *If* $\tilde{M}$ *is a G.Q.K manifold then any integral curve of the structure vector field* ξ *is a geodesic i.e.* $\tilde{\nabla}_\xi \xi = 0$. *Again* $d\Phi = 0$ *iff* ξ *is a Killing vector field.*

Proof From equation (2.5) putting $X = Y = \xi$ we can easily prove this assertion.

Next, we derive

$$\begin{aligned} 3d\Phi(X, Y, Z) &= \tilde{g}((\tilde{\nabla}_X \phi)Z, Y) + \tilde{g}((\tilde{\nabla}_Z \phi)Y, X) + \tilde{g}((\tilde{\nabla}_Y \phi)X, Z) \\ &+ \eta(X)(\tilde{g}(Y, \tilde{\nabla}_{\phi Z}\xi) + \tilde{g}(\phi Z, \tilde{\nabla}_Y \xi)) \\ &+ \eta(Y)(\tilde{g}(Z, \tilde{\nabla}_{\phi X}\xi) + \tilde{g}(\phi X, \tilde{\nabla}_Z \xi)) \\ &+ \eta(Z)(\tilde{g}(X, \tilde{\nabla}_{\phi Y}\xi) + \tilde{g}(\phi Y, \tilde{\nabla}_X \xi)) = 0. \end{aligned}$$

Therefore, if ξ is a killing vector field then $d\Phi = 0$.

Conversely, Suppose $d\Phi = 0$. taking into account $X = \xi, \eta(Y) = \eta(Z) = 0$, the last equation implies

$$\tilde{g}(Y, \tilde{\nabla}_{\phi Z}\xi) + \tilde{g}(\phi Z, \tilde{\nabla}_Y \xi = 0$$

Now substituting $Z = \phi Z$ and $Y = Y - \eta(Y)\xi$ we get,

$$\tilde{g}(Y - \eta(Y)\xi, \tilde{\nabla}_{\phi^2 Z}\xi) + \tilde{g}(\phi^2 Z, \tilde{\nabla}_{Y-\eta(Y)\xi}\xi = 0$$

This implies ξ is a killing vector field. □

Let $\tilde{M}$ be a G.Q.K manifold and considering an m-dimensional submanifold M, isometrically immersed in $\tilde{M}$. Assuming g, ∇, are the induced metric and levi-Civita connevtion on M respectively. Let $\nabla^{\perp}$ and h be the normal connection induced by $\tilde{\nabla}$ on the normal bundle $TM^{\perp}$ and the second fundamental form of M, respectively.

Therefore, we can decompose the tangent bundle as

$$T\tilde{M} = TM \oplus TM^{\perp}.$$

The Gauss and Weingarten formulae are characterized by the equations

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.7}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.8}$$

where A_N is the Weingarten map w.r.t the normal section N and satisfies

$$g(A_N X, Y) = g(h(X, Y), N) \quad X, Y \in \Gamma(TM), N \in \Gamma(TM^{\perp}). \tag{2.9}$$

Now we shall give the definition of semi-invariant $\xi^{\perp}$-submanifold. According to Bejancu ([4]) M is a semi-invariant $\xi^{\perp}$-submanifold if there exists two orthogonal distributions, D and $D^{\perp}$ in TM such that

$$TM = D \oplus D^{\perp}, \phi D = D, \phi D^{\perp} \subset TM^{\perp}, \tag{2.10}$$

where $\oplus$ denotes the orthogonal sum.

If $D^{\perp} = \{0\}$, then M is an invariant $\xi^{\perp}$-submanifold.The normal bundle can also be decomposed as

$$TM^{\perp} = \phi D^{\perp} \oplus \mu,$$

where $\phi\mu \subset \mu$. Hence μ contains ξ.

§3. Integrability of Distributions on a Semi-Invariant $\xi^{\perp}$-Submanifolds

Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.K manifold $\tilde{M}$. We denote by P and Q the projections of TM on D and $D^{\perp}$ respectively, namely for any $X \in \Gamma(TM)$.

$$X = PX + QX, \tag{3.1}$$

Again, for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$ we put

$$\phi X = tX + \omega X, \tag{3.2}$$

$$\phi N = BN + CN, \tag{3.3}$$

with $tX \in \Gamma(D), BN \in \Gamma(TM)$ and $\omega X, CN \in \Gamma(TM^{\perp})$. Again, for $X \in \Gamma(TM)$, the decomposition is

$$FX = \alpha X + \beta X, \alpha X \in \Gamma(D), \beta X \in \Gamma(TM^{\perp}). \tag{3.4}$$

This section deals with the study of the integrability of both distributions D and $D^{\perp}$. We have the following proposition:

Proposition 3.1 *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.K manifold $\tilde{M}$. Then we obtain*

$$(\nabla_X t)Y = A_{\omega Y}X + Bh(X,Y) \tag{3.5a}$$

$$(\nabla_X \omega)Y = Ch(X,Y) - h(X,tY) - g(FX,Y)\xi. \tag{3.5b}$$

Proof Notice that

$$\begin{aligned}(\tilde{\nabla}_X \phi)YZ &= -\phi\tilde{\nabla}_X Y + \tilde{\nabla}_X \phi Y \\ &= -\phi(\nabla_X Y + h(X,Y)) + \tilde{\nabla}_X TY + \tilde{\nabla}_X tY + \tilde{\nabla}_X \omega Y.\end{aligned}$$

Using (3.3) and (3.4) and the Gauss and Weingarten formula we get

$$(\tilde{\nabla}_X \phi)Y = (-t\nabla_X Y + \nabla_X tY) + (-\omega\nabla_X Y + \nabla^{\perp}_X \omega Y) - Bh(X,Y) - Ch(X,Y) + h(X,tY) - A_{\omega Y}X.$$

After some brief calculations we deduce

$$(\tilde{\nabla}_X \phi)Y = (\nabla_X t)Y + (\nabla_X \omega)Y - Bh(X,Y) - Ch(X,Y) + h(X,tY) - A_{\omega Y}X.$$

Again,

$$(\tilde{\nabla}_X \phi)Y = \tilde{g}(\tilde{\nabla}_{\phi^2 X}\xi, Y)\xi - \eta(Y)\tilde{\nabla}_{\phi^2 X}\xi,$$

Using (2.1) and some steps of calculations, we obtain

$$(\tilde{\nabla}_X \phi)Y = -g(\tilde{\nabla}_X \xi, Y)\xi,$$

as$(\eta(Y) = 0, g(\xi, Y) = 0$ as $\xi \perp D, D^{\perp})$. Hence,

$$(\tilde{\nabla}_X \phi)Y = -g(FX,Y)\xi.$$

On comparing the tangential and normal components we shall obtain the results. □

Taking into the consideration the decomposition of $TM^{\perp}$, we can prove that:

Proposition 3.2 *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.K manifold $\tilde{M}$. Then for nay $N \in \Gamma(TM^{\perp})$, there are*

(1) $BN \in D^{\perp}$;
(2) $CN \in \mu$.

Proof Let $N \in \Gamma(TM^{\perp})$,

$$\phi N = BN + CN,$$

We know $TM^{\perp} = \phi D^{\perp} \oplus \mu$. Therefore we have

$$BN \in \phi D^{\perp} \subseteq D^{\perp}, CN \in \mu. \qquad \square$$

Proposition 3.3 *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.K manifold $\tilde{M}$, then*

$$A_{\omega X}Y = A_{\omega Y}X,$$

for any $X, Y \in \Gamma(TM^{\perp})$.

Proof From equation (2.9) we have

$$\begin{aligned} g(A_{\omega X}Y, Z) &= g(h(Y,Z), \omega X) = g(\tilde{\nabla}_Z Y, \omega X) \\ &= -g(\omega\tilde{\nabla}_Z Y, X) = g(\omega Y, h(Z,X)) \\ &= g(h(X,Z), \omega Y) = g(A_{\omega Y}X, Z). \end{aligned}$$

Hence the result. $\square$

Proposition 3.4 *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.K manifold $\tilde{M}$. Then the distribution $D^{\perp}$ is integrable.*

Proof Let $Z, X \in \Gamma(D^{\perp})$. Then

$$\begin{aligned} \nabla_Z tX &= (\nabla_Z t)X + t\nabla_Z X \\ \nabla_Z tX &= A_{X\omega}Z + Bh(Z,X) + t\nabla_Z X. \end{aligned}$$

Therefore, (i)

$$t\nabla_Z X = \nabla_Z tX - A_{\omega X}Z - Bh(X,Z)$$

Interchanging X and Z we have (ii)

$$t\nabla_X Z = \nabla_X tZ - A_{\omega Z}X - Bh(Z,X).$$

Subtracting equation (ii) from (i) and using Proposition (3.3), we obtain

$$t([Z,X]) = \nabla_Z tX - \nabla_X tZ. \qquad \square$$

Theorem 3.1 *If M is a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.K manifold $\tilde{M}$, then the* *distribution D is integrable if and only if*

$$h(Z,tW)-h(W,tZ)=(\boldsymbol{L}_\xi\tilde{g})(Z,W)\xi, \quad X,Y\in\Gamma(D).$$

Proof From the covariant derivative we have

$$\nabla_Z\omega W=(\tilde{\nabla}_Z\omega W)+\omega\nabla_Z W,$$

$$\nabla_Z\omega W=Ch(Z,W)-h(Z,tW)-g(FZ,W)\xi+\omega(\nabla_Z W+h(Z,W))$$

for $Z,W\in\Gamma(D)$. Again using Weingarten formulae we have

$$\nabla_Z\omega W=-A_{\omega W}Z+\nabla^\perp_Z\omega W.$$

Comparing both the equations we get

$$-A_{\omega W}Z+\nabla^\perp_Z\omega W=Ch(Z,W)-h(Z,tW)-g(FZ,W)\xi+\omega(\nabla_Z W+h(Z,W)).$$

On a simplification we obtain

$$\omega\nabla_W Z=\nabla^\perp_Z\omega W-A_{\omega W}Z-Ch(Z,W)+h(W,tZ)-g(FW,Z)\xi.$$

Interchanging W and Z in the above equation, we get

$$\omega\nabla_Z W=\nabla^\perp_W\omega Z-A_{\omega Z}W-Ch(W,Z)+h(Z,tW)-g(FZ,W)\xi.$$

Subtracting the above two equations and using Proposition 3.3 we get

$$\omega[Z,W]=h(Z,tW)-h(W,tZ)-g(FZ,W)\xi+g(FW,Z)\xi.$$

We also know that

$$(\mathbf{L}_\xi\tilde{g})(Z,W)\xi=g(FZ,W)\xi-g(FW,Z)\xi.$$

Therefore the distribution D is integrable if

$$h(Z,tW)-h(W,tZ)=(\mathbf{L}_\xi\tilde{g})(Z,W)\xi.$$ □

Proposition 3.5 *Let M be a semi-invariant $\xi^\perp$-submanifold of a G.Q.K manifold $\tilde{M}$. Then* $\alpha X=A_\xi X$ *and* $\beta X=-\nabla^\perp_X\xi \qquad X\in\Gamma(TM)$.

Proof From Weingarten formulae we get

$$\tilde{\nabla}_X\xi=-A_\xi X+\nabla^\perp_X\xi.$$

Again we know from (3.4),

$$\tilde{\nabla}_X\xi=-FX=-\alpha X-\beta X.$$

Comparing these formulae we get $\alpha X = A_\xi X$ and $\beta X = -\nabla^\perp_X \xi$ by assuming

$$\{e_i, \phi e_i, e_{2p+j}\}, i = \{1, \cdots, p\}, j = \{1, \cdots, q\}$$

being an adapted orthonormal local frame on M, where $q = dim D^\perp$ and $2p = dim D$. □

Similarly, the following theorem is obtained.

Theorem 3.2 *If M is a $\xi^\perp$-semiinvariant submanifold of a G.Q.K manifold $\tilde{M}$ one has*

$$\eta(H) = \frac{1}{m}\ trace\ (A_\xi); m = 2p + q.$$

Proof From the mean curvature formula

$$H = \frac{1}{m}\sum_{a=1}^{s} trace(A_{\xi_a})\xi_a,$$

where $\{\xi_1, \cdots, \xi_s\}$ is an orthonormal basis in $TM^\perp$,

$$\begin{aligned} \eta(H) &= \frac{1}{m}\sum_{a=1}^{s} trace(A_{\xi_a}) \cdot 1, \\ \eta(H) &= \frac{1}{m} trace(A_\xi). \end{aligned}$$ □

Corollary 3.1 *If the leaves of the integrable distribution D are totally geodesic in M then the structures vector field ξ is D-killing, i.e. $(\boldsymbol{L}_\xi g)(X, Y) = 0$, $X, Y \in \Gamma(D)$.*

Proof We know that

$$\begin{aligned} (\mathbf{L}_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= g(\nabla_X Y, \xi) + g(\xi, \nabla_Y X) = 0, \quad X, Y \in \Gamma(D). \end{aligned}$$

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