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Neighborhood Total 2-Domination in Graphs

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Abstract: Let $G = (V, E)$ be a graph without isolated vertices. A set $S \subseteq V$ is called the neighborhood total 2-dominating set (nt2d-set) of a graph G if every vertex in $V - S$ is adjacent to at least two vertices in S and the induced subgraph $< N(S) >$ has no isolated vertices. The minimum cardinality of a nt2d-set of G is called the neighborhood total 2-domination number of G and is denoted by $\gamma_{2nt}(G)$. In this paper we initiate a study of this parameter.

Key Words: Neighborhood total domination, neighborhood total 2-domination.

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§1. Introduction

The graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3] and Haynes et.al [5-6].

Let $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$ then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $\text{pn}[u, S] = \{v : N[v] \cap S = \{u\}\}$. The chromatic number $\chi(G)$ of a graph G is defined to be the minimum number of colours required to colour all the vertices such that no two adjacent vertices receive the same colour. $H(m_1, m_2, \cdots, m_n)$ denotes the graph obtained from the graph H by attaching m_i edges to the vertex $v_i \in V(H)$, $1 \leq i \leq n$. $H(P_{m_1}, P_{m_2}, \cdots, P_{m_n})$ is the graph obtained from the graph H by attaching the end vertex of P_{m_i} to the vertex v_i in H, $1 \leq i \leq n$.

A subset S of V is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al., Fink and Jacobson [4] introduced the concept of k-domination in graphs. A dominating set S of G is called a k- dominating set if every vertex in $V - S$ is adjacent to at least k vertices in S. The minimum cardinality of a k-dominating set is called k-domination number of G and is denoted by $\gamma_k(G)$. F. Harary and T.W. Haynes [4] introduced the concept of double

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domination in graphs. A dominating set S of G is called a double dominating set if every vertex in $V-S$ is adjacent to at least two vertices in S and every vertex in S is adjacent to at least one vertex in S. The minimum cardinality of a double dominating set is called double domination number of G and is denoted by $dd(G)$. S. Arumugam and C. Sivagnanam [1], [2] introduced the concept of neighborhood connected domination and neighborhood total domination in graphs. A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $< N(S) >$ is connected. The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$. A dominating set S of a graph G without isolate vertices is called the neighborhood total dominating set (ntd-set) if the induced subgraph $\langle N(S)\rangle$ has no isolated vertices. The minimum cardinality of a ntd-set of G is called the neighborhood total domination number of G and is denoted by $\gamma_{nt}(G)$. Sivagnanam et.al [8] studied the concept of neighborhood connected 2-domination in graphs. A set $S \subseteq V$ is called a neighborhood conneccted 2-dominating set (nc2d-set) of a connected graph G if every vertex in $V-S$ is adjacent to at least two vertices in S and the induced subgraph $\langle N(S)\rangle$ is connected. The minimum cardinality of a nc2d-set of G is called the neighborhood connected 2-domination number of G and is denoted by $\gamma_{2nc}(G)$. In this paper we introduce the concept of neighborhood total 2-domination and initiate a study of the corresponding parameter.

Through out this paper we assume the graph G has no isolated vertices.

§2. Neighborhood Total 2-Dominating Sets

Definition 2.1 *A set $S \subseteq V$ is called the neighborhood total 2-dominating set (nt2d-set) of a graph G if every vertex in $V-S$ is adjacent to at least two vertices in S and the induced subgraph $< N(S) >$ has no isolated vertices. The minimum cardinality of a nt2d-set of G is called the neighborhood total 2-domination number of G and is denoted by $\gamma_{2nt}(G)$.*

Remark 2.2 (i) Clearly $\gamma_{2nt}(G) \geq \gamma_{nt}(G) \geq \gamma(G)$, $\gamma_{2nt}(G) \leq \gamma_{2nc}(G)$ and $\gamma_{2nt}(G) \geq \gamma_2(G)$.

(ii) A graph G has $\gamma_{2nt}(G) = 2$ if and only if there exist two vertices $u, v \in V$ such that $(a)\deg u = \deg v = n-1$ or $(b)\deg u = \deg v = n-2$, $uv \notin E(G)$ with $G - \{u, v\}$ has no isolated vertices. Thus $\gamma_{2nt}(G) = 2$ if and only if G is isomorphic to either $H + K_2$ for some graph H or $H + \overline{K_2}$ for some graph H with $\delta(H) \geq 1$.

Examples A

(1) $\gamma_{2nt}(K_n) = 2, n \geq 2$;

(2) $\gamma_{2nt}(K_{1,n-1}) = n$;

(3) Let $K_{r,s}$ be a complete bipartite graph and not a star then

$$\gamma_{2nt}(K_{r,s}) = \begin{cases} 3 & \text{r or s} = 2 \\ 4 & \text{r,s} \geq 3 \end{cases}$$

(4) $\gamma_{2nt}(W_n) = \left\lceil \dfrac{n}{3} \right\rceil + 1$.

Theorem 2.3 *For any non trivial path P_n,*

$$\gamma_{2nt}(P_n) = \begin{cases} \lceil \frac{3n}{5} \rceil + 1 & \text{if} \quad n \equiv 0, 3 \pmod 5 \\ \lceil \frac{3n}{5} \rceil & \textit{otherwise} \end{cases}$$

Proof Let $P_n = (v_1, v_2, \cdots, v_n)$ and $n = 5k + r$, where $0 \leq r \leq 4$, $S = \{v_i \in V : i = 5j+1, 5j+3, 5j+4,\ 0 \leq j \leq k\}$ and

$$S_1 = \begin{cases} S & \text{if} \quad n \equiv 1, 4 \pmod 5 \\ S \cup \{v_n\} & \text{if} \quad n \equiv 0, 2 \pmod 5 \\ S \cup \{v_{n-1}\} & \text{if} \quad n \equiv 3 \pmod 5 \end{cases}$$

Clearly S_1 is a nt2d-set of P_n and hence

$$\gamma_{2nt}(P_n) \leq \begin{cases} \lceil \frac{3n}{5} \rceil + 1 & \text{if} \quad n \equiv 0, 3 \pmod 5 \\ \lceil \frac{3n}{5} \rceil & \text{otherwise} \end{cases}$$

Let S be any γ_{2nt}-set of P_n. Since any 2-dominating set D of order either $\lceil \frac{3n}{5} \rceil, n \equiv 0, 3 (mod\, 5)$ or $\lceil \frac{3n}{5} \rceil - 1, n \equiv 1, 2, 4 (mod\, 5)$, $N(D)$ contains isolated vertices, we have

$$|S| \geq \begin{cases} \lceil \frac{3n}{5} \rceil + 1 & \text{if} \quad n \equiv 0, 3 \pmod 5 \\ \lceil \frac{3n}{5} \rceil & \text{otherwise} \end{cases}$$

Hence,

$$\gamma_{2nt}(P_n) = \begin{cases} \lceil \frac{3n}{5} \rceil + 1 & \text{if} \quad n \equiv 0, 3 \pmod 5 \\ \lceil \frac{3n}{5} \rceil & \text{otherwise.} \end{cases} \qquad \square$$

Theorem 2.4 *For the cycle C_n on n vertices $\gamma_{2nt}(C_n) = \lceil \frac{3n}{5} \rceil$.*

Proof Let $C_n = (v_1, v_2, \cdots, v_n, v_1)$, $n = 5k + r$, where $0 \leq r \leq 4$, $S = \{v_i : i = 5j+1,\ 5j+3,\ 5j+4,\ 0 \leq j \leq\ k\}$ and

$$S_1 = \begin{cases} S \cup \{v_n\} & \text{if} \quad n \equiv 2 (\text{mod } 5) \\ S & \text{otherwise} \end{cases}$$

Clearly S_1 is a nt2d-set of C_n and hence $\gamma_{2nt}(C_n) \leq \lceil \frac{3n}{5} \rceil$. Now, let S be any γ_{2nt}-set of C_n. Since any 2-dominating set D of order $\lceil \frac{3n}{5} \rceil - 1$, $N(D)$ contains isolated vertices, we have

$|S| \geq \lceil \frac{3n}{5} \rceil$. Hence,

$$\gamma_{2nt}(C_n) = \lceil \frac{3n}{5} \rceil. \qquad \square$$

We now proceed to obtain a characterization of minimal nt2d-sets.

Lemma 2.5 *A superset of a nt2d-set is a nt2d-set.*

Proof Let S be a nt2d-set of a graph G and let $S_1 = S \cup \{v\}$, where $v \in V - S$. Clearly $v \in N(S)$ and S_1 is a 2-dominating set of G. Suppose there exists an isolated vertex y in $\langle N(S_1) \rangle$. Then $N(y) \subseteq S - N(S)$ and hence y is an isolated vertex in $\langle N(S) \rangle$, which is a contradiction. Hence $\langle N(S_1) \rangle$ has no isolated vertices and S_1 is a nt2d-set. $\square$

Theorem 2.6 *A nt2d-set S of a graph G is a minimal nt2d-set if and only if for every $u \in S$, one of the following holds.*

(1) $|N(u) \cap S| \leq 1$;

(2) *there exists a vertex* $v \in (V - S) \cap N(u)$ *such that* $|N(v) \cap S| = 2$;

(3) *there exists a vertex* $x \in N(S - \{u\}))$ *such that* $N(x) \cap N(S - \{u\}) = \phi$.

Proof Let S be a minimal nt2d-set and let $u \in S$. Let $S_1 = S - \{u\}$. Then S_1 is not a nt2d-set. This gives either S_1 is not a 2-dominating set or $\langle N(S_1) \rangle$ has an isolated vertex. If S_1 is not a 2-dominating set then there exists a vertex $v \in V - S_1$ such that $|N(v) \cap S_1| \leq 1$. If $v = u$ then $|N(u) \cap (S - \{u\})| \leq 1$ which gives $|N(u) \cap S| \leq 1$. Suppose $v \neq u$. If $|N(v) \cap S_1| < 1$ then $|N(v) \cap S| \leq 1$ and hence S is not a 2-dominating set which is a contradiction. Hence $|N(v) \cap S_1| = 1$. Thus $v \in N(u)$. So $v \in (V - S) \cap N(u)$ such that $|N(v) \cap S| = 2$. If S_1 is a 2-dominating set and if $x \in N(S_1)$ is an isolated vertex in $\langle N(S_1) \rangle$ then $N(x) \cap N(S_1) = \phi$. Thus $N(x) \cap N(S - \{u\}) = \phi$. Conversely, if S is a nt2d-set of G satisfying the conditions of the theorem, then S is 1-minimal nt2d-set and hence the result follows from Lemma 2.5. $\square$

Remark 2.7 Any nt2d-set contains all the pendant vertices of the graph.

Remark 2.8 Since any nt2d-set of a spanning subgraph H of a graph G is a nt2d-set of G, we have $\gamma_{2nt}(G) \leq \gamma_{2nt}(H)$.

Remark 2.9 If G is a disconnected graph with k components $G_1, G_2, \ldots, G_k$ then $\gamma_{2nt}(G) = \gamma_{2nt}(G_1) + \gamma_{2nt}(G_2) + \cdots + \gamma_{2nt}(G_k)$.

Theroem 2.10 *Let G be a connected graph on $n \geq 2$ vertices. Then $\gamma_{2nt}(G) \leq n$ and equality holds if and only if G is a star.*

Proof The inequality is obvious. Let G be a connected graph on n vertices and $\gamma_{2nt}(G) = n$. If $n = 2$ then nothing to prove. Let us assume $n \geq 3$. Suppose G contains a cycle C. Let $x \in V(C)$. Then $V(G) - x$ is a nt2d-set of G, which is a contradiction. Hence G is a tree.

Let u be a vertex such that $degu = \Delta$. Let v be a vertex such that $d(u, v) \geq 2$. Let $(u, x_1, x_2, \ldots, x_k, v), k \geq 1$ be the shortest $u - v$ path. Then $S_1 = V - \{x_k\}$ is a nt2d-set of G which is a contradiction. Hence $d(u, v) = 1$ for all $v \in V(G)$. Thus G is a star. The converse is obvious. □

Corollary 2.11 *Let G be a disconnected graph with $\gamma_{2nt}(G) = n$. Then G is a galaxy.*

Theorem 2.12 *Let T be a tree with $n \geq 3$ vertices. Then $\gamma_{2nt}(T) = n - 1$ if and only if T is a bistar $B(n-3, 1)$ or a tree obtained from a bistar by subdividing the edge of maximum degree once.*

Proof Let u be a support with maximum degree. Suppose there exists a vertex $v \in V(T)$ such that $d(u, v) \geq 4$. Let $(u, x_1, x_2, \cdots, x_k, v)$, $k \geq 3$ be the shortest $u - v$ path then $S_1 = V - \{u, x_k\}$ is a nt2d-set of T which is a contradiction. Hence $d(u, v) \leq 3$ for all $v \in V(T)$.

Case 1. $d(u, v) = 3$ for some $v \in V(T)$.

Suppose there exists an vertex $w \in V(T)$, $w \neq v$ such that $d(u, v) = d(u, w) = 3$. Let P_1 be the $u - v$ path and P_2 be the $u - w$ path. Let $P_1 = (u, v_1, v_2, v)$ and $P_2 = (u, w_1, w_2, w)$. If $v_1 \neq w_1$ then $V - \{v_1, w_1\}$ is a nt2d-set of T which is a contradiction. If $v_1 = w_1$ and $v_2 \neq w_2$ then $V - \{v_2, w_2\}$ is a nt2d-set of T which is a contradiction. Hence all the pendant vertices w such that $d(u, w) = 3$ are adjacent to the same support. Let it be x. Let $P = (u, v_1, x)$ be the unique $u - x$ path in T. Let $y \in N(u) - \{v_1\}$. If $\deg y \geq 2$ then $V - \{x, y\}$ is a nt2d-set of T which is a contradiction. Hence T is a tree obtained from a bistar by subdividing the edge of maximum degree once.

Case 2. $d(u, v) \leq 2$ for all $v \in V(T)$.

If $d(u, v) = 1$ for all $v \in V(T)$ and $v \neq u$ then T is a star, which is a contradiction.Hence $d(u, v) = 2$ for some $v \in V(T)$.Suppose there exist two vertices v and w such that $d(u, v) = d(u, w) = 2$. Let P_1 be the $u - v$ path and P_2 be the $u - w$ path. Let $P_1 = (u, v_1, v)$ and $P_2 = (u, w_1, w)$. If $v_1 \neq w_1$ then $V - \{v_1, w_1\}$ is a nt2d-set of T which is a contradiction. If $v_1 = w_1$ then $V - \{u_1, v_1\}$ is a nt2d-set of T which is a contradiction. Hence exactly one vertex $v \in V$ such that $d(u, v) = 2$. Hence T is isomorphic to $B(n-3, 1)$. The converse is obvious. □

Theorem 2.13 *Let G be an unicyclic graph. Then $\gamma_{2nt}(G) = n-1$ if and only if G is isomorphic to C_3 or C_4 or $K_3(n_1, 0, 0), n_1 \geq 1$.*

Proof Let G be an unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_r, v_1)$. If $G = C$ then by theorem 2.4, $G = C_3$ or C_4. Suppose $G \neq C$. Let A be the set of all pendant vertices in G. Clearly A is a subset of any γ_{2nt}-set of G.

Claim 1. Vertices of C of degree more than two or non adjacent.

Let v_i and v_j be the vertices of degree more than two in C. If v_i and v_j are adjacent then $V - \{v_i, v_j\}$ is a nt2d-set of G which is a contradiction. Hence vertices of C of degree more than two or non adjacent.

Claim 2. $d(C, w) = 1$ for all $w \in A$.

Suppose $d(C, w) \geq 2$ for some $w \in A$. Let $(v_1, w_1, w_2, \cdots, w_k, w)$ be the unique $v_1 - w$ path in G, $k \geq 1$. Then $V - \{w_1, v_2\}$ is a nt2d-set of G which is a contradiction. Hence $d(C, w) = 1$

for all $w \in A$.

Claim 3. $r = 3$.

Suppose $r \geq 5$. Let $v_1 \in V(C)$ such that $\deg v_1 \geq 3$. Then $V - \{v_1, v_3\}$ is a nt2d-set of G which is a contradiction. If $r = 4$ then $V - \{v_2, v_4\}$ is a nt2d-set of G which is a contradiction. Hence $r = 3$ and G is isomorphic to $K_3(n_1, 0, 0), n_1 \geq 1$. The converse is obvious. □

Problem 2.14 *Characterize the class of graphs for which $\gamma_{2nt}(G) = n - 1$.*

Theorem 2.15 *Let G be a graph with $\delta(G) \geq 2$ then $\gamma_{2nt}(G) \leq 2\beta_1(G)$.*

Proof Let G be a graph with $\delta(G) \geq 2$ and M be a maximum set of independent edges in G. Let S be the vertices in the set of edges of M. Since $V - S$ is an independent set, each $v \in V - S$ must have at least two neighbors in S. Also since S contains no isolated vertices, $\langle N(S) \rangle = G$ and hence $\langle N(S) \rangle$ contains no isolated vertices.Hence S is a nt2d-set of G. Thus $\gamma_{2nt}(G) \leq 2\beta_1(G)$. □

Problem 2.16 *Characterize the class of graphs for which $\gamma_{2nt}(G) = 2\beta_1(G)$.*

Notation 2.17 The graph G^* is a graph with the vertex set can be partition into two sets V_1 and V_2 satisfying the following conditions:

(1) $\langle V_1 \rangle = K_2 \cup \overline{K_s}$;
(2) $\langle V_2 \rangle$ is totally disconnected;
(3) $degw = 2$ for all $w \in V_2$;
(4) $\langle V_2 \cup \{u, v\} \rangle$, where $u, v \in V_1$ with $deg_{\langle V_1 \rangle} u = deg_{\langle V_1 \rangle} v = 1$, has no isolated vertices.

Theorem 2.18 *For any graph G, $\gamma_{2nt}(G) \geq \frac{2n+1-m}{2}$ and the equality holds if and only if G is isomorphic to $B(2,2)$ or $K_3(1,1,0)$ or $K_4 - e$ or $K_2 + \overline{K_{n-2}}$ or G^*.*

Proof Let S be a γ_{2nt}-set of G. Then each vertex of $V - S$ is adjacent to at least two vertices in S and since $\langle N(S) \rangle$ has no isolated vertices either $V - S$ or S contains at least one edge. Hence the number of edges $m \geq 2|V - S| + 1 = 2n - 2\gamma_{2nt} + 1$. Then $\gamma_{2nt} \geq \frac{2n+1-m}{2}$.

Let G be a graph with $\gamma_{2nt}(G) = \frac{2n+1-m}{2}$ and let S be the $\gamma_{2nt}-$ set of G. Suppose $|E(\langle S \rangle \cup \langle V - S \rangle)| \geq 2$. Then $m \geq 2(|V - S|) + 2$ and hence $\gamma_{2nt}(G) \geq \frac{2n+2-m}{2}$ which is a contradiction. Hence either $|E(\langle S \rangle)| = 1$ or $|E(\langle V - S \rangle)| = 1$. Suppose $|E(\langle V - S \rangle)| = 1$ then $|E\langle S \rangle)| = 0$ and hence $V - S = N(S)$. Since $\langle N(S) \rangle$ has no isolated vertices, $V - S = K_2$. Let $V - S = \{u, v\}$. If $degu \geq 3$ or $degv \geq 3$ then $m \geq 2(V - S) + 2$. Hence $\gamma_{2nt} \geq \frac{2n+2-m}{2}$ which is a contradiction. Hence $degu = 2$ and $degv = 2$. Then $|S| \leq 4$. If $|S| = 4$ then G is isomorphic to $B(2,2)$. If $|S| = 3$ then G is isomorphic to $K_3(1,1,0)$. If $|S| = 2$ then G is isomorphic to $K_4 - e$.

Suppose $|E\langle S \rangle| = 1$ then $|E\langle V - S \rangle| = 0$. Let $|S| = 2$. Since every vertex in $V - S$ is adjacent to both the vertices in S we have G is isomorphic to $K_2 + \overline{K_{n-2}}$. If $|S| \geq 3$ then G is isomorphic to G^*. The converse is obvious. □

Corollary 2.19 *For a tree T, $\gamma_{2nt}(T) \geq \frac{n+2}{2}$.*

Problem 2.20 *Characterize the class of trees for which* $\gamma_{2nt}(T) = \frac{n+2}{2}$.

Theorem 2.21 *For any graph* G, $\gamma_{2nt}(G) \geq \frac{2n}{(\Delta+2)}$

Proof Let S be a minimum nt2d-set and let k be the number of edges between S and $V-S$. Since the degree of each vertex in S is at most Δ, $k \leq \Delta\gamma_{2nt}$. But since each vertex in $V-S$ is adjacent to at least 2 vertices in S, $k \geq 2(n-\gamma_{2nt})$ combining these two inequalities produce $\gamma_{2nt}(G) \geq \frac{2n}{\Delta+2}$. □

Problem 2.22 *Characterize the class of graphs for which* $\gamma_{2nt}(G) = \frac{2n}{\Delta+2}$.

§3. Neighborhood Total 2-Domination Numbers and Chromatic Numbers

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J. Paulraj Joseph and S. Arumugam [7] proved that $\gamma(G) + \chi(G) \leq n+1$. They also characterized the class of graphs for which the upper bound is attained. In the following theorems we find an upper bound for the sum of the neighborhood total 2-domination number and chromatic number of a graph, also we characterized the corresponding extremal graphs.

We define the following graphs:

(1) G_1 is the graph obtained from $K_4 - e$ by attaching a pendant vertex to any one of the vertices of degree two by an edge.

(2) G_2 is the graph obtained from $K_4 - e$ by attaching a pendant vertex to any one of the vertices of degree three by an edge.

(3) G_3 is the graph obtained from $(K_4 - e) \cup K_1$ by joining a vertex of degree three, vertex of degree two to the vertex of degree zero by an edge.

(4) G_4 is the graph obtained from $C_5 + e$ by adding an edge between two non adjacent vertices of degree two.

(5) G_5 is the graph obtained from K_4 by subdividing an edge once.

(6) G_6 is the graph obtained from $C_5 + e$ by adding an edge between two non adjacent vertices with one has degree three and another has degree two.

(7) $G_7 = K_5 - Y_1$ where Y_1 is a maximum matching in K_5.

Theorem 3.1 *For any connected graph* G, $\gamma_{2nt}(G) + \chi(G) \leq 2n$ *and equality holds if and only if* G *is isomorphic to* K_2.

Proof The inequality is obvious. Now we assume that $\gamma_{2nt}(G) + \chi(G) = 2n$. This implies $\gamma_{2nt}(G) = n$ and $\chi(G) = n$. Then G is a complete graph and a star. Hence G is isomorphic to K_2. The converse is obvious. □

Theorem 3.2 *Let* G *be a connected graph. Then* $\gamma_{2nt}(G) + \chi(G) = 2n-1$ *if and only if* G *is isomorphic to* K_3 *or* P_3.

Proof Let us assume $\gamma_{2nt}(G) + \chi(G) = 2n-1$. This is possible only if (i) $\gamma_{2nt}(G) = n$

and $\chi(G) = n-1$ or (ii) $\gamma_{2nt}(G) = n-1$ and $\chi(G) = n$. Let $\gamma_{2nt}(G) = n$ and $\chi(G) = n-1$. Then G is a star and hence $n = 3$.Thus G is isomorphic to P_3. Suppose (ii) holds. Then G is a complete graph with $\gamma_{2nt}(G) = n-1$. Then $n = 3$ and hence G is isomorphic to K_3. The converse is obvious. □

Theorem 3.3 *For any connected graph G, $\gamma_{2nt}(G) + \chi(G) = 2n-2$ if and only if G is isomorphic to K_4 or $K_{1,3}$ or $K_3(1,0,0)$.*

Proof Let us assume $\gamma_{2nt}(G) + \chi(G) = 2n-2$. This is possible only if $\gamma_{2nt}(G) = n$ and $\chi(G) = n-2$ or $\gamma_{2nt}(G) = n-1$ and $\chi(G) = n-1$ or $\gamma_{2nt}(G) = n-2$ and $\chi(G) = n$.

Let $\gamma_{2nt}(G) = n$ and $\chi(G) = n-2$. Since $\gamma_{2nt}(G) = n$ we have G is a star with $\chi(G) = n-2$. Hence $n = 4$. Thus G isomorphic to $K_{1,3}$.

Suppose $\gamma_{2nt}(G) = n-1$ and $\chi(G) = n-1$. Since $\chi(G) = n-1$, G contains a complete subgraph K on $n-1$ vertices. Let $V(K) = \{v_1, v_2, \cdots, v_{n-1}\}$ and $V(G) - V(K) = \{v_n\}$. Then v_n is adjacent to v_i for some vertex $v_i \in V(K)$. If $\deg(v_n) = 1$ and $n \geq 4$ then $\{v_i, v_j, v_n\},\ i \neq j$ is a γ_{2nt}-set of G. Hence $n = 4$ and $K = K_3$. Thus G is isomorphic to $K_3(1,0,0)$. If $\deg v_n = 1$ and $n = 3$ then G is isomorphic to P_3 which is a contradiction to $\gamma_{2nt} = n-1$.If $\deg(v_n) > 1$ then $\gamma_{2nt} = 2$. Then $n = 3$ which gives G is isomorphic to K_3 which is a contradiction to $\chi(G) = n-1$.

Suppose $\gamma_{2nt}(G) = n-2$ and $\chi(G) = n$. Since $\chi(G) = n$, G is isomorphic to K_n. But $\gamma_{2nt}(K_n) = 2$ we have $n = 4$. Hence G is isomorphic to K_4. The converse is obvious. □

Theorem 3.4 *Let G be a connected graph. Then $\gamma_{2nt}(G) + \chi(G) = 2n-3$ if and only if G is isomorphic to C_4 or $K_{1,4}$ or P_4 or K_5 or $K_3(2,0,0)$ or $K_4(1,0,0,0)$ or $K_4 - e$.*

Proof Let $\gamma_{2nt}(G) + \chi(G) = 2n-3$. This is possible only if (i) $\gamma_{2nt}(G) = n,\ \chi(G) = n-3$ or (ii) $\gamma_{2nt}(G) = n-1,\ \chi(G) = n-2$ or (iii) $\gamma_{2nt}(G) = n-2,\ \chi(G) = n-1$ or (iv) $\gamma_{2nt}(G) = n-3,\ \chi(G) = n$.

Suppose (i) holds. Then G is a star with $\chi(G) = n-3$.Then $n = 5$. Hence G is isomorphic to $K_{1,4}$. Suppose (ii) holds. Since $\chi(G) = n-2$, G is either $C_5 + K_{n-5}$ or G contains a complete subgraph K on $n-2$ vertices. If $G = C_5 + K_{n-5}$ then $\gamma_{2nt}(G) + \chi(G) \neq 2n-3$.Thus G contains a complete subgraph K on $n-2$ vertices. Let $X = V(G) - V(K) = \{v_1, v_2\}$ and $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$.

Case 1. $\langle X \rangle = K_2$.

Since G is connected, without loss of generality we assume v_1 is adjacent to v_3. If $|N(v_1) \cap N(v_2)| \geq 2$ then $\gamma_{2nt}(G) = 2$ and hence $n = 3$ which is a contradiction. So $|N(v_1) \cap N(v_2)| \leq 1$. Then $\{v_2, v_3, v_4\}$ is a γ_{2nt}-set of G and hence $n = 4$. If $|N(v_1) \cap N(v_2)| = 1$ then G is either $K_4 - e$ or $K_3(1,0,0)$. For these graphs $\chi(G) = 3$ which is a contradiction. If $N(v_1) \cap N(v_2) = \phi$. Then G is isomorphic to P_4 or C_4 or $K_3(1,0,0)$. Since $\chi[K_3(1,0,0)] = 3$, we have G is isomorphic to P_4 or C_4.

Case 2. $\langle X \rangle = \overline{K_2}$.

Since G is connected v_1 and v_2 are adjacent to at least one vertex in K. If $\deg v_1 =$

$\deg v_2 = 1$ and $N(v_1) \cap N(v_2) \neq \phi$ then $|V(K)| \neq 1$. So $|V(K)| \geq 2$. If $|V(K)| = 2$ then G is isomorphic to $K_{1,3}$ which is a contradiction.Hence $|V(K)| \geq 3$.Then $\{v_1, v_2, v_3, v_4\}$ is a γ_{2nt}-set of G. Hence $n = 5$. Thus G is isomorphic to $K_3(2,0,0)$. If $\deg v_1 = \deg v_2 = 1$ and $N(v_1) \cap N(v_2) = \phi$ then $|V(K)| \geq 2$. If $|V(K)| = 2$ then G is isomorphic to P_4. If $V(K) \geq 3$ then $\{v_1, v_2, v_3, v_4\}$ is a γ_{2nt}-set of G. Hence $n = 5$. Thus G is isomorphic to $K_3(1,1,0)$. But $\gamma_{2nt}[K_3(1,1,0)] = 3$ which is a contradiction. Suppose $\deg v_1 \geq 2$ and $|N(v_1) \cap N(v_2)| \leq 1$ then $\{v_2, v_3, v_4\}$ where $v_3, v_4 \in N(v_1)$ is a γ_{2nt}-set of G. Hence $n = 4$. Then G is isomorphic to $K_3(1,0,0)$. For this graph $\gamma_{2nt}(G) = 3$ and $\chi(G) = 3$ which is a contradiction. If $\deg v_1 \geq 2$ and $|N(v_1) \cap N(v_2)| \geq 2$ then $\{v_3, v_4\}$ where $v_3, v_4 \in N(v_1) \cap N(v_2)$ is a γ_{2nt}-set of G. Then $n = 3$ which gives a contradiction.

Suppose (3) holds. Since $\chi(G) = n - 1$, G contains a clique K on $n - 1$ vertices. Let $X = V(G) - V(K) = \{v_1\}$. If $\deg v_1 \geq 2$ then $\gamma_{2nt}(G) = 2$. Hence $n = 4$. Thus G is isomorphic to $K_4 - e$. If $\deg v_1 = 1$ then $|V(K)| \geq 3$ and hence $\{v_1, v_2, v_3\}$ where $v_2 \in N(v_1)$ is a γ_{2nt}-set of G and hence $n = 5$. Thus G is isomorphic to $K_4(1,0,0,0)$.

Suppose (iv) holds. Since $\chi(G) = n$, G is a complete graph. Then $\gamma_{2nt}(G) = 2$ and hence $n = 5$. Therefore G is isomorphic to K_5. The converse is obvious. □

Theorem 3.5 *Let G be a connected graph. Then $\gamma_{2nt}(G)+\chi(G) = 2n-4$ if and only if G is isomorphic to one of the following graphs $P_5,\ K_6,\ C_5,\ K_{1,5},\ B(2,1),\ K_5(1,0,0,0,0),\ K_4(2,0,0,0),\ K_4(1,1,0,0),\ K_3(1,1,0), K_3(3,0,0),\ C_5+e,\ 2K_2+K_1$ and K_5-Y, where Y is the set of edges incident to a vertex with $|Y| = 1$ or 2, G_i, $\quad 1 \leq i \leq 7$.*

Proof Let $\gamma_{2nt}(G) + \chi(G) = 2n - 4$. This is possible only if

(1) $\gamma_{2nt}(G) = n, \quad \chi(G) = n - 4$, or
(2) $\gamma_{2nt}(G) = n - 1, \quad \chi(G) = n - 3$, or
(3) $\gamma_{2nt}(G) = n - 2, \quad \chi(G) = n - 2$, or
(4) $\gamma_{2nt}(G) = n - 3, \quad \chi(G) = n - 1$, or
(5) $\gamma_{2nt}(G) = n - 4, \quad \chi(G) = n$.

Case 1. $\gamma_{2nt}(G) = n, \quad \chi(G) = n - 4$.

Then G is a star and hence G is isomorphic to $K_{1,5}$.

Case 2. $\gamma_{2nt}(G) = n - 1, \quad \chi(G) = n - 3$.

Since $\chi(G) = n - 3$, G contains a clique K on $n - 3$ vertices. Let $X = V(G) - V(K) = \{v_1, v_2, v_3\}$ and let $V(G) = \{v_1, v_2, \cdots, v_n\}$.

Subcase 2.1 $\langle X \rangle = \overline{K_3}$.

Let all v_1, v_2 and v_3 be pendant vertices and $|V(K)| = 1$ then G is a star which is a contradiction. So we assume that v_1, v_2 and v_3 be the pendant vertices and $|V(K)| = 2$. If all v_1, v_2 and v_3 are adjacent to same vertices in K then G is isomorphic to $K_{1,4}$ which is a contradiction. If $N(v_1) \cap N(v_2) = \{v_4\}$ and $N(v_3) = \{v_5\}$ then G is isomorphic to $B(2,1)$. Let v_1, v_2 and v_3 be the pendant vertices and $|V(K)| = 3$. If all v_1, v_2 and v_3 are adjacent to same vertices in K then G is isomorphic to $K_3(3,0,0)$. If $N(v_1) \cap N(v_i) = \phi, i \neq 1$ then $\gamma_{2nt} \neq n - 1$

which is a contradiction. If $|V(K)| \geq 4$ then $\gamma_{2nt} \leq n-2$ which is a contradiction.

Suppose $\deg v_1 \geq 2$ then $\{v_2, v_3, v_4, v_5\}$ where $v_4, v_5 \in N(v_1)$ is a nt2d-set then $\gamma_{2nt}(G) \leq 4$. Hence $n \leq 5$. Since $\deg v_1 \geq 2$, $n = 5$. Then G contains K_3 and hence $\chi(G) \neq n-3$ which is a contradiction.

Subcase 2.2 $\langle X \rangle = K_2 \cup K_1$.

Let $v_1v_2 \in E(G)$ and $\deg v_3 = 1$. Suppose $\deg v_2 = 1$ and $\deg v_1 = 2$ and $N(v_1) \cap N(v_3) = \{v_4\}$. If $\deg v_4 = 2$ then G is isomorphic to P_4 and $\gamma_{2nt}(G) + \chi(G) \neq 2n-4$ which is a contradiction. If $\deg v_4 \geq 3$ then $\{v_2, v_3, v_4, v_5\}$ is γ_{2nt}-set of G. Therefore $n = 5$. Then G is isomorphic to a bistar $B(2,1)$. If $N(v_1) \cap N(v_3) = \phi$ then K contains at least 2 vertices. If $|V(K)| \geq 3$ then $\gamma_{2nt}(G) = 4$ and hence $n = 5$ which is a contradiction. So $|V(K)| = 2$ and hence G is isomorphic to P_5.

Suppose $\deg v_3 = 1$, $\deg v_2 = 1$ and $\deg v_1 \geq 3$. Then $\gamma_{2nt}(G) \leq 4$ and hence $n = 5$. This gives $|V(K)| = 2$. Then G is isomorphic to $K_3(1,1,0)$ and hence $\gamma_{2nt}(G) = 3$ which is a contradiction.

Suppose $\deg v_1 \geq 3$, $\deg v_2 \geq 2$ and $deg\, v_3 = 1$. Then $\gamma_{2nt}(G) \leq 4$ and hence $n = 5$. Then G is isomorphic to the either $K_4(1,0,0,0)$ or a graph obtained from $K_4 - e$ by attaching a pendant vertex to one of the vertices of degree 2. For this graphs $\chi(G) \neq n-3$ which is a contradiction. If $deg\, v_1 \geq 3$, $deg\, v_2 \geq 3$ and $deg\, v_3 \geq 2$ then $\gamma_{2nt}(G) \leq 4$ and hence $n = 5$. Then G is isomorphic to the graph which is obtained from $K_4 \cup K_1$ by including two edges between a vertex of degree zero and any two vertices of degree three. For this graph $\chi(G) = 4$ which is a contradiction.

Subcase 2.3. $\langle X \rangle = P_3$.

Let $\langle X \rangle = (v_1, v_2, v_3)$. Since G is connected at least one vertex of $\langle X \rangle$ is adjacent to K. Let $N(v_1) \cap V(K) \neq \phi$ and $N(v_i) \cap V(K) = \phi$ for $i = 2, 3$. Let $|N(v_1) \cap V(K)| = 1$ then $\{v_1, v_3, v_4, v_5\}$ is a γ_{2nt}-set of G. Hence $n = 5$. Therefore G is isomorphic to P_5. If $|N(v_1) \cap V(K)| \geq 2$ then $\gamma_{2nt}(G) \leq 4$. Hence $n = 5$. Then G is isomorphic to $K_3(P_3, P_1, P_1)$. For this graph $\gamma_{2nt}(G) = 3$ which is a contradiction.

Suppose $N(v_2) \cap V(K) \neq \phi$ and $N(v_i) \cap V(K) = \phi$ for $i = 1, 3$. If $|N(v_2) \cap V(K)| = 1$ then G is isomorphic to $B(2,1)$. If $|N(v_2) \cap V(K)| \geq 2$ then $\gamma_{2nt}(G) = 4$ and hence $n = 5$. Hence G is isomorphic to $K_3(2,0,0)$. For this graph $\chi(G) = 3 \neq n-3$ which is a contradiction. If $|N(v_1) \cap V(K)| \geq 2$, $|N(v_2) \cap V(K)| = 1$ and $N(v_3) \cap V(K) = \phi$ then G is a graph obtained from $K_4 - e$ by attaching a pendant vertex to one of the vertices of degree 2. For this graph $\chi(G) \neq n-3$ which is a contradiction.

If $|N(v_1) \cap V(K)| \geq 2$ and $|N(v_2) \cap V(K)| \geq 2$ and $N(v_3) \cap V(K) = \phi$then G is isomorphic to $K_4(1,0,0,0)$. For this graph $\chi(G) = 4$ which is a contradiction. If $|N(v_i) \cap V(K)| \geq 1$ for all $i = 1, 2, 3$ then $\gamma_{2nt}(G) \leq 4$. Hence $n = 5$. Then G is isomorphic to any of the following graphs:

(i) the graph obtained from $K_4 - e$ by attaching a pendant vertex to any one of the vertices of degree 3;

(ii) the graph obtained from $K_4 - e$ by subdividing the edge with the end vertices having degree 3 once;

(iii) $C_5 + e$.

For these graphs either $\gamma_{2nt}(G) \neq n-1$ or $\chi(G) \neq n-3$ which is a contradiction.

Subcase 2.4 $\langle X \rangle = K_3$.

Then any two vertices from X and two vertices from $V - X$ form a nt2d-set and hence $\gamma_{2nt}(G) \leq 4$. Then $n \leq 5$. For these graphs $\chi(G) \geq 3$ which is a contradiction.

Case 3. $\gamma_{2nt}(G) = n-2$ and $\chi(G) = n-2$.

Since $\chi(G) = n-2$, G is either $C_5 + K_{n-5}$ or G contains a clique K on $n-2$ vertices. If $G = C_5 + K_{n-5}$ and $n \geq 6$ then $\gamma_{2nt}(G) + \chi(G) \neq 2n-4$ which is a contradiction.Hence $n = 5$. Thus $G = C_5$. Let G contains a clique K on $n-2$ vertices. Let $X = V(G) - V(K) = \{v_1, v_2\}$.

Subcase 3.1 $\langle X \rangle = \overline{K_2}$.

Since G is connected v_1 and v_2 are adjacent to at least one vertex in K. If $\deg v_1 = \deg v_2 = 1$ and $N(v_1) \cap N(v_2) \neq \phi$ then $V(K) = 4$. Hence G is isomorphic to $K_4(2,0,0,0)$. If $\deg v_1 = \deg v_2 = 1$ and $N(v_1) \cap N(v_2) = \phi$ then G is isomorphic to $K_3(1,1,0)$ or $K_4(1,1,0,0)$.

Suppose $deg\, v_1 \geq 2$ and $|N(v_1) \cap N(v_2)| \leq 1$ then $\{v_2, v_3, v_4\}$ where $v_3, v_4 \in N(v_1)$ is a γ_{2nt}-set of G. Hence $n = 5$. Then G is isomorphic to G_1 or G_2 or G_3. If $\deg v_1 \geq 2$ and $|N(v_1) \cap N(v_2)| \geq 2$ then $\gamma_{2nt}(G) = 2$. Hence $n = 4$ with $\chi(G) = 3$ which is a contradiction.

Subcase 3.2 $\langle X \rangle = K_2$.

Since G is connected, without loss of generality we assume v_1 is adjacent to v_3. If $|N(v_1) \cap N(v_2)| \geq 2$ then $\gamma_{2nt}(G) = 2$ and hence $n = 4$. Thus G is K_4 which is a contradiction. So $|N(v_1) \cap N(v_2)| \leq 1$. Then $\{v_2, v_3, v_4\}$ is a γ_{2nt}-set of G and hence $n = 5$. If $\deg v_2 = 1$ then G is isomorphic to $K_3(P_3, P_1, P_1)$ or the graph obtained from $K_4 - e$ by attaching a pendant vertex to any one of the vertices of degree 2. If $\deg (v_2) \geq 2$ then G is isomorphic to $C_5 + e$ or $2K_2 + K_1$ or G_4 or G_5 or G_6 or G_7.

Case 4. $\gamma_{2nt}(G) = n-3$ and $\chi(G) = n-1$.

Then G contains a clique K on $n-1$ vertices. Let $X = V(G) - V(K) = \{v_1\}$. If $\deg v_1 \geq 2$ then $\gamma_{2nt}(G) = 2$. Hence $n = 5$. Thus G is isomorphic to $K_5 - Y$ where Y is the set of edges incident to a vertex with $|Y| = 1$ or 2. If $\deg v_1 = 1$ then $\{v_1, v_2, v_3\}$ be the γ_{2nt}-set of G. Hence $n = 6$. Thus G is isomorphic to $K_5(1,0,0,0,0)$.

Case 5. $\gamma_{2nt}(G) = n-4$ and $\chi(G) = n$

Then G is a complete graph. Hence $n = 6$. Therefore G is isomorphic to K_6. The converse is obvious. □

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