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Number of Spanning Trees for Shadow of Some Graphs

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Abstract: In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this paper we derive simple formulas of the complexity, number of spanning trees of shadow of some graphs, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

Key Words: Complexity of graphs, number of spanning trees, shadow graphs, Chebyshev polynomials.

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§1. Introduction

In this work we deal with simple and finite undirected graphs $G = (V, E)$, where V is the vertex set and E is the edge set. For a graph G, a spanning tree in G is a tree which has the same vertex set as G. The number of spanning trees in G, also called, the complexity of the graph, denoted by $\tau(G)$, is a well-studied quantity (for long time). A classical result of Kirchhoff [16] can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = \{v_1, v_2, \cdots, v_n\}$, then the Kirchhoff matrix H defined as $n \times n$ characteristic matrix $H = D - A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, $H = [a_{ij}]$ defined as follows:

(1) $a_{ij} = -1$,v_i and v_j are adjacent and $i \neq j$;
(2) a_{ij} equals the degree of vertex v_i if $i = j$, and
(3) $a_{ij} = 0$ otherwise. All of co-factors of H are equal to $\tau(G)$.

There are other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ denote the eignvalues of H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [15] shown that $\tau(G) = 1/p \prod_{k=1}^{p-1} \mu_k$. The formula for the number of

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spanning trees in a d-regular graph G can be expressed as $\tau(G) = 1/p\prod_{k=1}^{p-1}(d-\lambda_k)$ where $\lambda_0 = d, \lambda_1, \lambda_2, \cdots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exists simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such result is due to Cayley [3] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2} \quad, n \geq 2$. Another result, $\tau(K_{p,q}) = p^{q-1}q^{p-1}, \quad p, q \geq 1$,where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in e.g.,([4],[18]). Another result is due to Sedlacek [19] who derived a formula for the wheel on n+1 vertices,W_{n+1}, he showed that $\tau(W_{n+1}) = (3+\sqrt{5}/2)^n + (3-\sqrt{5}/2)^n - 2$ for $n \geq 3$. Sedlacek [20] also later derived a formula for the number of spanning trees in a Mobius ladder.The Mobius ladder M_n,$\tau(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$ for $n \geq 2$. Another class of graphs for which an explicit formula has been derived is based on a prism. Boesch, et al.[1] and [2]. Douad,([5]-[14]) later derived formulas for the number of spanning trees for many graphs. Now, we can introduce the following lemma:

Lemma 1.1([5]) $\tau(G) = \frac{1}{n^2}det(nI - \bar{D} + \bar{A})$, *where* $\bar{A}$, $\bar{D}$ *are the adjacency and degree matrices of* $\bar{G}$, *the complement of* G, *respectively and* I *is the* $n \times n$ *unit matrix.*

The advantage of these formula is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

§2. Chebyshev Polynomials

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, Yuanping, et al. [21].

Let $A_n(x)$ be $n \times n$ matrix such that

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & & \\ -1 & 2x & -1 & 0 & \\ 0 & \ddots & \ddots & & \ddots \\ & \ddots & \ddots & \ddots & -1 \\ & & 0 & -1 & 2x \end{pmatrix},$$

where all other elements are zeros. Further we recall that the Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x)\ . \tag{2.1}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)}.\ . \tag{2.2}$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \ . \tag{2.3}$$

It can then be shown from this recursion that by expanding $\det A_n(x)$ one gets

$$U_n(x) = \det(A_n(x)), n \geq 1 \ . \tag{2.4}$$

Furthermore, by using standard methods for solving the recursion (2.3), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}], n \geq 1, \ . \tag{2.5}$$

where the identity is true for all complex x (except at $x = \pm 1$ where the function can be taken as the limit). The definition of $U_n(x)$ easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x) = 2^{n-1}\prod_{j=1}^{n-1}(x - cos(\frac{j\pi}{n})).$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1}U_{n-1}(x)$$

These two results yield another formula for $U_n(x)$,

$$U_{n-1}^2(x) = 4^{n-1}\prod_{j=1}^{n-1}(x^2 - cos^2(\frac{j\pi}{n})).$$

Finally, simple manipulation of the above formula yields the following, which also will be extremely useful to us latter:

$$U_{n-1}^2(\sqrt{\frac{x+2}{4}}) = \prod_{j=1}^{n-1}(x - 2cos(\frac{2j\pi}{n})).$$

Furthermore one can show that

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}[1 - T_{2n}] = \frac{1}{2(1-x^2)}[1 - T_n(2x^2-1)].$$

and

$$T_n(x) = \frac{1}{2}[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n].$$

Now let $B_n(x)$,$C_n(x)$,$D_n(x)$ and $E_n(x)$ be $n \times n$ matrices.

Lemma 2.1([18])

(*i*)

$$B_n(x) = \begin{pmatrix} x & -1 & 0 & & \\ -1 & 1+x & -1 & 0 & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & -1 & x+1 & -1 \\ & & 0 & -1 & x \end{pmatrix} \Longrightarrow \det(B_n(x)) = (x-1)U_{n-1}(\frac{1+x}{2}).$$

(ii)

$$C_n(x)=\begin{pmatrix} x & 0 & 1 & & \\ 0 & 1+x & 0 & \ddots & \\ 1 & 0 & \ddots & \ddots & 1 \\ & \ddots & \ddots & x+1 & 0 \\ & & 1 & 0 & x \end{pmatrix} \Longrightarrow \det(C_n(x))=(n+x-2)U_{n-1}(\frac{x}{2}), n\geq 3, x>2.$$

(iii)

$$D_n(x)=\begin{pmatrix} x & 0 & 1 & \cdots & 0 \\ 0 & x & 0 & \ddots & 1 \\ 1 & 0 & x & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 0 & x \end{pmatrix} \Longrightarrow \det(D_n(x))=\frac{2(x+n-3)}{x-3}[T_n(\frac{x-1}{2})], n\geq 3, x\geq 2.$$

(iv)

$$E_n(x)=\begin{pmatrix} x & 1 & 1 & \cdots & \cdots & 1 \\ 1 & x & 1 & \ddots & & \vdots \\ 1 & \ddots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & & \ddots & \ddots & x & 1 \\ 1 & \cdots & \cdots & 1 & 1 & x \end{pmatrix} \Longrightarrow \det(E_n(x))=(x+n-1)(x-1)^{n-1}.$$

Lemma 2.2([17]) *Let $A\in F^{n\times n}$, $B\in F^{n\times m}$, $C\in F^{m\times n}$ and $D\in F^{m\times m}$ and assume that D is nonsingular matrix. Then*

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}=(-1)^{nm}\det(A-BD^{-1}C)\det D.$$

This formula gives some sort of symmetry in some matrices which failitates our calculation of determinants.

§3. Complexity of Some Graphs

A shadow graph $D_2(G)$ of a graph G is obtained by taking two copies of G say G_1 and G_2 and join each vertex u_i in G_1 to the neighbors of the corresponding vertex v_i in G_2.

Theorem 3.1 *Let P_n be a path graph of order n. Then*

$$\tau(D_2(P_n)) = 2^{3n-4}; n \geq 2.$$

Proof Applying Lemma 1.1, we have

$$\begin{aligned}
\tau(D_2(P_n)) &= \frac{1}{(2n)^2}\det(2nI - \bar{D} + \bar{A}) \\
&= \frac{1}{(2n)^2}\det\begin{pmatrix}
3 & 0 & 1 & \cdots & \cdots & 1 & 1 & 0 & 1 & \cdots & \cdots & 1 \\
1 & 5 & 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & 5 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & \cdots & \cdots & 1 & 0 & 3 & 1 & \cdots & \cdots & \cdots & 0 & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 3 & 0 & 1 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & \vdots & 0 & 5 & 0 & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 5 & 0 \\
1 & \cdots & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 3
\end{pmatrix} \\
&= \frac{1}{(2n)^2}\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \frac{1}{(2n)^2}\det(A+B).\det(A-B), (AB = BA).
\end{aligned}$$

A straightforward induction using properties of determinants and above mentioned definition of Chebyshev polynomial in Lemma 2.1, we have

$$\begin{aligned}
\tau(D_2(P_n)) &= \frac{1}{(2n)^2}\det\begin{pmatrix}
4 & 0 & 2 & \cdots & 2 \\
0 & 6 & 0 & \ddots & \vdots \\
2 & 0 & \ddots & \ddots & 2 \\
\vdots & \ddots & \ddots & 6 & 0 \\
2 & \cdots & 2 & 0 & 4
\end{pmatrix} \times \det\begin{pmatrix}
2 & 0 & \cdots & \cdots & 0 \\
0 & 4 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 4 & 0 \\
0 & \cdots & \cdots & 0 & 2
\end{pmatrix} \\
&= \frac{1}{(2n)^2} \times 2^n n^2 \times 2^2 \times 4^{n-2} = 2^{3n-4}.
\end{aligned}$$

□

Theorem 3.2 *Let C_n be a cycle graph of order n. Then*

$$\tau(D_2(C_n)) = n2^{3n-2}, n \geq 3.$$

Proof Applying Lemma 1.1, we have

$$\tau(D_2(C_n)) = \frac{1}{(2n)^2}\det(2nI - \bar{D} + \bar{A})$$

$$= \frac{1}{(2n)^2}\det\begin{pmatrix} 5 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 5 & 0 & 1 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 5 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 5 \end{pmatrix}$$

$$= \frac{1}{(2n)^2}\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \frac{1}{(2n)^2}\det(A+B).\det(A-B), (AB = BA).$$

A straightforward induction using properties of determinants and above mentioned definition of Chebyshev polynomial in Lemma 2.1, we have

$$\tau(D_2(C_n)) = \frac{1}{(2n)^2} \times 2^n n^3 \times 4^n = n2^{3n-2}. \qquad \square$$

Theorem 3.3 *Let K_n be a complete graph of order n. Then*

$$\tau(D_2(K_n)) = 2^{2n-2}n^{n-2}(n-1)^n, n \geq 2.$$

Proof Applying Lemma 1.1, we have

$$\tau(D_2(K_n)) = \frac{1}{(2n)^2}\det(2nI - \bar{D} + \bar{A}) = \frac{1}{(2n)^2}\det\begin{pmatrix} A & I \\ I & A \end{pmatrix}$$

$$= \frac{1}{(2n)^2}\det(A+I) \times \det(A-I),$$

where

$$A = \begin{pmatrix} 2n-1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2n-1 \end{pmatrix}.$$

Thus,

$$\begin{aligned}
\tau(D_2(K_n)) &= \frac{1}{(2n)^2}\det\begin{pmatrix} 2n & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2n \end{pmatrix} \\
&\quad \times \det\begin{pmatrix} 2n-2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2n-2 \end{pmatrix} \\
&= \frac{1}{(2n)^2}\times (2n)^n(2n-2)^n = 2^{2n-2}n^{n-2}(n-1)^n.
\end{aligned}$$

□

Theorem 3.4 *Let $K_{n,m}$ be a complete bipartite graph. Then*

$$\tau(D_2(K_{n,m})) = 2^{n+m-2}n^{2m-1}m^{2n-1}$$

Proof Applying Lemma 1.1, we have

$$\begin{aligned}
\tau(D_2(K_{n,m})) &= \frac{1}{(2(m+n))^2}\det(2(n+m)I-\bar{D}+\bar{A}) \\
&= \frac{1}{(2(n+m))^2}\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A+B)\times\det(A-B), (AB=BA) \\
&= \frac{1}{(2(n+m))^2} \\
&\times \quad \det\begin{pmatrix} 2m+2 & 2 & \cdots & 2 & 0 & \cdots & \cdots & 0 \\ 2 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \vdots \\ 2 & \cdots & 2 & 2m+2 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 2n+2 & 2 & \cdots & 2 \\ \vdots & \ddots & \ddots & \vdots & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 2 \\ 0 & \cdots & \cdots & 0 & 2 & \cdots & 2 & 2n+2 \end{pmatrix}
\end{aligned}$$

$$\times \det\begin{pmatrix} 2m & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2m & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 2n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 2n \end{pmatrix}$$

$$= \frac{1}{(2(n+m))^2} \det\begin{pmatrix} 2m+2 & 2 & \cdots & 2 \\ 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2 & \cdots & 2 & 2m+2 \end{pmatrix}$$

$$\times \det\begin{pmatrix} 2n+2 & 2 & \cdots & 2 \\ 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2 & \cdots & 2 & 2n+2 \end{pmatrix}$$

$$\times \det\begin{pmatrix} 2m & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2m \end{pmatrix} \times \det\begin{pmatrix} 2n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2n \end{pmatrix}$$

$$= \frac{1}{(2(m+n))^2} \times 2^{2n+2m}(m+n)^2(m)^{n-1}(n)^{m-1}m^n n^m$$

$$= 4^{m+n-1}(m)^{2n-1}(n)^{2m-1}. \qquad \square$$

Theorem 3.5 *Let F_n be the the fan graph of order n. Then*

$$\tau(D_2(F_n)) = \frac{16 \times 6^{n-2} n}{\sqrt{5}}((3+\sqrt{5})^n - (3-\sqrt{5})^n), n \geq 2.$$

Proof Applying Lemma 1.1, we have

$$\tau(D_2(F_n)) = \frac{1}{(2(n+1))^2} \det(2n+1)I - \bar{D} + \bar{A})$$

$$= \frac{1}{(2(n+1))^2} \times$$

$$\times \det\begin{pmatrix}
2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
0 & 5 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & \cdots & 1\\
\vdots & 0 & 7 & \ddots & \ddots & \ddots & \vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1\\
\vdots & \vdots & \ddots & \ddots & \ddots & 7 & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & 1 & \cdots & \cdots & 1 & 0 & 5 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1\\
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
0 & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 5 & 0 & 1 & \cdots & \cdots & 1\\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 0 & 7 & \ddots & \ddots & \ddots & \vdots\\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & 7 & 0\\
0 & 1 & \cdots & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 5
\end{pmatrix}$$

$$= \frac{1}{(2(n+1))^2}\det(A+B)\times\det(A-B), (AB=BA)$$

$$= \frac{1}{(2(n+1))^2}\det\begin{pmatrix}
2n+2 & 0 & \cdots & \cdots & \cdots & 0\\
2 & 6 & 0 & 2 & \cdots & 2\\
\vdots & 0 & 8 & \ddots & \ddots & \vdots\\
\vdots & 2 & \ddots & \ddots & \ddots & 2\\
\vdots & \vdots & \ddots & \ddots & 8 & 0\\
0 & 2 & \cdots & 2 & 0 & 6
\end{pmatrix}\times\det\begin{pmatrix}
2n & 0 & \cdots & \cdots & \cdots & 0\\
0 & 4 & 0 & \cdots & \cdots & 0\\
\vdots & 0 & 6 & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & 6 & 0\\
0 & 0 & \cdots & \cdots & 0 & 4
\end{pmatrix}$$

$$= \frac{1}{(2(n+1))^2}\times(2^{n+1}(n+1))\det\begin{pmatrix}
3 & 0 & 1 & \cdot & 1\\
0 & 4 & 0 & \ddots & \vdots\\
1 & 0 & \ddots & \ddots & 1\\
\vdots & \ddots & \ddots & 4 & 0\\
1 & \cdots & 1 & 0 & 3
\end{pmatrix}$$

$$\times(2^{n+1}n)\det\begin{pmatrix}
2 & 0 & \cdots & \cdot & 0\\
0 & 3 & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & 3 & 0\\
0 & \cdots & \cdots & 0 & 2
\end{pmatrix}.$$

A straightforward induction using properties of determinants and above mentioned definition of Chebyshev polynomial in Lemma 2.1, we have

$$\tau(D_2(F_n)) = \frac{1}{n+1}\times 2^{2n}\times 6^{n-2}\times n\times (n+1)U_{n-1}(\frac{3}{2}) = \frac{16\times 6^{n-2}n}{\sqrt{5}}((3+\sqrt{5})^n-(3-\sqrt{5})^n). \quad \Box$$

Theorem 3.6 *Let W_n be the wheel graph. Then*

$$\tau(D_2(W_n)) = (6^n\times n)[(3+\sqrt{5})^n+(3-\sqrt{5})^n-2^{n+1}], n\geq 3.$$

Proof Applying Lemma 1.1, we have

$$\tau(D_2(W_n)) = \frac{1}{(2(n+1))^2}\det(2(n+1)I-\bar{D}+\bar{A}) = \frac{1}{2(n+1))^2}$$

$$\times\det\begin{pmatrix}
2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
0 & 7 & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & 0 & 1 & \cdots & 1 & 0\\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 1\\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1\\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \cdots & 1 & 1 & \ddots & \ddots & 0\\
0 & 0 & 1 & \cdots & 1 & 0 & 7 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1\\
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
0 & 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 7 & 0 & 1 & \cdots & 1 & 0\\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1\\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1\\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0\\
0 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & 0 & 1 & \cdots & 1 & 0 & 7
\end{pmatrix}$$

$$= \frac{1}{(2n)^2}\det\begin{pmatrix} A & B\\ B & A\end{pmatrix} = \frac{1}{(2(n+1))^2}\det(A+B).\det(A-B), (AB=BA)$$

$$= \frac{1}{(2(n+1))^2}\det\begin{pmatrix}
2n+2 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
0 & 8 & 0 & 2 & \cdots & 2 & 0\\
\vdots & 0 & 8 & \ddots & \ddots & \ddots & 2\\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2\\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 0\\
0 & 0 & 2 & \cdots & 2 & 0 & 8
\end{pmatrix}\times$$

$$\times \det \begin{pmatrix} 2n & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 6 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 6 \end{pmatrix}$$

$$= \frac{1}{(2(n+1))^2}(2^{n+1}(n+1)) \times \det \begin{pmatrix} 4 & 0 & 1 & \cdot & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 4 \end{pmatrix}$$

$$\times (2^{n+1}n) \times \det \begin{pmatrix} 3 & 0 & \cdots & \cdot & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 3 \end{pmatrix}.$$

A straightforward induction using properties of determinants and above mentioned definition of Chebyshev polynomial in Lemma 2.1, we have

$$\begin{aligned} \tau(D_2(W_n)) &= \frac{1}{n+1} \times 2^{2n} \times 3^n \times n \times (n+1)[T_n(\frac{3}{2}) - 1] \\ &= (n \times 6^n)[((3+\sqrt{5})^n + (3-\sqrt{5})^n - 2^{n+1}]. \end{aligned}$$ □

§4. **Conclusion**

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. Its evaluation is not only interesting from a mathematical perspective, but also important for reliability of a network and designing electrical circuits. Some computationally hard problems such as the traveling salesman problem can be solved approximately by using spanning trees. Due to the high dependence in network design and reliability on graph theory, we obtained theorems with proofs in this paper.

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Corrigendum

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