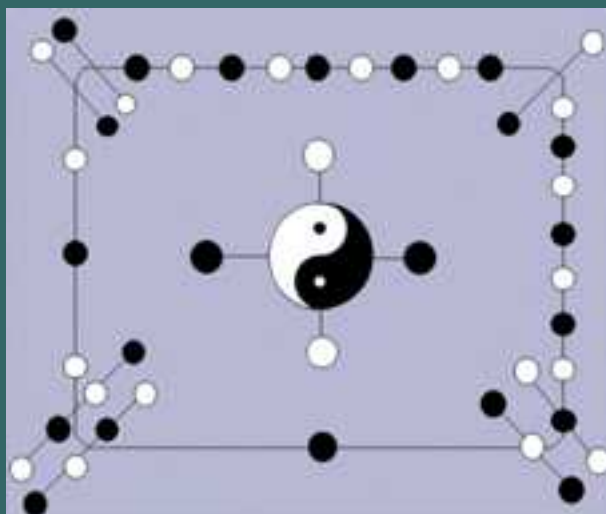




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**Famous Words:**

*The physicists say that I am a mathematician, and the mathematicians say that I am a physicist. I am a completely isolated man and though everybody knows me, there are very few people who really know me.*

By Albert Einstein, an American theoretical physicist.

## Finite Forms of Reciprocity Theorem of Ramanujan and its Generalizations

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**Abstract:** In his lost notebook, Ramanujan has stated a beautiful two variable reciprocity theorem. Its three and four variable generalizations were recently, given by Kang. In this paper, we give new and an elegant approach to establish all the three reciprocity theorems via their finite forms. Also we give some applications of the finite forms of reciprocity theorems.

**Key Words:**  $q$ -series, reciprocity theorems, bilateral extension,  $q$ -gamma,  $q$ -beta, eta-functions.

**AMS(2010):** 33D15, 33D05, 11F20

### §1. Introduction

In his lost notebook [16], Ramanujan has stated the following beautiful two variable reciprocity theorem.

**Theorem 1.1** *If  $a, b$  are complex numbers other than 0 and  $-q^{-n}$ , then*

$$\rho(a, b) - \rho(b, a) = \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(aq/b, bq/a, q)_{\infty}}{(-aq, -bq)_{\infty}}, \quad (1.1)$$

where

$$\rho(a, b) = \left( 1 + \frac{1}{b} \right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n},$$

and as usual

$$(a)_{\infty} := (a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n),$$

$$(a)_n := (a; q)_n := \frac{(a)_{\infty}}{(aq^n)_{\infty}}, \quad n \text{ is an integer.}$$

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In what follows, we assume  $|q| < 1$  and employ the following notations

$$(a_1, a_2, a_3, \dots, a_m)_n = (a_1)_n (a_2)_n (a_3)_n \dots (a_m)_n,$$

$$(a_1, a_2, a_3, \dots, a_n)_\infty = (a_1)_\infty (a_2)_\infty (a_3)_\infty \dots (a_n)_\infty.$$

The first proof of (1.1) was given by Andrews [4] using his identity, which he has derived using many summation and transformation formulae for basic hypergeometric series and the well-known Jacobi's triple product identity, which in fact, is a special case of (1.1). Somashekara and Fathima [19] used Ramanujan's  ${}_1\psi_1$  summation formula and Heine's transformation formula to establish an equivalent version of (1.1). Bhargava, Somashekara and Fathima [9] provided another proof of (1.1). Kim, Somashekara and Fathima [15] gave a proof of (1.1) using only  $q$  - binomial theorem. Guruprasad and Pradeep [11] also have devised a proof of (1.1) using  $q$  - binomial theorem. Adiga and Anitha [1] devised a proof of (1.1) along the lines of Ismail's proof of Ramanujan's  ${}_1\psi_1$  summation formula. Berndt, Chan, Yeap and Yee [8] found the three different proofs of (1.1). The first one is similar to that of Somashekara and Fathima [19]. The second proof depends on Rogers-Fine identity and the third proof is combinatorial. Kang [14] constructed a proof of (1.1) along the lines of Venkatachaleingar's proof of Ramanujan's  ${}_1\psi_1$  summation formula. Recently, Somashekara and Narasimha Murthy [21] have given a proof of (1.1) using Abel's lemma on summation by parts and Jacobi's triple product identity. For more details one may refer the book by Andrews and Berndt [5].

Kang, in her paper [14] has obtained the following three and four variable generalizations of (1.1).

**Theorem 1.2** *If  $|c| < |a| < 1$  and  $|c| < |b| < 1$ , then*

$$\rho_3(a, b; c) - \rho_3(b, a; c) = \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(c, aq/b, bq/a, q)_\infty}{(-c/a, -c/b, -aq, -bq)_\infty}, \quad (1.2)$$

where

$$\rho_3(a, b; c) := \left( 1 + \frac{1}{b} \right) \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b)_{n+1}}.$$

**Theorem 1.3** *If  $|c|, |d| < |a|, |b| < 1$ , then*

$$\rho_4(a, b; c, d) - \rho_4(b, a; c, d) = \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(c, d, cd/ab, aq/b, bq/a, q)_\infty}{(-c/a, -c/b, -d/a, -d/b, -aq, -bq)_\infty}, \quad (1.3)$$

where

$$\rho_4(a, b; c, d) := \left( 1 + \frac{1}{b} \right) \sum_{n=0}^{\infty} \frac{(c, d, cd/ab)_n \left( 1 + \frac{cdq^{2n}}{b} \right) (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b, -d/b)_{n+1}}.$$

In fact, to derive (1.2), Kang [14] has employed Ramanujan's  ${}_1\psi_1$  summation formula and Jackson's transformation of  ${}_2\phi_1$  and  ${}_2\phi_2$  series. Later, Adiga and Guruprasad [2] have given a proof of (1.2) using  $q$  - binomial theorem and Gauss summation formula. Somashekara and Mamta [20] have obtained (1.2) using (1.1) by parameter augmentation method. One more proof of (1.2) was given by Zhang [23].

Kang [14] has established the four variable reciprocity theorem(1.3) by employing Andrews generalization of  ${}_1\psi_1$  summation formula [4, Theorem 6], Sears transformation of  ${}_3\phi_2$  series and a limiting case of Watson's transformation for a terminating very well-poised  ${}_8\phi_7$  series. Adiga and Guruprasad [3] have derived (1.3) using an identity of Andrew's [4, Theorem 1], Ramanujan's  ${}_1\psi_1$  summation formula and the Watson's transformation.

The main objective of this paper is to give finite forms of the reciprocity theorems (1.1), (1.2) and (1.3). To obtain our results, we begin with a known finite unilateral summation and then shift the summation index, say  $k$  ( $0 \leq k \leq 2n$ ) by  $n$  :

$$\sum_{k=0}^{2n} A(k) = \sum_{k=-n}^n A(k+n).$$

After some manipulations, we employ some well-known transformation formulae for the basic hypergeometric series. The same method has been extensively utilized by Bailey [6]-[7], Slater [18], Schlosser [17] and Jouhet and Schlosser [13].

We recall some standard definitions which we use in this paper. The  $q$ -gamma function  $\Gamma_q(x)$ , was introduced by Thomae [22] and later by Jackson [12] as

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty} (1-q)^{1-x}, \quad 0 < q < 1. \quad (1.4)$$

A  $q$ -Beta function is defined by

$$B_q(x, y) = (1-q) \sum_{n=0}^{\infty} \frac{(q^{n+1})_\infty}{(q^{n+y})_\infty} q^{nx}.$$

A relation between  $q$ -Beta function and  $q$ -gamma function is given by

$$B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \quad (1.5)$$

The Dedekind eta function is defined by

$$\begin{aligned} \eta(\tau) &:= e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \quad \text{Im}(\tau) > 0 \\ &:= q^{1/24} (q; q)_\infty, \quad \text{where } e^{2\pi i \tau} = q. \end{aligned} \quad (1.6)$$

In Section 2, we state some standard identities for basic hypergeometric series which we use for our purpose. In Section 3, we establish the finite forms of two, three and four variable reciprocity theorems 1.1, 1.2 and 1.3. In Section 4, we give some applications of the finite forms of reciprocity theorems.

## §2. Some Standard Identities for Basic Hypergeometric Series

In this section, we list some standard summation and transformation formulae for the basic hypergeometric series which will be used in the remainder of this paper. Some identities involving  $q$  - shifted factorials are

$$(a)_{-n} = \frac{1}{(aq^{-n})_n} = \frac{(-q/a)_n}{(q/a)_n} q^{\binom{n}{2}}, \quad (2.1)$$



$$(a)_{k+n} = (a)_n (aq^n)_k, \quad (2.2)$$

$$(aq^{-n})_n = (q/a)_n \left( \frac{-a}{q} \right)^n q^{-\binom{n}{2}}, \quad (2.3)$$

$$(aq^{-kn})_n = \frac{(q/a)_{kn}}{(q/a)_{(k-1)n}} (-a)^n q^{\binom{n}{2} - kn^2}. \quad (2.4)$$

$q$  - Chu- Vandermonde's Sum [10, equation (II.7), p.354]

$$\sum_{k=0}^n \frac{(q^{-n}, A)_k}{(q, C)_k} (Cq^n/A)^k = \frac{(C/A)_n}{(C)_n}. \quad (2.5)$$

$q$  - Pfaff- Saalschütz's Summation formula [10, equation (II.12), p.355]

$$\sum_{k=0}^n \frac{(q^{-n}, A, B)_k}{(q, C, ABq^{1-n}/C)_k} q^k = \frac{(C/A, C/B)_n}{(C, C/AB)_n}. \quad (2.6)$$

Jackson's  $q$  - analogue of Dougall's  ${}_7F_6$  Sum [10, equation (II.22), p.356]

$$\begin{aligned} \sum_{k=0}^n \frac{(A, qA^{1/2}, -qA^{1/2}, B, C, D, E, q^{-n})_k}{(q, A^{1/2}, -A^{1/2}, Aq/B, Aq/C, Aq/D, Aq/E, Aq^{n+1})_k} q^k \\ = \frac{(Aq, Aq/BC, Aq/BD, Aq/CD)_n}{(Aq/B, Aq/C, Aq/D, Aq/BCD)_n}, \end{aligned} \quad (2.7)$$

where  $A^2q = BCDEq^{-n}$ .

Sear's terminating transformation formula [10, equation (III.13), p.360]

$$\sum_{k=0}^n \frac{(q^{-n}, B, C)_k}{(q, D, E)_k} (DEq^n/BC)^k = \frac{(E/C)_n}{(E)_n} \sum_{k=0}^n \frac{(q^{-n}, C, D/B)_k}{(q, D, Cq^{1-n}/E)_k} q^k, \quad (2.8)$$

Watson's transformation for a terminating very well poised  ${}_8\phi_7$  series [10, equation (III.19), p.361]

$$\begin{aligned} \sum_{k=0}^n \frac{(q^{-n}, A, B, C)_k}{(q, D, E, F)_k} q^k &= \frac{(D/B, D/C)_n}{(D, D/BC)_n} \\ &\times \sum_{k=0}^n \frac{(\sigma, q\sigma^{1/2}, -q\sigma^{1/2}, B, C, E/A, F/A, q^{-n})_k}{(q, \sigma^{1/2}, -\sigma^{1/2}, E, F, EF/AB, EF/AC, EFq^n/A)_k} (EFq^n/BC)^k, \end{aligned} \quad (2.9)$$

where  $DEF = ABCq^{1-n}$  and  $\sigma = EF/Aq$ .

Bailey's terminating  ${}_{10}\phi_9$  transformation formula [10, equation (III.28), p.363]

$$\begin{aligned} \sum_{k=0}^n \frac{(A, qA^{1/2}, -qA^{1/2}, B, C, D, E, F, \lambda Aq^{n+1}/EF, q^{-n})_k}{(q, A^{1/2}, -A^{1/2}, Aq/B, Aq/C, Aq/D, Aq/E, Aq/F, EFq^{-n}/\lambda, Aq^{n+1})_k} q^k \\ = \frac{(Aq, Aq/EF, \lambda q/E, \lambda q/F)_n}{(Aq/E, Aq/F, \lambda q/EF, \lambda q)_n} \\ \times \sum_{k=0}^n \frac{(\lambda, q\lambda^{1/2}, -q\lambda^{1/2}, B/A, C/A, D/A, E, F, \lambda Aq^{n+1}/EF, q^{-n})_k}{(q, \lambda^{1/2}, -\lambda^{1/2}, Aq/B, Aq/C, Aq/D, \lambda q/E, \lambda q/F, EFq^{-n}/A, \lambda q^{n+1})_k} q^k, \end{aligned} \quad (2.10)$$

where  $\lambda = qA^2/BCD$ .

### §3. Main Identities

In this section, we establish the finite forms of reciprocity theorems.

**Theorem 3.1** *If  $a, b$  are complex numbers other than 0 and  $-q^{-n}$ , then*

$$\begin{aligned} & \left(1 + \frac{1}{b}\right) \sum_{k=0}^n \frac{(q^{-n}, -bq^n)_k}{(q^{1+n}, -aq)_k} (aq^{1+n}/b)^k \\ & - \left(1 + \frac{1}{a}\right) (1 - q^n) \sum_{k=0}^{n-1} \frac{(q^{-n+1}, -aq^{n+1})_k}{(q^{1+n})_{k+1} (-bq)_k} (bq^n/a)^k \\ & = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(aq/b)_n (bq/a)_{n-1} (q)_n}{(-aq)_n (-bq)_{n-1} (q^{1+n})_n}. \end{aligned} \quad (3.1)$$

*Proof* Replace  $n$  by  $2n$  in (2.5) to obtain

$$\sum_{k=0}^{2n} \frac{(q^{-2n}, A)_k}{(q, C)_k} (Cq^{2n}/A)^k = \frac{(C/A)_{2n}}{(C)_{2n}}. \quad (3.2)$$

Shift the summation index  $k$  by  $n$ , so that the sum runs from  $-n$  to  $n$  and (3.2) takes the form

$$\frac{(q^{-2n}, A)_n}{(q, C)_n} (Cq^{2n}/A)^n \sum_{k=-n}^n \frac{(q^{-n}, Aq^n)_k}{(q^{1+n}, Cq^n)_k} (Cq^{2n}/A)^k = \frac{(C/A)_{2n}}{(C)_{2n}}.$$

This can be written as

$$\sum_{k=-n}^n \frac{(q^{-n}, Aq^n)_k}{(q^{1+n}, Cq^n)_k} (Cq^{2n}/A)^k = \frac{(q, C)_n (C/A)_{2n}}{(A, q^{-2n})_n (C)_{2n}} (Cq^{2n}/A)^{-n}. \quad (3.3)$$

Now, replacing  $A$  by  $-b$  and  $C$  by  $-aq^{1-n}$  in (3.3), then using (2.2) and (2.3) in the resulting identity, we obtain

$$\sum_{k=-n}^n \frac{(q^{-n}, -bq^n)_k}{(q^{1+n}, -aq)_k} (aq^{1+n}/b)^k = \frac{(\frac{1}{b} - \frac{1}{a})}{(1 + \frac{1}{b})} \frac{(q)_n (aq/b)_n (bq/a)_{n-1}}{(-aq)_n (-bq)_{n-1} (q^{1+n})_n}.$$

This can be written as

$$\begin{aligned} & \left(1 + \frac{1}{b}\right) \sum_{k=0}^n \frac{(q^{-n}, -bq^n)_k}{(q^{1+n}, -aq)_k} (aq^{1+n}/b)^k + \left(1 + \frac{1}{b}\right) \sum_{k=1}^n \frac{(q^{-n}, -1/a)_k}{(q^{1+n}, -q^{1-n}/b)_k} q^k \\ & = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(aq/b)_n (bq/a)_{n-1} (q)_n}{(-aq)_n (-bq)_{n-1} (q^{1+n})_n}. \end{aligned} \quad (3.4)$$

Now, the first term on left side of (3.4) is same as the first term on the left side of (3.1). Therefore, to complete the proof, it suffices to show that the second term on the left side of (3.4) is same as the second term on left side of (3.1). To this end, we change  $n \rightarrow n-1$  and then set  $B = -aq^{n+1}$ ,  $C = q$ ,  $D = q^{2+n}$  and  $E = -bq$  in (2.8) to obtain

$$\sum_{k=0}^{n-1} \frac{(q^{-n+1}, -aq^{n+1})_k}{(q^{n+2}, -bq)_k} (bq^n/a)^k = \frac{(-b)_{n-1}}{(-bq)_{n-1}} \sum_{k=0}^{n-1} \frac{(q^{-n+1}, -q/a)_k}{(q^{n+2}, -q^{2-n}/b)_k} q^k. \quad (3.5)$$

Multiply (3.5) throughout by  $\frac{(1 + \frac{1}{a})(1 - q^{-n})}{(1 - q^{n+1})(1 + \frac{q^{1-n}}{b})}q$ , to obtain

$$\begin{aligned} & \frac{(1 + \frac{1}{a})(1 - q^{-n})}{(1 - q^{n+1})(1 + \frac{q^{1-n}}{b})}q \sum_{k=0}^{n-1} \frac{(q^{-n+1}, -aq^{n+1})_k}{(q^{n+2}, -bq)_k} (bq^n/a)^k \\ &= \frac{(1+b)}{(1+bq^{n-1})} \sum_{k=0}^{n-1} \frac{(q^{-n}, -1/a)_{k+1}}{(q^{n+1}, -q^{1-n}/b)_{k+1}} q^{k+1}. \end{aligned}$$

This on simplification yields

$$\begin{aligned} & \left(1 + \frac{1}{b}\right) \sum_{k=1}^n \frac{(q^{-n}, -1/a)_k}{(q^{n+1}, -q^{1-n}/b)_k} q^k \\ &= -\left(1 + \frac{1}{a}\right) (1 - q^n) \sum_{k=0}^{n-1} \frac{(q^{-n+1}, -aq^{n+1})_k}{(q^{1+n})_{k+1}(-bq)_k} (bq^n/a)^k, \end{aligned}$$

completing the proof of (3.1).  $\square$

**Theorem 3.2** *If  $|c| < |a| < 1$  and  $|c| < |b| < 1$ , then*

$$\begin{aligned} & \left(1 + \frac{1}{b}\right) \left(1 + \frac{cq^n}{b}\right) \\ & \times \sum_{k=0}^n \frac{(q^{-n}, c, -cq^n/a, -bq^n)_k}{(q^{1+n}, -aq)_k(-c/b, cq^{2n})_{k+1}} (1 - cq^{2k+n}) \left(\frac{aq^{1+n}}{b}\right)^k - \left(1 + \frac{1}{a}\right) (1 - q^n) \\ & \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, c, -aq^{n+1})_k(-cq^n/b)_{k+1}}{(q^{1+n}, -c/a, cq^{2n})_{k+1}(-bq)_k} (1 - cq^{2k+n+1}) \left(\frac{bq^n}{a}\right)^k \\ &= \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c)_{2n}(aq/b, -bq^n, q)_n (bq/a)_{n-1}}{(-c/a, -c/b, -aq, q^{1+n})_n (-bq)_{2n-1}}. \end{aligned} \quad (3.6)$$

*Proof* Replace  $n$  by  $2n$  in (2.6) to obtain

$$\sum_{k=0}^{2n} \frac{(q^{-2n}, A, B)_k}{(q, C, ABq^{1-2n}/C)_k} q^k = \frac{(C/A, C/B)_{2n}}{(C, C/AB)_{2n}}. \quad (3.7)$$

Shift the summation index  $k$  by  $n$ , so that the sum runs from  $-n$  to  $n$  and (3.7) takes the form

$$\sum_{k=-n}^n \frac{(q^{-n}, Aq^n, Bq^n)_k}{(q^{1+n}, Cq^n, ABq^{1-n}/C)_k} q^k = \frac{(C/A, C/B)_{2n}(q, C, ABq^{1-2n}/C)_n}{(C, C/AB)_{2n}(A, B, q^{-2n})_n} q^{-n}. \quad (3.8)$$

Now, we replace  $A$  by  $-c/b$ ,  $B$  by  $-q^{-n}/a$  and  $C$  by  $-cq^{-n}/a$  in (3.8), and then use (2.2) and (2.3) in the resulting identity, to obtain

$$\sum_{k=-n}^n \frac{(q^{-n}, -1/a, -cq^n/b)_k}{(q^{1+n}, -c/a, -q^{1-n}/b)_k} q^k = \frac{(\frac{1}{b} - \frac{1}{a})}{(1 + \frac{1}{b})} \frac{(c)_{2n}(aq/b, -bq^n, q)_n (bq/a)_{n-1}}{(-c/a, -c/b, -aq, q^{1+n})_n (-bq)_{2n-1}}.$$

This can be written as

$$\begin{aligned} & \left(1 + \frac{1}{b}\right) \sum_{k=0}^n \frac{(q^{-n}, -aq/c, -bq^n)_k}{(q^{1+n}, -aq, -bq^{1-n}/c)_k} q^k + \left(1 + \frac{1}{b}\right) \sum_{k=1}^n \frac{(q^{-n}, -1/a - cq^n/b)_k}{(q^{1+n}, -c/a, -q^{1-n}/b)_k} q^k \\ &= \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c)_{2n} (aq/b, -bq^n, q)_n (bq/a)_{n-1}}{(-c/a, -c/b, -aq, q^{1+n})_n (-bq)_{2n-1}}. \end{aligned} \quad (3.9)$$

Now, set  $A = -aq/c$ ,  $B = q$ ,  $C = -bq^n$ ,  $D = -bq^{1-n}/c$ ,  $E = q^{n+1}$  and  $F = -aq$  in (2.9) and multiply the resulting identity throughout by  $(1 + b^{-1})$ , to obtain

$$\begin{aligned} & \left(1 + \frac{1}{b}\right) \sum_{k=0}^n \frac{(q^{-n}, -aq/c, -bq^n)_k}{(q^{1+n}, -aq, -bq^{1-n}/c)_k} q^k \\ &= \left(1 + \frac{1}{b}\right) \left(1 + \frac{cq^n}{b}\right) \sum_{k=0}^n \frac{(q^{-n}, c, -cq^n/a, -bq^n)_k}{(q^{1+n}, -aq)_k (-c/b, cq^{2n})_{k+1}} (1 - cq^{2k+n}) \left(\frac{aq^{1+n}}{b}\right)^k. \end{aligned} \quad (3.10)$$

Next, change  $n \rightarrow n-1$  in (2.9) and then set  $A = -q/a$ ,  $B = q$ ,  $C = -cq^{n+1}/b$ ,  $D = -q^{2-n}/b$ ,  $E = q^{n+2}$  and  $F = -cq/a$  to obtain

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(q^{-n+1}, -q/a, -cq^{n+1}/b)_k}{(q^{2+n}, -cq/a, -q^{2-n}/b)_k} q^k = \frac{(-q^{1-n}/b, q^{1-2n}/c)_{n-1}}{(-q^{2-n}/b, q^{-2n}/c)_{n-1}} \\ & \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, c, -aq^{n+1}, -cq^{1+n}/b)_k}{(q^{2+n}, -cq/a, -bq, cq^{2n+1})_k} \frac{(1 - cq^{2k+n+1})}{(1 - cq^{n+1})} \left(\frac{bq^n}{a}\right)^k. \end{aligned} \quad (3.11)$$

Multiply (3.11) throughout by  $\frac{(1 + \frac{1}{b})(1 - q^{-n})(1 + \frac{1}{a})(1 + \frac{cq^n}{b})}{(1 - q^{1+n})(1 + \frac{c}{a})(1 + \frac{q^{1-n}}{b})} q$  to obtain

$$\begin{aligned} & \left(1 + \frac{1}{b}\right) \sum_{k=0}^{n-1} \frac{(q^{-n}, -1/a, -cq^n/b)_{k+1}}{(q^{1+n}, -c/a, -q^{1-n}/b)_{k+1}} q^{k+1} \\ &= \frac{(1 + \frac{1}{a})(1 - q^{-n}) \left(1 + \frac{cq^n}{b}\right) \left(1 - \frac{q^{-n-1}}{c}\right)}{(1 - q^{1+n}) \left(1 + \frac{c}{a}\right) \left(1 - \frac{q^{-2n}}{c}\right)} \\ & \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, c, -aq^{n+1}, -cq^{1+n}/b)_k}{(q^{2+n}, -cq/a, -bq, cq^{2n+1})_k} \frac{(1 - cq^{2k+n+1})}{(1 - cq^{n+1})} \left(\frac{bq^n}{a}\right)^k. \end{aligned} \quad (3.12)$$

On simplification (3.12) yields

$$\begin{aligned} & \left(1 + \frac{1}{b}\right) \sum_{k=1}^n \frac{(q^{-n}, -1/a, -cq^n/b)_k}{(q^{1+n}, -c/a, q^{1-n}/b)_k} q^k = - \left(1 + \frac{1}{a}\right) (1 - q^n) \\ & \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, c, -aq^{n+1})_k (-cq^n/b)_{k+1}}{(q^{1+n}, -c/a, cq^{2n})_{k+1} (-bq)_k} (1 - cq^{2k+n+1}) \left(\frac{bq^n}{a}\right)^k. \end{aligned} \quad (3.13)$$

Using (3.10) and (3.13) in (3.9), we obtain (3.6).  $\square$

**Theorem 3.3** If  $|c|, |d| < |a|, |b| < 1$ , then

$$\begin{aligned}
& \frac{(1+1/b)(1-aq^{n+1}/b)(1+cdq^{2n-1}/a)}{(1+q^n)} \\
& \times \sum_{k=0}^n \frac{(q^{-n}, c, d, cd/ab, aq^{1-n}/b, cdq^{2n}/b)_k}{(q^{1-2n}, -aq)_k(-c/b, -d/b, -cdq^{n-1}/a, -cdq^n/b)_{k+1}} \left(1 + \frac{cdq^{2k}}{b}\right) q^k \\
& - \frac{(1+\frac{1}{a})(1-aq^{n+1}/b)}{(1+q^n)(1+cdq^{n-1}/a)} \\
& \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, c, d, cd/ab, bq^{-n}/a)_k(cdq^{2n-1}/a)_{k+1}}{(q^{1-2n}, -bq, -cdq^{n+1}/a)_k(-c/a, -d/a, -cdq^n/b)_{k+1}} \left(1 + \frac{cdq^{2k}}{a}\right) q^k \\
& = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c, d, cd/ab)_{2n}(aq/b)_{n+1}(q)_n}{(-c/a, -d/a, -aq)_n(-c/b, -d/b)_{n+1}(q^{1+n})_n} \\
& \times \frac{(bq/a)_{n-1}(-cd/aq)_{3n}}{(-bq)_{n-1}(-cdq^n/b)_n(-cd/aq, -cdq^{n-1}/a)_{2n}}. \tag{3.14}
\end{aligned}$$

*Proof* Replace  $n$  by  $2n$  in (2.7) to obtain

$$\begin{aligned}
& \sum_{k=0}^{2n} \frac{(q^{-2n}, A, B, C, D, A^2q^{2n+1}/BCD)_k}{(q, Aq/B, Aq/C, Aq/D, BCDq^{-2n}/A, Aq^{2n+1})_k} \frac{(1-Aq^{2k})}{(1-A)} q^k \\
& = \frac{(Aq, Aq/BC, Aq/BD, Aq/CD)_{2n}}{(Aq/B, Aq/C, Aq/D, Aq/BCD)_{2n}}. \tag{3.15}
\end{aligned}$$

Shift the summation index  $k$  by  $n$ , so that the sum runs from  $-n$  to  $n$  and (3.15) takes the form

$$\begin{aligned}
& \sum_{k=-n}^n \frac{(q^{-n}, Aq^n, Bq^n, Cq^n, Dq^n, A^2q^{3n+1}/BCD)_k}{(q^{1+n}, Aq^{1+n}/B, Aq^{1+n}/C, Aq^{1+n}/D, BCDq^{-n}/A, Aq^{3n+1})_k} \frac{(1-Aq^{2k+2n})}{(1-A)} q^k \\
& = \frac{(Aq, Aq/BC, Aq/BD, Aq/CD)_{2n}}{(Aq/B, Aq/C, Aq/D, Aq/BCD)_{2n}} \\
& \times \frac{(q, Aq/B, Aq/C, Aq/D, BCDq^{-2n}/A, Aq^{2n+1})_n}{(q^{-2n}, A, B, C, D, A^2q^{2n+1}/BCD)_n} q^{-n}. \tag{3.16}
\end{aligned}$$

Replacing  $A, B, C, D$  respectively by  $Aq^{-2n}, Bq^{-n}, Cq^{-n}, Dq^{-n}$  in (3.16) and simplifying using (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned}
& \sum_{k=0}^n \frac{(q^{-n}, Aq^{-n}, B, C, D, A^2q^{2n+1}/BCD)_k}{(q^{1+n}, Aq/B, Aq/C, Aq/D, BCDq^{-2n}/A, Aq^{n+1})_k} (1-Aq^{2k}) q^k \\
& - A \sum_{k=1}^n \frac{(q^{-n}, q^{-n}/A, B/A, C/A, D/A, Aq^{2n+1}/BCD)_k}{(q^{1+n}, q/B, q/C, q/D, BCDq^{-2n}/A^2, q^{n+1}/A)_k} \left(1 - \frac{q^{2k}}{A}\right) q^k \\
& = (1-A) \frac{(q, Aq, q/A)_n(Aq/BC, Aq/BD, Aq/CD)_{2n}}{(q/B, q/C, q/D, Aq/B, Aq/C, Aq/D, q^{1+n} A^2q^{2n+1}/BCD)_n} \\
& \times \frac{(Aq/BCD)_{3n}}{(Aq/BCD, Aq^{1+n}/BCD)_{2n}}. \tag{3.17}
\end{aligned}$$

Setting  $A = aq/b$ ,  $B = -q/b$ ,  $C = -aq/c$  and  $D = -aq/d$  in (3.17) and then simplifying, we obtain

$$\begin{aligned}
& \sum_{k=0}^n \frac{(q^{-n}, aq^{1-n}/b, -q/b, -aq/c, -aq/d, -cdq^{2n}/b)_k}{(q^{1+n}, -aq, -cq/b, -dq/b, -aq^{2-2n}/cd, aq^{n+2}/b)_k} \left(1 - \frac{aq^{2k+1}}{b}\right) q^k \\
& - \frac{aq}{b} \sum_{k=1}^n \frac{(q^{-n}, bq^{-n-1}/a, -1/a, -b/c, -b/d, -cdq^{2n-1}/a)_k}{(q^{1+n}, -b, -c/a, -d/a, -bq^{1-2n}/cd, bq^n/a)_k} \left(1 - \frac{bq^{2k-1}}{a}\right) q^k \\
& = \left(1 - \frac{aq}{b}\right) \frac{(q, aq^2/b, b/a)_n (c, d, cd/ab)_{2n}}{(-b, -c/a, -d/a, -cq/b, -dq/b, -aq, q^{1+n}, -cdq^n/b)_n} \\
& \quad \times \frac{(-cd/aq)_{3n}}{(-cd/aq, -cdq^{n-1}/a)_{2n}}. \tag{3.18}
\end{aligned}$$

Multiply (3.18) throughout by  $\frac{(1 + \frac{1}{b})}{(1 + \frac{c}{b})(1 + \frac{d}{b})}$  to obtain

$$\begin{aligned}
& \left(1 + \frac{1}{b}\right) \sum_{k=0}^n \frac{(q^{-n}, aq^{1-n}/b, -q/b, -aq/c, -aq/d, -cdq^{2n}/b)_k}{(q^{1+n}, -aq, -aq^{2-2n}/cd, aq^{n+2}/b)_k (-c/b, -d/b)_{k+1}} \left(1 - \frac{aq^{2k+1}}{b}\right) q^k \\
& - \frac{(1 + \frac{1}{b}) \frac{aq}{b}}{(1 + \frac{c}{b})(1 + \frac{d}{b})} \\
& \times \sum_{k=1}^n \frac{(q^{-n}, bq^{-n-1}/a, -1/a, -b/c, -b/d, -cdq^{2n-1}/a)_k}{(q^{1+n}, -b, -c/a, -d/a, -bq^{1-2n}/cd, bq^n/a)_k} \left(1 - \frac{bq^{2k-1}}{a}\right) q^k \\
& = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c, d, cd/ab)_{2n} (aq/b)_{n+1} (q)_n}{(-c/a, -d/a, -aq)_n (-c/b, -d/b)_{n+1} (q^{1+n})_n} \\
& \quad \times \frac{(bq/a)_{n-1} (-cd/aq)_{3n}}{(-bq)_{n-1} (-cdq^n/b)_n (-cd/aq, -cdq^{n-1}/a)_{2n}}. \tag{3.19}
\end{aligned}$$

Now, set  $A = aq/b$ ,  $B = -q/b$ ,  $C = -aq/c$ ,  $D = -aq/d$ ,  $E = aq^{1-n}/b$ ,  $F = q$  and  $\lambda = -cd/b$  in (2.10), to obtain

$$\begin{aligned}
& \sum_{k=0}^n \frac{(q^{-n}, aq^{1-n}/b, -q/b, -aq/c, -aq/d, -cdq^{2n}/b)_k}{(q^{1+n}, -aq, -cq/b, -dq/b, -aq^{2-2n}/cd, aq^{n+2}/b)_k} \left(1 - \frac{aq^{2k+1}}{b}\right) q^k \\
& = \frac{(1 - aq^{n+1}/b)(1 - q^n)(1 + cdq^{2n-1}/a)}{(1 - q^{2n})(1 + cdq^{n-1}/a)(1 + cdq^n/b)} \\
& \quad \times \sum_{k=0}^n \frac{(q^{-n}, c, d, cd/ab, aq^{1-n}/b, -cdq^{2n}/b)_k}{(q^{1-2n}, -aq, -cq/b, -dq/b, -cdq^n/a, cdq^{n+1}/b)_k} \left(1 + \frac{cdq^{2k}}{b}\right) q^k. \tag{3.20}
\end{aligned}$$

Multiply (3.20) throughout by  $\frac{(1 + \frac{1}{b})}{(1 + \frac{c}{b})(1 + \frac{d}{b})}$  to obtain

$$\begin{aligned}
& \left(1 + \frac{1}{b}\right) \sum_{k=0}^n \frac{(q^{-n}, aq^{1-n}/b, -q/b, -aq/c, -aq/d, -cdq^{2n}/b)_k}{(q^{1+n}, -aq, -aq^{2-2n}/cd, aq^{n+2}/b)_k (-c/b, -d/b)_{k+1}} \left(1 - \frac{aq^{2k+1}}{b}\right) q^k \\
& = \frac{(1 + \frac{1}{b})(1 - aq^{n+1}/b)(1 + cdq^{2n-1}/a)}{(1 + q^n)}
\end{aligned}$$

$$\times \sum_{k=0}^n \frac{(q^{-n}, c, d, cd/ab, aq^{1-n}/b, -cdq^{2n}/b)_k}{(q^{1-2n}, -aq)_k(-c/b, -d/b, -cdq^{n-1}/a, -cdq^n/b)_{k+1}} \left(1 + \frac{cdq^{2k}}{b}\right) q^k. \quad (3.21)$$

Next, change  $n \rightarrow n-1$  in (2.10) and then set  $A = bq/a$ ,  $B = -q/a$ ,  $C = -bq/c$ ,  $D = -bq/d$ ,  $E = bq^{-n}/a$ ,  $F = q$  and  $\lambda = -cd/a$  to obtain

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(q^{-n+1}, bq^{-n}/a, -q/a, -bq/c, -bq/d, -cdq^{2n}/a)_k}{(q^{2+n}, -bq, -cq/a, -dq/a, -bq^{2-2n}/cd, bq^{n+1}/a)_k} \left(1 - \frac{bq^{2k+1}}{a}\right) q^k \\ &= \frac{(1 - bq^n/a)(1 - q^{n+1})(1 + cdq^{2n-1}/b)}{(1 - q^{2n})(1 + cdq^{n-1}/a)(1 + cdq^n/b)} \\ & \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, c, d, cd/ab, bq^{-n}/a, -cdq^{2n}/a)_k}{(q^{1-2n}, -bq, -cq/a, -dq/a, -cdq^{n+1}/a, -cdq^{n+1}/b)_k} \left(1 + \frac{cdq^{2k}}{a}\right) q^k. \end{aligned} \quad (3.22)$$

Multiplying (3.22) throughout by

$$\frac{(1 + \frac{1}{a})(1 + \frac{b}{c})(1 + \frac{b}{d})(1 - q^{-n})\left(1 - \frac{bq^{-n-1}}{a}\right)\left(1 + \frac{cdq^{2n-1}}{a}\right)(1 + \frac{1}{b})q}{(1+b)(1 + \frac{c}{a})(1 + \frac{d}{a})(1 - q^{n+1})\left(1 + \frac{bq^{1-2n}}{cd}\right)\left(1 - \frac{bq^n}{a}\right)(1 + \frac{c}{b})(1 + \frac{d}{b})},$$

we obtain

$$\begin{aligned} & \frac{(1 + \frac{1}{b})}{(1 + \frac{c}{b})(1 + \frac{d}{b})} \sum_{k=0}^{n-1} \frac{(q^{-n}, bq^{-n-1}/a, -1/a, -b/c, -b/d, -cdq^{2n-1}/a)_{k+1}}{(q^{1+n}, -b, -c/a, -d/a, -bq^{1-2n}/cd, bq^n/a)_{k+1}} \\ & \times \left(1 - \frac{bq^{2k+1}}{a}\right) q^{k+1} = \frac{(1 + \frac{1}{a})(1 - aq^{n+1}/b)(b/aq)}{(1 + q^n)(1 + cdq^{n-1}/a)} \\ & \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, c, d, cd/ab, bq^{-n}/a)_k(-cdq^{2n-1}/a)_{k+1}}{(q^{1-2n}, -bq, -cdq^{n+1}/a)_k(-c/a, -d/a, -cdq^n/b)_{k+1}} (1 + cdq^{2k}/a) q^k. \end{aligned} \quad (3.23)$$

Now, (3.23) can be written as

$$\begin{aligned} & \frac{(1 + \frac{1}{b}) \frac{aq}{b}}{(1 + \frac{c}{b})(1 + \frac{d}{b})} \sum_{k=1}^n \frac{(q^{-n}, bq^{-n-1}/a, -1/a, -b/c, -b/d, -cdq^{2n-1}/a)_k}{(q^{1+n}, -b, -c/a, -d/a, -bq^{1-2n}/cd, bq^n/a)_k} \\ & \times \left(1 - \frac{bq^{2k-1}}{a}\right) q^k = \frac{(1 + \frac{1}{a})(1 - aq^{n+1}/b)}{(1 + q^n)(1 + cdq^{n-1}/a)} \\ & \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, c, d, cd/ab, bq^{-n}/a)_k(-cdq^{2n-1}/a)_{k+1}}{(q^{1-2n}, -bq, -cdq^{n+1}/a)_k(-c/a, -d/a, -cdq^n/b)_{k+1}} \times (1 + cdq^{2k}/a) q^k. \end{aligned} \quad (3.24)$$

On using (3.21) and (3.24) in (3.19), we obtain (3.14).  $\square$

**Remark 3.1** Letting  $n \rightarrow \infty$  in (3.1), (3.6) and (3.14), we obtain (1.1), (1.2) and (1.3) respectively.

#### §4. Some Applications of the Finite Forms of the Reciprocity Theorems

In this Section, we deduce finite forms of some  $q$ -series identities along with the  $q$ -gamma,  $q$ -beta and eta function identities from (3.1) and (3.6).

**Corollary 4.1** (*Finite form of Euler's identity*)

$$\sum_{k=0}^{n-1} \frac{(q^{-n+1}, -q^{n+2}/x)_k}{(q^{1+n}, q)_k} (-1)^k q^{nk-k} x^k = (-x)_{n-1}. \quad (4.1)$$

*Proof* Set  $b = -1$  and  $a = q/x$  in (3.1), and after some simplifications, we obtain (4.1). Let  $n \rightarrow \infty$  in (4.1) to obtain the well-known Euler's Identity

$$\sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{(q)_k} = (-x)_{\infty}. \quad \square$$

**Corollary 4.2** (*Finite form of  ${}_1\phi_1$ - series [10, equation (II.5), p.354]*)

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{(q^{-n+1}, x - xq^{n+2}/y)_k (xq^n)_{k+1}}{(q^{1+n}, xq^{2n})_{k+1} (q, y)_k} (1 - xq^{2n+k+1}) (y/x)^k q^{nk-k} x^k \\ = \frac{(xq^n)_{n-1} (y/x)_{n-1}}{(q^{1+n})_n (y)_{n-1}}. \end{aligned} \quad (4.2)$$

*Proof* Set  $b = -1, a = -xq/y$  and  $c = x$  in (3.6), and after some simplifications, we obtain (4.2). If we let  $n \rightarrow \infty$  in (4.2), gives

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} (x)_k (y/x)^k}{(q, y)_k} = \frac{(y/x)_{\infty}}{(y)_{\infty}}. \quad \square$$

**Corollary 4.3**

$$(1 + bq^{n+1}) \sum_{k=0}^n \frac{(q^{-n}, -q^n/b)_k}{(-bq, q^{2n+1})_k} (1 - q^{2n+k+1}) (-1)^k q^{nk+k} b^k = 1. \quad (4.3)$$

*Proof* Set  $a = -1, c = q$  and  $b = b^{-1}$  in (3.6), and after some simplifications, we obtain (4.3). If we let  $n \rightarrow \infty$  in (4.3), gives

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2} b^k}{(-bq)_{k+1}} = 1. \quad \square$$

If we set  $b = 1$  in (4.3), we obtain

$$(1 + q^{n+1}) \sum_{k=0}^n \frac{(q^{-n}, -q^n)_k}{(-q, q^{2n+1})_k} (1 - q^{2n+k+1}) (-1)^k q^{nk+k} = 1. \quad (4.4)$$

Letting  $n \rightarrow \infty$  in (4.4), we obtain

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(-q)_{k+1}} = 1.$$

We define

$$\Gamma_{q,n}(x) := \frac{(q)_n}{(q^x)_n} (1 - q)^{1-x},$$



and

$$B_{q,n}(x, y) := \frac{(q, q^{x+y})_n}{(q^x, q^y)_n} (1 - q).$$

Note that  $\Gamma_{q,n}(x) \rightarrow \Gamma_q(x)$  and  $B_{q,n}(x, y) \rightarrow B_q(x, y)$  as  $n \rightarrow \infty$ , which are define in (1.4) and (1.5).

**Corollary 4.4**

$$\begin{aligned} \Gamma_{q,n}(x) = & \frac{(-q^{1+x}, q^{1+n})_n (1-q)^{1-x}}{2(-q)_n (-q)_{n-1}} \left[ \sum_{k=0}^n \frac{(q^{-n}, q^{n+x})_k}{(q^{1+n}, -q^{1+x})_k} (-1)^k q^{nk+k} \right. \\ & \left. + (1+q^x)(1-q^n) \sum_{k=0}^{n-1} \frac{(q^{-n+1}, -q^{n+x+1})_k}{(q^{1+n})_{k+1} (q^{x+1})_k} (-1)^k q^{nk} \right]. \end{aligned} \quad (4.5)$$

*Proof* Set  $a = q^x$  and  $b = -q^x$  in (3.1), and after some simplifications, we obtain (4.5).  $\square$

**Corollary 4.5**

$$\begin{aligned} B_{q,n}(x, y) = & \frac{(1-q)(1-q^x)(q^{1+x}, q^{1+n}, q^y)_n}{(q^{1+x-y}, q^{y-x}, q^{x+y+n})_n} \\ & \times \left[ (1-q^{n+x}) \sum_{k=0}^n \frac{(q^{-n}, q^{x+y})_k (q^{n+y})_k^2}{(q^{1+n})_k (q^{2n+x+y})_{k+1} (q^x)_{k+1}^2} (1-q^{2k+n+x+y}) q^{nk+k+ky-kx} \right. \\ & \left. - q^{y-x} (1-q^n) \sum_{k=0}^{n-1} \frac{(q^{-n+1}, q^{x+y}, q^{n+x+1})_k (q^{n+x})_{k+1}}{(q^{1+n}, q^y, q^{2n+x+y}, q^y)_{k+1}} (1-q^{2k+n+x+y+1}) q^{nk+ky-kx} \right]. \end{aligned} \quad (4.6)$$

*Proof* Set  $a = -q^x, b = -q^y$  and  $c = q^{x+y}$  in (3.6), and after some simplifications, we obtain (4.6).  $\square$

**Corollary 4.6**

$$\sum_{k=0}^n \frac{(q^{-n})_k}{(-q)_{k+1}} (-1)^k q^{nk+k} + \sum_{k=0}^{n-1} \frac{(q^{-n+1}, -q^{n+2})_k}{(q, q^{n+1})_{k+1}} (-1)^k q^{nk} = \frac{2(-q)_{n-1}}{(1+q^{n+1})(q^{n+1})_n}. \quad (4.7)$$

*Proof* Set  $a = q$  and  $b = -q$  in (3.1), and after some simplifications, we obtain (4.5).  $\square$

Letting  $n \rightarrow \infty$  in (4.5), (4.6) and (4.7) and using (1.4), (1.5) and (1.6), we obtain respectively  $q$  - gamma,  $q$  - beta and eta function identities

$$\Gamma_q(x) = \frac{(-q^{1+x})_\infty (1-q)^{1-x}}{2(-q)_\infty^2} \left[ \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(-q^{x+1})_k} + (1+q^x) \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(q^{x+1})_k} \right],$$

$$\begin{aligned}
B_q(x, y) &= \frac{(1-q)(1-q^x)(q^{1+x}, q^y)_\infty}{(q^{1+x-y}, q^{y-x})_\infty} \\
&\quad \times \left[ \sum_{k=0}^n \frac{(-1)^k q^{k(k+1)/2} (q^{x+y})_k}{(q^x)_k^2} q^{k+kx-ky} \right. \\
&\quad \left. - q^{y-x} \sum_{k=0}^{n-1} \frac{(-1)^k q^{k(k+1)/2} (q^{x+y})_k}{(q^y)_k (q^y)_{k+1}} q^{ky-kx} \right], \\
\frac{\eta(2\tau)}{\eta(\tau)} &= \frac{q^{-1/24}}{2} \left[ \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(-q)_{k+1}} + \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(q)_{k+1}} \right].
\end{aligned}$$

**Conclusion** We see that the finite forms of reciprocity theorems are interesting and also preserve all the symmetries. A number of identities of the types (4.1) - (4.7) can be deduced from the finite forms of reciprocity theorems.

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### References

- [1] C.Adiga and N.Anitha, On a reciprocity theorem of Ramanujan, *Tamsui J. Math. Sci.*, **22** (2006), 9 - 15.
- [2] C.Adiga and P.S.Guruprasad, On a three variable reciprocity theorem, *South East Asian Journal of Mathematics and Mathematical Sciences*, Vol. **6**, 2(2008), 57 - 61.
- [3] C.Adiga and P.S.Guruprasad, A note on four variable reciprocity theorem, *Int. Jour. Math. Math Sci*, Vol. 2009 doi: 10, 1155/2009/370390.
- [4] G.E.Andrews, Ramanujans 'lost' notebook I, Partial  $\theta$ -functions, *Adv. in Math.*, **41** (1981), 137 - 172.
- [5] G.E.Andrews and B.C. Berndt, *Ramanujans Lost Notebook, Part II*, Springer, New York, 2009.
- [6] W.N.Bailey, Series of hypergeometric type which are infinite in both directions, *Quart. J. Math.*, **7** (1936), 105 - 115.
- [7] W.N.Bailey, On the basic bilateral hypergeometric series  ${}_2\psi_2$ , *Quart. J. Math. Oxford Ser.*, (2) **1** (1950), 194 - 198.
- [8] B.C.Berndt, S.H.Chan, B.P.Yeap, and A.J.Yee, A reciprocity theorem for certain q-series found in Ramanujan's lost notebook, *Ramanujan J.*, **13** (2007), 27 - 37.
- [9] S.Bhargava, D.D.Somashekara, and S.N.Fathima, Some q-gamma and q- beta function identities deducible from the reciprocity theorem of Ramanujan, *Adv. Stud. Contemp.*

- Math. (Kyungshang)*, **11** (2005), 227 - 234.
- [10] G.Gasper and M.Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics, Cambridge University press. Cambridge, 2004.
  - [11] P.S.Guruprasad and N.Pradeep, A simple proof of Ramanujans reciprocity theorem, *Proc. Jangjeon Math. Soc.*, **9** (2006), 121 - 124.
  - [12] F. H.Jackson, On q-definite integrals, *Quart. J. Pure and Appl. Math.*, **41** (1910), 193 - 203.
  - [13] F.Jouhet and M.Schlosser, Another proof of Bailey's  ${}_6\psi_6$  summation, *Aequationes Math.*, **70**, 1-2(2005), 43 - 50.
  - [14] S.-Y. Kang, Generalizations of Ramanujans reciprocity theorem and their applications, *J. London Math. Soc.*, **75** (2007), 18 - 34.
  - [15] T.Kim, D.D.Somashekara, and S.N.Fathima, On a generalization of Jacobis triple product identity and its application, *Adv. Stud. Contemp. Math. (Kyungshang)*, **9** (2004), 165 - 174.
  - [16] S.Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
  - [17] M.Schlosser, Abel.Ro, The type generalizations of Jacobi's triple product identity, In: Theory and Applications of Special Functions, M.E.H. Ismail and E. Koelink, Eds., *Dev. Math.*, **13** (2005) 383 - 400.
  - [18] L.J.Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.
  - [19] D.D.Somashekara and S.N.Fathima, An interesting generalization of Jacobi's triple product identity, *Far East J. Math. Sci.*, **9** (2003), 255 - 259.
  - [20] D.D.Somashekara and D.Mamta, On the three variable reciprocity theorem and its application, *Aus. jour. Math. Anal. Appl.*, Vol 9(1), 1-9, 2012.
  - [21] D.D.Somashekara and K.Narasimha Murthy, On a two variable reciprocity theorem Of Ramanujan *Bull. Pure Appl. Math.*, Vol. 5, No.1, 2011.
  - [22] J.Thomae, Beitrage zur Theorie der durch die Heinesche Reihe  $\dots$ , *J. Reine Angew. Math.*, **70**, (1869), 258 - 281.
  - [23] Z.Zhang, An identity related to Ramanujan's and its applications, *Indian J. Pure Appl. Math.*, to appear.

## The Jordan $\theta$ -Centralizers of Semiprime Gamma Rings with Involution

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**Abstract:** Let  $M$  be a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  satisfying a certain assumption and let  $\theta : M \rightarrow M$  be an endomorphism of  $M$ . We prove that if  $T : M \rightarrow M$  is an additive mapping such that  $2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$  holds for all  $x \in M$  and  $\alpha \in \Gamma$ , then  $T$  is a Jordan  $\theta$ -centralizer with involution.

**Key Words:** Semiprime  $\Gamma$ -ring, involution, semiprime  $\Gamma$ -ring with involution, centralizer,  $\theta$ -centralizer, Jordan  $\theta$ -centralizer.

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### §1. Introduction

An extensive generalized concept of classical ring set forth the notion of a gamma ring theory. As an emerging field of research, the research work of classical ring theory to the gamma ring theory has been drawn interest of many algebraists and prominent mathematicians over the world to determine many basic properties of gamma ring and to enrich the world of algebra. The different researchers on this field have been doing a significant contributions to this field from its inception. In recent years, a large number of researchers are engaged to increase the efficacy of the results of gamma ring theory over the world.

Let  $M$  and  $\Gamma$  be additive abelian groups. If there exists a mapping  $(x, \alpha, y) \rightarrow x\alpha y$  of  $M \times \Gamma \times M \rightarrow M$ , which satisfies the conditions

- (i)  $x\alpha y \in M$ ;
- (ii)  $(x + y)\alpha z = x\alpha z + y\alpha z$ ,  $x(\alpha + \beta)z = x\alpha z + x\beta z$ ,  $x\alpha(y + z) = x\alpha y + x\alpha z$ .
- (iii)  $(x\alpha y)\beta z = x\alpha(y\beta z)$  for all  $x, y, z \in M$  and  $\alpha, \beta \in \Gamma$ , then  $M$  is called a  $\Gamma$ -ring.

Every ring  $M$  is a  $\Gamma$ -ring with  $M = \Gamma$ . However a  $\Gamma$ -ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa[11]. Bernes[2] weakened slightly the conditions in the definition of  $\Gamma$ -ring in the sense of Nobusawa.

Let  $M$  be a  $\Gamma$ -ring. Then an additive subgroup  $U$  of  $M$  is called a left (right) ideal of  $M$  if  $M\Gamma U \subset U$  ( $U\Gamma M \subset U$ ). If  $U$  is both a left and a right ideal, then we say  $U$  is an ideal of  $M$ .

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Suppose again that  $M$  is a  $\Gamma$ -ring. Then  $M$  is said to be a 2-torsion free if  $2x=0$  implies  $x=0$  for all  $x \in M$ . An ideal  $P_1$  of a  $\Gamma$ -ring  $M$  is said to be prime if for any ideals  $A$  and  $B$  of  $M$ ,  $A\Gamma B \subseteq P_1$  implies  $A \subseteq P_1$  or  $B \subseteq P_1$ . An ideal  $P_2$  of a  $\Gamma$ -ring  $M$  is said to be semiprime if for any ideal  $U$  of  $M$ ,  $UTU \subseteq P_2$  implies  $U \subseteq P_2$ . A  $\Gamma$ -ring  $M$  is said to be prime if  $a\Gamma M\Gamma b=(0)$  with  $a, b \in M$ , implies  $a=0$  or  $b=0$  and semiprime if  $a\Gamma M\Gamma a=(0)$  with  $a \in M$  implies  $a=0$ . Furthermore,  $M$  is said to be commutative  $\Gamma$ -ring if  $x\alpha y=y\alpha x$  for all  $x, y \in M$  and  $\alpha \in \Gamma$ . Moreover, the set  $Z(M) = \{x \in M : x\alpha y = y\alpha x \text{ for all } \alpha \in \Gamma, y \in M\}$  is called the centre of the  $\Gamma$ -ring  $M$ .

If  $M$  is a  $\Gamma$ -ring, then  $[x, y]_\alpha = x\alpha y - y\alpha x$  is known as the commutator of  $x$  and  $y$  with respect to  $\alpha$ , where  $x, y \in M$  and  $\alpha \in \Gamma$ . We make the basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \text{ and } [x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$$

for all  $x, y, z \in M$  and  $\alpha, \beta \in \Gamma$ . We consider the following assumption:

$$(A) x\alpha y\beta z = x\beta y\alpha z \text{ for all } x, y, z \in M \text{ and } \alpha, \beta \in \Gamma.$$

According to the assumption (A), the above two identifies reduce to

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta \text{ and } [x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta,$$

which we extensively used.

An additive mapping  $T : M \rightarrow M$  is a left(right) centralizer if  $T(x\alpha y) = T(x)\alpha y(T(x\alpha y) = x\alpha T(y))$  holds for all  $x, y \in M$  and  $\alpha \in \Gamma$ . A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed  $a \in M$  and  $\alpha \in \Gamma$ , the mapping  $T(x) = a\alpha x$  is a left centralizer and  $T(x) = x\alpha a$  is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers. An additive mapping  $D : M \rightarrow M$  is called a derivation if  $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$  holds for all  $x, y \in M$ , and  $\alpha \in \Gamma$  and is called a Jordan derivation if  $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$  for all  $x \in M$  and  $\alpha \in \Gamma$ .

An additive mapping  $T : M \rightarrow M$  is Jordan left(right) centralizer if

$$T(x\alpha x) = T(x)\alpha x(T(x\alpha x) = x\alpha T(x))$$

for all  $x \in M$  and  $\alpha \in \Gamma$ . Every left centralizer is a Jordan left centralizer but the converse is not ingeneral true.

An additive mappings  $T : M \rightarrow M$  is called a Jordan centralizer if  $T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x)$  for all  $x, y \in M$  and  $\alpha \in \Gamma$ . Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Bernes[2], Luh [10] and Kyuno[9] studied the structure of  $\Gamma$ -rings and obtained various generalizations of corresponding parts in ring theory.

Borut Zalar [15] worked on centralizers of semiprime rings and proved that Jordan centralizers and centralizers of this rings coincide. Joso Vukman[12, 13, 14] developed some remarkable results using centralizers on prime and semiprime rings.

Y.Ceven [3] worked on Jordan left derivations on completely prime  $\Gamma$ -rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime  $\Gamma$ -ring that makes the

$\Gamma$ -ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime  $\Gamma$ -ring is a left derivation on it.

In [4], M. F. Hoque and A.C. Paul have proved that every Jordan centralizer of a 2-torsion free semiprime  $\Gamma$ -ring is a centralizer. There they also gave an example of a Jordan centralizer which is not a centralizer.

In [5], M. F. Hoque and A.C. Paul have proved that if  $M$  is a 2-torsion free semiprime  $\Gamma$ -ring satisfying the assumption (A) and if  $T : M \rightarrow M$  is an additive mapping such that  $T(x\alpha y\beta x) = x\alpha T(y)\beta x$  for all  $x, y \in M$  and  $\alpha, \beta \in \Gamma$ , then  $T$  is a centralizer. Also, they have proved that  $T$  is a centralizer if  $M$  contains a multiplicative identity 1.

Our research works are inspired by the works of [1], [5], [7] and [8] and we obtain the results in  $\Gamma$ -rings with involution by assuming an assumption (A).

## §2. The $\theta$ -Centralizers of Semiprime Gamma Rings with Involution

**Definition 2.1** Let  $M$  be a 2-torsion free semiprime  $\Gamma$ -ring and let  $\theta$  be an endomorphism of  $M$ . An additive mapping  $T : M \rightarrow M$  is a left(right)  $\theta$ -centralizer if  $T(x\alpha y) = T(x)\alpha\theta(y)$  ( $T(x\alpha y) = \theta(x)\alpha T(y)$ ) holds for all  $x, y \in M$  and  $\alpha \in \Gamma$ . If  $T$  is a left and a right  $\theta$ -centralizer, then it is natural to call  $T$  a  $\theta$ -centralizer.

**Definition 2.2** Let  $M$  be a  $\Gamma$ -ring and let  $a \in M$  and  $\alpha \in \Gamma$  be fixed element. Let  $\theta : M \rightarrow M$  be an endomorphism. Define a mapping  $T : M \rightarrow M$  by  $T(x)a\alpha\theta(x)$ . Then it is clear that  $T$  is a left  $\theta$ -centralizer. If  $T(x) = \theta(x)\alpha a$  is defined, then  $T$  is a right  $\theta$ -centralizer.

**Definition 2.3** An additive mapping  $T : M \rightarrow M$  is Jordan left(right)  $\theta$ -centralizer if  $T(x\alpha x) = T(x)\alpha\theta(x)$  ( $T(x\alpha x) = \theta(x)\alpha T(x)$ ) holds for all  $x \in M$  and  $\alpha \in \Gamma$ .

It is obvious that every left  $\theta$ -centralizer is a Jordan left  $\theta$ -centralizer but in general Jordan left  $\theta$ -centralizer is not a left  $\theta$ -centralizer [8, Example-2.1].

**Definition 2.4** Let  $M$  be a  $\Gamma$ -ring and let  $\theta$  be an endomorphism on  $M$ . An additive mapping  $T : M \rightarrow M$  is called a Jordan  $\theta$ -centralizer if  $T(x\alpha y + y\alpha x) = T(x)\alpha\theta(y) + \theta(y)\alpha T(x)$ , for all  $x, y \in M$  and  $\alpha \in \Gamma$ .

It is clear that every  $\theta$ -centralizer is a Jordan  $\theta$ -centralizer but the converse is not in general a  $\theta$ -centralizer [8, Example-2.2 and 2.3].

**Definition 2.5** An additive mapping  $D : M \rightarrow M$  is called a  $(\theta, \theta)$ -derivation if  $D(x\alpha y) = D(x)\alpha\theta(y) + \theta(x)\alpha D(y)$  holds for all  $x, y \in M$  and  $\alpha \in \Gamma$  and is called a Jordan  $(\theta, \theta)$ -derivation if  $D(x, x) = D(x)\alpha\theta(x) + \theta(x)\alpha D(x)$  holds for all  $x \in M$  and  $\alpha \in \Gamma$ .

We have given two examples in [8] which ensure that a  $\theta$ -centralizer and a Jordan  $\theta$ -centralizer exist in  $\Gamma$ -ring.

**Definition 2.6** Let  $M$  be a  $\Gamma$ -ring. Then the mapping  $I : M \rightarrow M$  is called an involution if

- (i)  $II(a) = a;$
- (ii)  $I(a + b) = I(a) + I(b);$
- (iii)  $I(a\alpha b) = I(b)\alpha I(a)$

for all  $a, b \in M$  and  $\alpha \in \Gamma$ .

**Example 2.1** Let  $R$  be a ring with involution  $I$  containing the unity element 1. Let  $M = M_{1,2}(R)$  and  $\Gamma = \left\{ \begin{pmatrix} n_{1.1} \\ n_{2.1} \end{pmatrix} : n_1, n_2 \in Z \right\}$ . Then  $M$  is a  $\Gamma$ -ring. We define an involution  $I : M \rightarrow M$  by

$$I(a, b) = (I(a), I(b))$$

$$II(a, b) = (II(a), II(b)) = (a, b)$$

$$\begin{aligned} I((a, b) + (c, d)) &= I(a + c, b + d) \\ &= (I(a + c), I(b + d)) \\ &= (I(a) + I(c), I(b) + I(d)) \\ &= (I(a), I(b)) + (I(c), I(d)) \\ &= I(a, b) + I(c, d) \end{aligned}$$

Now

$$\begin{aligned} I \left( (a, b) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} (c, d) \right) &= I((an_1 + bn_2)(c, d)) \\ &= I(an_1c + bn_2c, an_1d + bn_2d) \\ &= (I(an_1c + bn_2c), I(an_1d + bn_2d)) \\ &= (I(an_1c) + I(bn_2c), I(an_1d) + I(bn_2d)) \\ &= (I(c)n_1I(a) + I(c)n_2I(b), I(d)n_1I(a) + I(d)n_2I(b)) \\ &= (I(c), I(d)) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} (I(a), I(b)) \\ &= I(c, d)\alpha I(a, b), \end{aligned}$$

where  $\alpha = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ .

**Definition 2.7** Let  $M$  be a 2-torsion free semiprime  $\Gamma$  ring with involution  $I$  and let  $\theta : M \rightarrow M$  be an endomorphism of  $M$ . An additive mapping  $T : M \rightarrow M$  is called a left(right) Jordan  $\theta$ -centralizer with involutio if for all  $x \in M$ ,  $\alpha \in \Gamma$ ,

$$T(x\alpha x) = T(x)\alpha\theta(I(x))(T(x\alpha x) = \theta(I(x))\alpha T(x)).$$

If  $T$  is both left and right Jordan  $\theta$ -centralizer of  $M$  with involution, then it is called Jordan  $\theta$ -centralizer of  $M$  with involution.

First, we need the following Lemmas, for proving our main results:

**Lemma 2.1** Suppose  $M$  is a semiprime  $\Gamma$ -ring satisfying the assumption (A). Suppose that the relation  $x\alpha\beta y + y\alpha\beta z = 0$  holds for all  $a \in M$ , some  $x, y, z \in M$  and  $\alpha, \beta \in \Gamma$ . Then  $(x + z)\alpha\beta y = 0$  for all  $a \in M$  and  $\alpha, \beta \in \Gamma$ .

**Lemma 2.2** Suppose  $M$  is a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  and satisfying the assumption (A). Let  $T : M \rightarrow M$  be an additive mapping such that

$$2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all  $x \in M$ ,  $\alpha \in \Gamma$  and  $\theta$  is an endomorphism on  $M$ . Then

$$\begin{aligned} 2T(x\alpha y + y\alpha x) &= T(x)\alpha\theta(I(y)) + T(y)\alpha\theta(I(x)) \\ &\quad + \theta(I(x))\alpha T(y) + \theta(I(y))\alpha T(x) \end{aligned}$$

for all  $x, y \in M$ .

*Proof* We have  $2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$ . By linearizing, the above relation becomes

$$\begin{aligned} 2T(x\alpha y + y\alpha x) &= T(x)\alpha\theta(I(y)) + T(y)\alpha\theta(I(x)) \\ &\quad + \theta(I(x))\alpha T(y) + \theta(I(y))\alpha T(x). \end{aligned} \tag{1}$$

This completes the proof.  $\square$

**Lemma 2.3** Suppose  $M$  is a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  and satisfying the assumption (A). Let  $T : M \rightarrow M$  be an additive mapping such that

$$2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all  $x \in M$ ,  $\alpha \in \Gamma$  and  $\theta$  is an endomorphism on  $M$ . Then

$$\begin{aligned} 8T(x\alpha y\beta x) &= T(x)\alpha\theta(I(x)\beta I(y) + 3I(y)\beta I(x)) + \theta(I(y)\beta I(x)) \\ &\quad + 3I(x)\beta I(y)\alpha T(x) + 2\theta(I(x))\beta T(y)\alpha\theta(I(x)) \\ &\quad - \theta(I(x)\alpha I(x)\beta T(y) - T(y)\beta\theta(I(x)\alpha I(x))) \end{aligned}$$

for all  $x, y \in M$ .

*Proof* Putting  $2(x\beta y + y\beta x)$  for  $y$  in (1) and using Lemma 2.2, we get

$$\begin{aligned} &4T(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) \\ &= T(x)\alpha\theta(2I(x)\beta I(y) + 3I(y)\beta I(x)) + \theta(3I(x)\beta I(y)) \\ &\quad + 2I(y)\beta I(x)\alpha T(x) + \theta(I(x))\alpha T(x)\beta\theta(I(y)) \\ &\quad + \theta(I(x)\alpha I(x)\beta T(y) + T(y)\beta\theta(I(x)\alpha I(x))) \\ &\quad + 2\theta(I(x))\beta T(y)\alpha\theta(I(x)) + \theta(I(y)\beta T(x)\alpha\theta(I(x))) \end{aligned} \tag{2}$$



On the other hand

$$\begin{aligned} 4T(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) &= 4T(x\alpha x\beta y + y\beta x\alpha x) \\ &\quad + 8T(x\alpha y\beta x) \end{aligned}$$

Now, using hypothesis, we obtain

$$\begin{aligned} 4T(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) &= T(x)\alpha\theta(I(x)\beta I(y)) + \theta(I(y)\beta I(x))\alpha T(x) + \\ &\quad \theta(I(x))\alpha T(x)\beta\theta(I(y)) + \theta(I(y))\beta T(x)\alpha\theta(I(x)) \\ &\quad + 2\theta(I(x)\alpha I(x))\beta T(y) + 2T(y)\beta\theta(I(x)\alpha I(x)) + 8T(x\alpha y\beta x) \end{aligned} \quad (3)$$

Then from (2) and (3), we have

$$\begin{aligned} 8T(x\alpha y\beta x) &= T(x)\alpha\theta(I(x)\beta I(y) + 3I(y)\beta I(x)) + \theta(I(y)\beta I(x)) \\ &\quad + 3I(x)\beta I(y))\alpha T(x) + 2\theta(I(x))\beta T(y)\alpha\theta(I(x)) \\ &\quad - \theta(I(x)\alpha I(x))\beta T(y) - T(y)\beta\theta(I(x)\alpha I(x)) \end{aligned} \quad (4)$$

for all  $x, y \in M$ . This completes the proof.  $\square$

**Lemma 2.4** *Suppose  $M$  is a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  and satisfying the assumption (A). Let  $T : M \rightarrow M$  be an additive mapping such that*

$$2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

*holds for all  $x \in M$ ,  $\alpha \in \Gamma$  and  $\theta$  is an endomorphism on  $M$ . Then*

$$\begin{aligned} 0 &= T(x)\alpha\theta(I(x)\gamma I(y)\beta I(x) - 2I(y)\gamma I(x)\beta I(x) \\ &\quad - 2I(x)\beta(x)\gamma(y)) + \theta(I(x)\gamma I(y)\beta I(x) \\ &\quad - 2I(y)\gamma I(x)\beta I(x) - 2I(x)\beta I(x)\gamma I(y))\alpha T(x) \\ &\quad + \theta(I(x)\alpha T(x)\beta\theta(I(x)\gamma I(y) + I(y)\gamma I(x)) \\ &\quad + \theta(I(x)\gamma I(y) + I(y)\gamma I(x))\beta T(x)\alpha\theta(I(x)) \\ &\quad + \theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y)) \\ &\quad + \theta(I(y))\beta T(x)\gamma\theta(I(x)I(x)) \end{aligned}$$

*for all  $x, y \in M$  and  $\alpha, \beta, \gamma \in \Gamma$ .*

*Proof* Putting  $8(x\beta y\gamma x)$  for  $y$  in (1) and using lemma (2.3), we obtain

$$\begin{aligned}
16T(x\alpha x\beta y\gamma x + x\beta y\gamma x\alpha x) &= T(x)\alpha\theta(9I(x)\gamma I(y)\beta I(x) + 3I(y)\gamma I(x)\beta I(x)) \\
&+ \theta(9I(x)\gamma I(y)\beta I(x) + 3I(x)\beta I(x)\gamma I(y))\alpha T(x) \\
&+ \theta(I(x))\alpha T(x)\beta\theta(I(x)\gamma I(y) + 3I(y)\gamma I(x)) \\
&+ \theta(I(y)\gamma I(x) + 3I(x)\gamma I(y))\beta T(x)\alpha\theta(I(x)) \\
&- T(y)\gamma\theta(I(x)\beta I(x)\alpha I(x)) \\
&+ \theta(I(x)I(x))\gamma T(y)\beta\theta(I(x)) \\
&+ \theta(I(x))\gamma T(y)\beta\theta(I(x)\alpha I(x)) \\
&- \theta(I(x)\alpha I(x)\beta I(x))\gamma T(y). \tag{5}
\end{aligned}$$

On the other hand using (4) and then after collecting some terms using Lemma 2.2, we obtain

$$\begin{aligned}
16T(x\alpha x\beta y\gamma x + x\beta y\gamma x\alpha x) &= T(x)\alpha\theta(2I(x)\beta I(x)\gamma I(y) \\
&+ 5I(y)\beta I(x)\gamma I(x) + 8I(x)\gamma I(y)\beta I(x)) \\
&+ \theta(2I(y)\beta I(x)\gamma I(x) + 5I(x)\beta I(x)\gamma I(y)) \\
&+ 8I(x)\gamma I(y)\beta I(x))\alpha T(x) \\
&+ 2\theta(I(x))\gamma T(x)\beta\theta(I(y)\alpha I(x)) \\
&+ 2\theta(I(x)\gamma I(y))\beta T(x)\alpha\theta(I(x)) \\
&+ \theta(I(x)\alpha I(x))\gamma T(y)\beta\theta(I(x)) \\
&+ \theta(I(x))\gamma T(y)\beta\theta(I(x)\alpha I(x)) \\
&- \theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y)) \\
&- \theta(I(y))\beta T(x)\gamma\theta(I(x)\alpha I(x)) \\
&- \theta(I(x)\beta I(x)\alpha I(x))\gamma T(y) \\
&- T(y)\gamma\theta(I(x)\alpha I(x)\beta I(x)). \tag{6}
\end{aligned}$$

By comparing (5) and (6), we get

$$\begin{aligned}
0 &= T(x)\alpha\theta(I(x)\gamma I(y)\beta I(x) - 2I(y)\gamma I(x)\beta I(x) - 2I(x)\beta(x)\gamma(y)) \\
&+ \theta(I(x)\gamma I(y)\beta I(x) - 2I(y)\gamma I(x)\beta I(x) - 2I(x)\beta I(x)\gamma I(y))\alpha T(x) \\
&+ \theta(I(x)\alpha T(x)\beta\theta(I(x)\gamma I(y) + I(y)\gamma I(x)) + \theta(I(x)\gamma I(y) \\
&+ I(x)\gamma I(y))\beta T(x)\alpha\theta(I(x)) + \theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y)) \\
&+ \theta(I(y))\beta T(x)\gamma\theta(I(x)I(x)), \tag{7}
\end{aligned}$$

for  $x, y \in M$  and  $\alpha, \beta, \gamma \in \Gamma$ .  $\square$

**Lemma 2.5** Suppose  $M$  is a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  and satisfying the assumption (A). Let  $T : M \rightarrow M$  be an additive mapping such that

$$2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all  $x \in M$ ,  $\alpha \in \Gamma$  and  $\theta$  is an endomorphism on  $M$ . Then

$$\begin{aligned}
0 &= \theta(I(x)\gamma I(y))\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\
&\quad + 2\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + 2\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + \theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\
&\quad + \theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma\theta(I(x)\alpha I(x))
\end{aligned}$$

for all  $x, y \in M$  and  $\alpha, \beta, \gamma \in \Gamma$ .

*Proof* Replacing  $y$  by  $x\alpha y$  in (7), we have

$$\begin{aligned}
0 &= T(x)\alpha\theta(I(x)\gamma I(y)\alpha I(x)\beta I(x) - 2I(y)\alpha I(x)\gamma I(x)\beta I(x) \\
&\quad - 2I(x)\beta(x)\gamma(y)\alpha I(x)) + \theta(I(x)\gamma I(y)\alpha I(x)\beta I(x) \\
&\quad - 2I(y)\alpha I(x)\gamma I(x)\beta I(x) - 2I(x)\beta I(x)\gamma I(y)\alpha I(x))\alpha T(x) \\
&\quad + \theta(I(x))\alpha T(x)\beta\theta(I(x)\gamma I(y)\alpha I(x) + I(y)\alpha I(x)\gamma I(x)) \\
&\quad + \theta(I(x)\gamma I(y)\alpha I(x) + I(y)\alpha I(x)\gamma I(x))\beta T(x)\alpha\theta(I(x)) \\
&\quad + \theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y)\alpha I(x)) \\
&\quad + \theta(I(y)\alpha I(x))\beta T(x)\gamma\theta(I(x)\alpha I(x)). \tag{8}
\end{aligned}$$

Right multiplication of (7) by  $\theta(I(x))$ , we get

$$\begin{aligned}
0 &= T(x)\alpha\theta(I(x)\gamma I(y)\beta I(x)\alpha(I(x)) - 2I(y)\gamma I(x)\beta I(x)\alpha I(x) \\
&\quad - 2I(x)\beta(x)\gamma(y)\alpha I(x)) + \theta(I(x)\gamma I(y)\beta I(x) \\
&\quad - 2I(y)\gamma I(x)\beta I(x) - 2I(x)\beta I(x)\gamma I(y))\alpha T(x)\alpha\theta(I(x)) \\
&\quad + \theta(I(x)\alpha T(x))\beta\theta(I(x)\gamma I(y) \\
&\quad + I(y)\gamma I(x))\alpha\theta(I(x)) + \theta(I(x)\gamma I(y) \\
&\quad + I(y)\gamma I(x))\beta T(x)\alpha\theta(I(x)\alpha I(x)) \\
&\quad + \theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y)\alpha I(x)) \\
&\quad + \theta(I(y))\beta T(x)\gamma\theta(I(x)\alpha I(x)\alpha I(x)). \tag{9}
\end{aligned}$$

Subtracting (9) from (8) and using assumption(A), we have

$$\begin{aligned}
0 &= \theta(I(x)\alpha I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \\
&\quad + 2\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + 2\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + \theta(I(x)\gamma I(y))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\
&\quad + \theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\
&\quad + \theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma\theta(I(x)\alpha I(x))
\end{aligned}$$

Now combining first and fourth term together this relation reduces as,

$$\begin{aligned}
0 &= \theta(I(x)\gamma I(y))\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\
&\quad + 2\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + 2\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + \theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha \theta(I(x)) \\
&\quad + \theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma \theta(I(x)\alpha I(x))
\end{aligned} \tag{10}$$

for all  $x, y \in M$  and  $\alpha, \beta, \gamma \in \Gamma$ .  $\square$

**Lemma 2.6** Suppose  $M$  is a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  and satisfying the assumption (A). Let  $T : M \rightarrow M$  be an additive mapping such that

$$2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all  $x \in M$ ,  $\alpha \in \Gamma$  and  $\theta$  is an endomorphism on  $M$ . Then

$$\begin{aligned}
0 &= [T(x), \theta(I(x))]_\alpha \gamma \theta(y)\beta[T(x), \theta(I(x)\alpha I(x))]_\alpha \\
&\quad - 2[T(x), \theta(I(x)\alpha I(x))]_\alpha \theta(y)\beta[T(x), \theta(I(x))]_\alpha
\end{aligned}$$

for all  $x, y \in M$  and  $\alpha, \beta, \gamma \in \Gamma$ .

*Proof* First replacing  $y$  by  $I(y)$  in (10), we have

$$\begin{aligned}
0 &= \theta(I(x)\gamma y)\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\
&\quad + 2\theta(I(x)\alpha I(x)\gamma y)\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + 2\theta(y\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + \theta(y\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha \theta(I(x)) \\
&\quad + \theta(y)\beta[\theta(I(x)), T(x)]_\alpha \gamma \theta(I(x)\alpha I(x))
\end{aligned}$$

Now putting  $\theta(y) = T(x)\alpha\theta(I(y))$

$$\begin{aligned}
0 &= \theta(I(x)\gamma T(x)\alpha\theta(I(y))\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\
&\quad + 2\theta(I(x)\alpha I(x)\gamma T(x)\alpha\theta(I(y))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + 2T(x)\alpha\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + T(x)\alpha\theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha \theta(I(x)) \\
&\quad + T(x)\alpha\theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma \theta(I(x)\alpha I(x)).
\end{aligned} \tag{11}$$

Left multiplication of (11) by  $T(x)\alpha$ , we get

$$\begin{aligned}
0 &= T(x)\alpha\theta(I(x)\gamma I(y))\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\
&\quad + 2T(x)\alpha\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + 2T(x)\alpha\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\
&\quad + T(x)\alpha\theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha \theta(I(x)) \\
&\quad + T(x)\alpha\theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma \theta(I(x)\alpha I(x)).
\end{aligned} \tag{12}$$

Subtracting (12) from (11), we arrive at

$$\begin{aligned} 0 &= [T(x), \theta(I(x))]_{\alpha} \gamma \theta(I(y)) \beta [T(x), \theta(I(x)I(x))]_{\alpha} - \\ &\quad 2[T(x), \theta(I(x)\alpha I(x))]_{\alpha} \gamma \theta(I(y)) \beta [T(x), \theta(I(x))]_{\alpha} \end{aligned}$$

Replacing  $y$  by  $I(y)$ , we get

$$\begin{aligned} 0 &= [T(x), \theta(I(x))]_{\alpha} \gamma \theta(y) \beta [T(x), \theta(I(x)\alpha I(x))]_{\alpha} \\ &\quad - 2[T(x), \theta(I(x)\alpha I(x))]_{\alpha} \gamma \theta(y) \beta [T(x), \theta(I(x))]_{\alpha} \end{aligned} \quad (13)$$

for all  $x, y \in M$  and  $\alpha, \beta, \gamma \in \Gamma$ .  $\square$

**Lemma 2.7** *Suppose  $M$  is a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  and satisfying the assumption (A). Let  $T : M \rightarrow M$  be an additive mapping such that*

$$2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

*holds for all  $x \in M$ ,  $\alpha \in \Gamma$  and  $\theta$  is an endomorphism on  $M$ . Then*

$$[T(x), \theta(I(x)\alpha I(x))]_{\alpha} = 0$$

*for all  $x \in M$ .*

*Proof* Now replacing  $\theta(y)$  by  $r$  and taking  $a = [T(x), \theta(I(x))]_{\alpha}$   $b = [T(x), \theta(I(x)\alpha I(x))]_{\alpha}$  and  $c = -2[T(x), \theta(I(x)\alpha I(x))]_{\alpha}$  in (13), we get  $a\gamma r\beta b + c\gamma r\beta a = 0$  for all  $r \in M$ . Hence using Lemma 2.1, we obtain that  $(c + b)\beta r\gamma a = 0$ , which implies that

$$[T(x), \theta(I(x)\alpha I(x))]_{\alpha} \beta r\gamma [T(x), \theta(I(x))]_{\alpha} = 0.$$

Using this relation, we arrive at

$$\begin{aligned} 0 &= [T(x), \theta(I(x)\alpha I(x))]_{\alpha} \beta r\gamma (\theta(I(x))\alpha [T(x), \theta(I(x))]_{\alpha} \\ &\quad + [T(x), \theta(I(x))]_{\alpha} \alpha \theta(I(x))) \end{aligned}$$

We therefore have

$$[T(x), \theta(I(x)\alpha I(x))]_{\alpha} \beta r\gamma [T(x), \theta(I(x)\alpha I(x))]_{\alpha} = 0$$

for all  $r \in M$ .

Hence by semiprimeness of  $M$ , we have

$$[T(x), \theta(I(x)\alpha I(x))]_{\alpha} = 0, \quad (14)$$

for all  $x \in M$ .  $\square$

**Theorem 2.1** *Suppose  $M$  is a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  and satisfying the assumption (A). Let  $T : M \rightarrow M$  be an additive mapping such that*

$$2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all  $x \in M$ ,  $\alpha \in \Gamma$  and  $\theta$  is an endomorphism on  $M$ . Then  $T$  is a Jordan  $\theta$ -centralizer.

*Proof* Linearizing the relation given in Lemma-2.7, we get

$$\begin{aligned} 0 &= [T(x), \theta(I(y)\alpha I(y))]_{\alpha} + [T(y), \theta(I(x)\alpha I(x))]_{\alpha} + [T(x), \theta(I(x)\alpha I(y) \\ &\quad + I(y)\alpha I(x))]_{\alpha} + [T(y), \theta(I(x)\alpha I(y) + I(y)\alpha I(x))]_{\alpha} \end{aligned}$$

Putting in above relation  $-x$  for  $x$  and comparing the relation so obtained with the above relation and by 2-torsion freeness of  $M$ ,

$$[T(x), \theta(I(x)\alpha I(y) + I(y)\alpha I(x))]_{\alpha} + [T(y), \theta(I(x)\alpha I(x))]_{\alpha} = 0 \quad (15)$$

Replacing  $2(x\beta y + y\beta x)$  for  $y$  and using Lemma 2.7, we obtain

$$\begin{aligned} 0 &= [T(x), \theta(I(x)\alpha 2I(x\beta y + y\beta x) + 2I(x\beta y + y\beta x)\alpha I(x))]_{\alpha} \\ &\quad + [T(x\beta y + y\beta x), \theta(I(x)\alpha I(x))]_{\alpha} \\ &= 2[T(x), \theta(I(x)\alpha I(x)\beta I(y) + I(y)\beta I(x)\alpha I(x) + 2I(x)\alpha I(y)\beta I(x))]_{\alpha} + \\ &\quad [T(x)\beta\theta(I(y)) + \theta(I(x))\beta T(y) + T(y)\beta\theta(I(x)) \\ &\quad + \theta(I(y))\beta T(x), \theta(I(x)\alpha I(x))]_{\alpha} \end{aligned}$$

Thus we have

$$\begin{aligned} 0 &= 2\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_{\alpha} + 2[T(x), \theta(I(y))]_{\alpha}\beta\theta(I(x)\alpha I(x)) \\ &\quad 4[T(x), \theta(I(x)\alpha I(y)\beta I(x))]_{\alpha} + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha} \\ &\quad + [\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha}\beta T(x) + \theta(I(x))\beta[T(y), \theta(I(x)\alpha I(x))]_{\alpha} + \\ &\quad [T(y), \theta(I(x)\alpha I(x))]_{\alpha}\beta\theta(I(x)). \end{aligned} \quad (16)$$

Hence in particular, we find that

$$\begin{aligned} 0 &= \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_{\alpha} + [T(x), \theta(I(x))]_{\alpha}\beta\theta(I(x)\alpha I(x)) \\ &\quad + 2[T(x), \theta(I(x)\beta I(x)\alpha I(x))]_{\alpha} \end{aligned}$$

In view of Lemma-2.7, this reduces to

$$\begin{aligned} 0 &= \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_{\alpha} + 3[T(x), \theta(I(x))]_{\alpha}\beta \\ &\quad \theta(I(x)\alpha I(x)). \end{aligned} \quad (17)$$

According to Lemma 2.7, we get

$$[T(x), \theta(I(x))]_{\alpha}\beta\theta(I(x)) + \theta(I(x))\beta[T(x), \theta(I(x))]_{\alpha} = 0$$

Hence using the later relation, we find that

$$\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_{\alpha} = [T(x), \theta(I(x))]_{\alpha}\beta\theta(I(x)\alpha I(x))$$

Further using this replacement in (17), we have

$$[T(x), \theta(I(x))]_{\alpha}\beta\theta(I(x)\alpha I(x)) = 0 \quad (18)$$

for all  $x \in M$  and  $\alpha, \beta \in \Gamma$ .

Similarly

$$\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_{\alpha} = 0 \quad (19)$$

for all  $x \in M$  and  $\alpha, \beta \in \Gamma$ .

We also have

$$\theta(I(x))\beta[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(x)) = 0 \quad (20)$$

for all  $x \in M$  and  $\alpha, \beta \in \Gamma$ .

From (15) we have

$$[T(x), \theta(I(x)\alpha I(y) + I(y)\alpha I(x))]_{\alpha} = -[T(y), \theta(I(x)\alpha I(x))]_{\alpha}$$

and combining this fact with (16), we arrive at

$$\begin{aligned} 0 &= 2\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_{\alpha} + 2[T(x), \theta(I(y))]_{\alpha}\beta\theta(I(x)\alpha I(x)) \\ &\quad + 4[T(x), \theta(I(x)\beta I(y)\alpha I(x))]_{\alpha} + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha} \\ &\quad + [\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha}\beta T(x) - \theta(I(x))\beta[T(x), \theta(I(x)\alpha I(y) \\ &\quad + I(y)\alpha I(x))]_{\alpha} - [T(x), \theta(I(x)\alpha I(y) + I(y)\alpha I(x))]_{\alpha}\beta\theta(I(x)) \\ &= 2\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_{\alpha} + 2[T(x), \theta(I(y))]_{\alpha}\beta\theta(I(x)\alpha I(x)) \\ &\quad + 4[T(x), \theta(I(x))\beta\theta(I(y)\alpha I(x))]_{\alpha} + 4\theta(I(x))\beta[T(x), \theta(I(y))]_{\alpha}\alpha\theta(I(x)) \\ &\quad + 4\theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]_{\alpha} + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha} \\ &\quad + [\theta(I(y), \theta(I(x)\alpha I(x))]_{\alpha}\beta T(x) - \theta(I(x))\beta[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(y)) \\ &\quad - \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_{\alpha} - \theta(I(x))\beta[T(x), \theta(I(y))]_{\alpha}\alpha\theta(I(x)) \\ &\quad - \theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]_{\alpha} - [T(x), \theta(I(x))]_{\alpha}\beta\theta(I(y)\alpha I(x)) \\ &\quad - \theta(I(x))\beta[T(x), \theta(I(y))]_{\alpha}\alpha\theta(I(x)) - [T(x), \theta(I(y))]_{\alpha}\beta\theta(I(x)\alpha I(x)) \\ &\quad - \theta(I(y))\beta[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(x)) \end{aligned}$$

Hence, we have

$$\begin{aligned} 0 &= \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_{\alpha} + [T(x), \theta(I(y))]_{\beta}\theta(I(x)\alpha I(x)) \\ &\quad + 3[T(x), \theta(I(x))]_{\alpha}\beta\theta(I(y)\alpha I(x)) + 3\theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]_{\alpha} \\ &\quad + 2\theta(I(x))\beta[T(x), \theta(I(y))]_{\alpha}\alpha\theta(I(x)) + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha} \\ &\quad + [\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha}\beta T(x) - \theta(I(x))\beta[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(y)) \\ &\quad - \theta(I(y))\beta[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(x)). \end{aligned} \quad (21)$$

Replacing  $y$  by  $x\gamma y$ , we have

$$\begin{aligned}
0 &= \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y)\gamma I(x))]_{\alpha} \\
&\quad + [T(x), \theta(I(y)\gamma I(x))]_{\alpha}\beta\theta(I(x)\alpha I(x)) \\
&\quad + 3[T(x), \theta(I(x))]_{\alpha}\beta\theta(I(y)\gamma I(x)\alpha I(x)) \\
&\quad + 3\theta(I(x)\alpha I(y)\gamma I(x))\beta[T(x), \theta(I(x))]_{\alpha} \\
&\quad + 2\theta(I(x))\beta[T(x), \theta(I(y)\gamma I(x))]_{\alpha}\alpha\theta(I(x)) \\
&\quad + T(x)\beta[\theta(I(y)\gamma I(x)), \theta(I(x)\alpha I(x))]_{\alpha} \\
&\quad + [\theta(I(y)\gamma I(x)), \theta(I(x)\alpha I(x))]_{\alpha}\beta T(x) \\
&\quad - \theta(I(x))\beta[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(y)\gamma I(x)) \\
&\quad - \theta(I(y)\gamma I(x))\beta[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(x))
\end{aligned}$$

This can be written as (also using assumption (A))

$$\begin{aligned}
0 &= \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_{\alpha}\gamma\theta(I(x)) \\
&\quad + \theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_{\alpha} \\
&\quad + [T(x), \theta(I(y))]_{\alpha}\beta\theta(I(x)\alpha I(x)\gamma I(x)) \\
&\quad + \theta(I(y))\beta[T(x), \theta(I(x))]_{\alpha}\gamma\theta(I(x)\alpha I(x)) \\
&\quad + 3[T(x), \theta(I(x))]_{\alpha}\beta\theta(I(y)\gamma I(x)\alpha I(x)) \\
&\quad + 3\theta(I(x)\alpha I(y)\gamma I(x))\beta[T(x), \theta(I(x))]_{\alpha} \\
&\quad + 2\theta(I(x))\beta[T(x), \theta(I(y))]_{\alpha}\gamma\theta(I(x)\alpha I(x)) \\
&\quad + 2\theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]_{\alpha}\gamma\theta(I(x)) \\
&\quad + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha}\gamma\theta(I(x)) \\
&\quad + [\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha}\gamma\theta(I(x))\beta T(x) \\
&\quad - \theta(I(x))\beta[T(x), \theta(I(x))]_{\alpha}\gamma\theta(I(y)I(x)) \\
&\quad - \theta(I(y)\alpha I(x))\beta[T(x), \theta(I(x))]_{\alpha}\gamma\theta(I(x))
\end{aligned}$$

In view of (18) and (20), we have

$$\begin{aligned}
0 &= \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_{\alpha}\gamma\theta(I(x)) \\
&\quad + \theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_{\alpha} \\
&\quad + [T(x), \theta(I(y))]_{\alpha}\beta\theta(I(x)\alpha I(x)\gamma I(x)) \\
&\quad + 3[T(x), \theta(I(x))]_{\alpha}\beta\theta(I(y)\gamma I(x)\alpha I(x)) \\
&\quad + 3\theta(I(x)\alpha I(y)\gamma I(x))\beta[T(x), \theta(I(x))]_{\alpha} \\
&\quad + 2\theta(I(x))\beta[T(x), \theta(I(y))]_{\alpha}\gamma\theta(I(x)\alpha I(x)) \\
&\quad + 2\theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]_{\alpha}\gamma\theta(I(x)) \\
&\quad + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha}\gamma\theta(I(x)) \\
&\quad + [\theta(I(y)), \theta(I(x)\alpha I(x))]_{\alpha}\gamma\theta(I(x))\beta T(x) \\
&\quad - \theta(I(x))\beta[T(x), \theta(I(x))]_{\alpha}\gamma\theta(I(y)\alpha I(x)). \tag{22}
\end{aligned}$$



Right multiplication of (21) by  $\gamma\theta(I(x))$  gives

$$\begin{aligned}
0 = & \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]\alpha\gamma\theta(I(x)) \\
& + [T(x), \theta(I(y))]\alpha\beta\theta(I(x)\alpha I(x)\gamma I(x)) \\
& + 3[T(x), \theta(I(x))]\alpha\beta\theta(I(y)\alpha I(x)\gamma I(x)) \\
& + 3\theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]\alpha\gamma\theta(I(x)) \\
& + 2\theta(I(x))\beta[T(x), \theta(I(y))]\alpha\alpha\theta(I(x)\gamma I(x)) \\
& + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]\alpha\gamma\theta(I(x)) \\
& + [\theta(I(y)), \theta(I(x)\alpha I(x))]\alpha\beta T(x)\gamma\theta(I(x)) \\
& - \theta(I(x))\beta[T(x), \theta(I(x))]\alpha\alpha\theta(I(y)\gamma I(x))
\end{aligned} \tag{23}$$

Subtracting (23) from (22), we get (also we using assumption (A))

$$\begin{aligned}
0 = & \theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]\alpha \\
& + 3\theta(I(x)\gamma I(y))\beta[\theta(I(x)), [T(x), \theta(I(x))]]\alpha \\
& + 2\theta(I(x)\gamma I(y))\beta[T(x), \theta(I(x))]\alpha\alpha\theta(I(x)) \\
& + [\theta(I(y)), \theta(I(x)\gamma I(x))]\alpha\beta[\theta(I(x)), T(x)]\alpha
\end{aligned}$$

Further in view of (19) this yields

$$\begin{aligned}
0 = & 2\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]\alpha \\
& + 3\theta(I(x)\gamma I(y)\alpha I(x))\beta[T(x), \theta(I(x))]\alpha \\
& - \theta(I(x)\gamma I(y))\beta[T(x), \theta(I(x))]\alpha\alpha\theta(I(x))
\end{aligned}$$

In view of Lemma 2.7, the above relation yields that

$$\begin{aligned}
0 = & \theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]\alpha \\
& + 2\theta(I(x)\gamma I(y)\alpha I(x))\beta[T(x), \theta(I(x))]\alpha
\end{aligned} \tag{24}$$

Further application of Lemma 2.7, (18), (19), (20) together with Lemma 2.5 yields that

$$0 = \theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]\alpha. \tag{25}$$

Hence combining (24) and (25) and by 2-torsion freeness of  $M$ , we have

$$\theta(I(x)\gamma I(y)\alpha I(x))\beta[T(x), \theta(I(x))]\alpha = 0$$

Now put  $y = I(y)$ , we have

$$\theta(I(x)\gamma y\alpha I(x))\beta[T(x), \theta(I(x))]\alpha = 0$$

Now replacing  $\theta(y)$  by  $[T(x), \theta(I(x))]\alpha\alpha\theta(y)$  in the later expression, we have (also using assumption (A))

$$\theta(I(x))\alpha[T(x), \theta(I(x))]\alpha\gamma\theta(y)\beta\theta(I(x))\alpha[T(x), \theta(I(x))]\alpha = 0$$

As  $\theta$  is an endomorphism, the semiprimeness of  $M$  gives

$$\theta(I(x))\alpha[T(x), \theta(I(x))]_{\alpha} = 0 \quad (26)$$

and hence in view of Lemma 2.7, we can write

$$[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(x)) = 0. \quad (27)$$

Linearizing (26) and using (27), we get

$$\begin{aligned} 0 &= \theta(I(x))\alpha[T(x), \theta(I(y))]_{\alpha} + \theta(I(x))\alpha[T(y), \theta(I(x))]_{\alpha} \\ &\quad + \theta(I(x))\alpha[T(y), \theta(I(y))]_{\alpha} + \theta(I(y))\alpha[T(x), \theta(I(x))]_{\alpha} \\ &\quad + \theta(I(y))\alpha[T(x), \theta(I(y))]_{\alpha} + \theta(I(y))\alpha[T(y), \theta(I(x))]_{\alpha} \end{aligned}$$

Putting in the above relation  $-x$  for  $x$  and comparing the relation so obtained with the above we get,

$$\begin{aligned} 0 &= \theta(I(x))\alpha[T(x), \theta(I(y))]_{\alpha} + \theta(I(x))\alpha[T(y), \theta(I(x))]_{\alpha} \\ &\quad + \theta(I(y))\alpha[T(x), \theta(I(x))]_{\alpha} \end{aligned}$$

Now, multiply the above relation by  $[T(x), \theta(I(x))]_{\alpha}\alpha$  from left and use (27), we have

$$[T(x), \theta(I(x))]_{\alpha}\alpha\theta(I(y))\alpha[T(x), \theta(I(x))]_{\alpha} = 0$$

This follows that,

$$[T(x), \theta(I(x))]_{\alpha} = 0. \quad (28)$$

Combining (28) with our hypothesis, we get

$$T(x\alpha x) = T(x)\alpha\theta(I(x)) \text{ for all } x \in M$$

and

$$T(x\alpha x) = \theta(I(x))\alpha T(x) \text{ for all } x \in M$$

This means that  $T$  is a left and right Jordan  $\theta$ -centralizer. This complete the proof of our theorem.  $\square$

**Corollary 2.1** Suppose that  $M$  be a 2-torsion free semiprime  $\Gamma$ -ring with involution  $I$  and satisfying the assumption(A). If  $T : M \rightarrow M$  be an additive mapping such that

$$2T(x\alpha x) = T(x)\alpha I(x) + I(x)\alpha T(x)$$

holds for all  $x \in M$  and  $\alpha \in \Gamma$ , then  $T$  is a Jordan centralizer with involution  $I$ .

## References

- [1] M.Ashraf and M.R. Mozumder, On Jordan  $\alpha$ -\* centralizers in semiprime rings with involution, *Int. J. Contemp. Math. Sciences*, Vol.7, 2012, No.23, 1103 - 1112.

- [2] W.E.Bernes, On the  $\Gamma$ -rings of Nobusawa, *Pacific J.Math.*, 18(1966), 411-422.
- [3] Y.Ceven, Jordan left derivations on completely prime gamma rings, C.U.Fen-Edebiyat Fakultesi,Fen Bilimleri Dergisi (2002)Cilt 23 Sayı 2.
- [4] M.F.Hoque and A.C.Paul, On centralizers of semiprime gamma rings, *International Mathematical Forum*, Vol.6(2011), No.13, 627-638.
- [5] M.F.Hoque and A.C.Paul, Centralizers on semiprime gamma rings, *Italian J. of Pure and Applied Mathematics*, Vol. 30(2013), 289-302.
- [6] M.F.Hoque and A.C.Paul, An equation related to centralizers in semiprime gamma rings, *Annals of Pure and Applied Mathematics*, Vol.1 (2012), 84-90.
- [7] M.F.Hoque, H.O.Roshid and A.C.Paul, An equation related to  $\theta$ -centralizers in semiprime gamma rings, *International J. Math. Combin.*, Vol. 4 (2012), 17-26.
- [8] M.F.Hoque and A.C.Paul, The  $\theta$ -centralizers of semiprime gamma rings, *Research Journal of Applied Sciences, Engineering and Technology*, 6(22), (2013), 4129-4137.
- [9] S.Kyuno, On prime gamma ring, *Pacific J.Math.*,75(1978), 185-190.
- [10] L.Luh, On the theory of simple gamma rings, *Michigan Math.J.*, 16(1969), 65-75.
- [11] N.Nobusawa, On the generalization of the ring theory, *Osaka J. Math.*,1(1964), 81-89.
- [12] J.Vukman and I. Kosi-Ulbl, On centralizers of semiprime rings with involution, *Studia Sci. Math. Hungarica*, 43,1(2006), 61-67.
- [13] J.Vukman, An identity related to centralizers in semiprime rings, *Comment. Math. Univ. Carolinae*, 40, 3(1999), 447-456.
- [14] J.Vukman, Centralizers on semiprime rings, *Comment. Math. Univ. Carolinae*, 42, 2(2001), 237-245.
- [15] B.Zalar, On centralizers of semiprime rings, *Comment.Math. Univ. Carolinae*, 32(1991), 609-614.

## First Approximate Exponential Change of Finsler Metric

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**Abstract:** The purpose of the present paper is to find the necessary and sufficient conditions under which a first approximate exponential change of Finsler metric becomes a Projective change. The condition under which a first approximate exponential change of Finsler metric of Douglas space becomes a Douglas space have been also found. The exponential change of Finsler metric has been studied [1].

**Key Words:** Exponential change, projective change, Douglas space.

**AMS(2010):** 53C60, 53B40

### §1. Introduction

Let  $F^n = (M^n, L)$  is a Finsler space, where  $L$  is Finsler function of  $x$  and  $y = \dot{x}$  and  $M^n$  is  $n$ -dimensional smooth manifold. In the paper [1] exponential change of Finsler metric, i.e. Finsler metric  $L$  changed to  $Le^{\beta/L}$  represented by  $\bar{L}$  where  $\beta = b_i(x)y^i$  is one form defined on the manifold  $M^n$ . The exponential change of Finsler metric is represented as

$$\bar{L} = L \left\{ 1 + \frac{\beta}{L} + \frac{1}{2!} \left( \frac{\beta}{L} \right)^2 + \frac{1}{3!} \left( \frac{\beta}{L} \right)^3 + \frac{1}{4!} \left( \frac{\beta}{L} \right)^4 + \dots \right\} \quad \text{for } |\beta| < |L|.$$

Neglecting powers of  $\beta$  higher than 2,  $\bar{L}$  approximates to  $L + \beta + \frac{\beta^2}{2L}$ , which will be called first approximate exponential change of Finsler metric  $L$ . That is,

$$\bar{L} = L + \beta + \frac{\beta^2}{2L} \quad (1.1)$$

Then Finsler space  $\bar{F}^n = (M^n, \bar{L})$  is said to be obtained from Finsler space  $F^n = (M^n, L)$  by first approximate exponential change. The quantities corresponding to  $\bar{F}^n$  is denoted by putting bar on those quantities.

Some basic tensor of  $F^n = (M^n, L)$  are given as follows:

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad l_i = \frac{\partial L}{\partial y^i} = L_i \quad \text{and} \quad h_{ij} = g_{ij} - l_i l_j,$$

where  $g_{ij}$  is fundamental metric tensor,  $l_i$  is normalized element of support and  $h_{ij}$  is angular metric tensor.

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Partial derivative with respect to  $x^i$  and  $y^i$  will be denoted as  $\partial_i$  and  $\dot{\partial}_i$  respectively and derivatives are written as

$$L_i = \frac{\partial L}{\partial y^i}, \quad L_{ij} = \frac{\partial^2 L}{\partial y^j \partial y^i} \quad \text{and} \quad L_{ijk} = \frac{\partial^3 L}{\partial y^k \partial y^j \partial y^i}. \quad (1.2)$$

The equation of geodesic of a Finsler space [2] is

$$\frac{d^2 x^i}{ds^2} + 2G^i \left( x, \frac{dx}{ds} \right) = 0,$$

where  $G^i$  is positively homogeneous function of degree two in  $y^i$  and is given by

$$2G^i = \frac{g^{ij}}{2} (y^r \dot{\partial}_j \partial_r L^2 - \partial_j L^2).$$

Berwald connection  $B\Gamma = (G_{jk}^i, G_j^i, 0)$  of Finsler space  $F^n = (M^n, L)$  is given by [2]

$$G_j^i = \frac{\partial G^i}{\partial y^j}, \quad G_{jk}^i = \frac{\partial G_j^i}{\partial y^k}.$$

Cartan connection  $C\Gamma = (F_{jk}^i, G_j^i, C_{jk}^i)$  is constructed from  $L$  with the help of following axioms [3]:

- (1) Cartan connection  $C\Gamma$  is  $v$ -metrical;
- (2) Cartan connection  $C\Gamma$  is  $h$ -metrical;
- (3) The  $(v)v$  torsion tensor field  $S^1$  of Cartan connection vanishes;
- (4) The  $(h)h$  torsion tensor field  $T$  of Cartan connection vanishes;
- (5) The deflection Tensor field  $D$  of Cartan connection vanishes,

Denote the  $h$  and  $v$ -covariant derivative with respect to Cartan connection by  $|_k$  and  $|_k$ .

Let

$$G^i = G^i + D^i, \quad (1.3)$$

where  $D^i$  is difference tensor homogeneous function of second degree in  $y^i$ . Then  $G_j^i = G_j^i + D_j^i$ ,  $G_{jk}^i = G_{jk}^i + D_{jk}^i$ , where  $D_j^i = \frac{\partial D^i}{\partial y^j}$  and  $D_{jk}^i = \frac{\partial D_j^i}{\partial y^k}$  are homogeneous function of degree 1 and 0 in  $y^i$  respectively.

## §2. Difference Tensor $D^j$

From (1.1) and (1.2) we have,

$$\bar{L}_i = \left( 1 - \frac{\beta^2}{2L^2} \right) L_i + \left( 1 + \frac{\beta}{L} \right) b_i, \quad (2.1)$$

$$\bar{L}_{ij} = \left( 1 - \frac{\beta^2}{2L^2} \right) L_{ij} + \frac{\beta^2}{L^3} L_i L_j - \frac{\beta}{L^2} (L_i b_j + L_j b_i) + \frac{1}{L} b_i b_j, \quad (2.2)$$

$$\bar{L}_{ijk} = \left( 1 - \frac{\beta^2}{2L^2} \right) L_{ijk} + \frac{\beta^2}{L^3} (L_{ij} L_k + L_{ik} L_j + L_{jk} L_i) - \frac{\beta}{L^2} (L_{ij} b_k + L_{ik} b_j + L_{jk} b_i)$$

$$+\frac{2\beta}{L^3}(L_i L_j b_k + L_i L_k b_j + L_j L_k b_i) - \frac{1}{L^2}(L_i b_j b_k + L_j b_i b_k + L_k b_j b_i) - \frac{3\beta^2}{L^4} L_i L_j L_k, \quad (2.3)$$

$$\partial_j \bar{L}_i = \left(1 - \frac{\beta^2}{2L^2}\right) \partial_j L_i + \frac{\beta}{L^3}(\beta L_i - L b_i) \partial_j L + \frac{1}{L^2}(L b_i - \beta L_i) \partial_j \beta + \left(1 + \frac{\beta}{L}\right) \partial_j b_i, \quad (2.4)$$

$$\begin{aligned} \partial_k \bar{L}_{ij} = & \left\{ \left(1 - \frac{\beta^2}{2L^2}\right) \partial_k L_{ij} + \left(\frac{\beta^2}{L^3} L_{ij} - \frac{3\beta^2}{L^4} L_i L_j + \frac{2\beta}{L^3}(L_i b_j + L_j b_i) - \frac{1}{L^2} b_i b_j\right) \partial_k L \right. \\ & - \left(\frac{1}{L^2}(L_i b_j + L_j b_i) + \frac{\beta}{L^2} L_{ij} - \frac{2\beta}{L^3} L_i L_j\right) \partial_k \beta + \frac{\beta}{L^3}(\beta L_i - L b_i) \partial_k L_j \\ & \left. + \frac{\beta}{L^3}(\beta L_j - L b_j) \partial_k L_i + \frac{1}{L^2}(L b_i - \beta L_i) \partial_k b_j + \frac{1}{L^2}(L b_j - \beta L_j) \partial_k b_i \right\}. \end{aligned} \quad (2.5)$$

Now in  $\bar{F}^n$  and  $F^n$ , we have

$$\bar{L}_{ij|k} = 0 \Rightarrow \quad \partial_k \bar{L}_{ij} - \bar{L}_{ijr} \bar{G}_k^r - \bar{L}_{ir} \bar{F}_{jk}^r - \bar{L}_{jr} \bar{F}_{ik}^r = 0, \quad (2.6)$$

$$L_{ij|k} = 0 \Rightarrow \quad \partial_k L_{ij} - L_{ijr} G_k^r - L_{ir} F_{jk}^r - L_{jr} F_{ik}^r = 0, \quad (2.7)$$

$$\bar{G}_k^r = G_k^r + D_k^r \quad \text{and} \quad \bar{F}_{ik}^r = F_{ik}^r + D_{ik}^{*r}.$$

Putting the value from (2.2), (2.3), (2.5) and (2.7) in (2.6) and contract the resulting equation by  $y^k$ , we have

$$\begin{aligned} & \left\{ \frac{1}{L^2}(L_i b_j + L_j b_i) + \frac{\beta}{L^2} L_{ij} - \frac{2\beta}{L^3} L_i L_j \right\} r_{00} - \frac{1}{L^2}(L b_i - \beta L_i)(r_{j0} + s_{j0}) \\ & - \frac{1}{L^2}(L b_j - \beta L_j)(r_{i0} + s_{i0}) + 2\bar{L}_{ijr} D^r + \bar{L}_{ir} D_j^r + \bar{L}_{jr} D_i^r = 0, \end{aligned} \quad (2.8)$$

where ‘0’ denotes contraction with  $y^k$ .

Now deal with following equations in  $\bar{F}^n$  and  $F^n$

$$\bar{L}_{i|j} = 0 \Rightarrow \quad \partial_j \bar{L}_i - \bar{L}_{ir} \bar{G}_j^r - \bar{L}_r \bar{F}_{ij}^r = 0, \quad (2.9)$$

$$L_{i|j} = 0 \Rightarrow \quad \partial_j L_i - L_{ir} G_j^r - L_r F_{ij}^r = 0. \quad (2.10)$$

Putting the value from (2.1), (2.2), (2.4) and (2.10) in (2.9), we have

$$\left(1 + \frac{\beta}{L}\right) b_{i|j} = \bar{L}_{ir} D_j^r + \bar{L}_r D_{ij}^{*r} + \frac{1}{L^2}(\beta L_i - L b_i)(r_{j0} + s_{j0}). \quad (2.11)$$

Since

$$2r_{ij} = b_{i|j} + b_{j|i}, \quad (2.12)$$

therefore putting the value from (2.11) in (2.12), we have

$$2\left(1 + \frac{\beta}{L}\right) r_{ij} = \bar{L}_{ir} D_j^r + \bar{L}_{jr} D_i^r + 2\bar{L}_r D_{ij}^{*r} + \frac{1}{L^2}(\beta L_i - L b_i)(r_{j0} + s_{j0}) + \frac{1}{L^2}(\beta L_j - L b_j)(r_{i0} + s_{i0}). \quad (2.13)$$

Subtract (2.8) from (2.13) and contract the resulting equation by  $y^i y^j$ , we get

$$\left(1 - \frac{\beta^2}{2L^2}\right) L_r D^r + \left(1 + \frac{\beta}{L}\right) b_r D^r = \frac{1}{2} \left(1 + \frac{\beta}{L}\right) r_{00}. \quad (2.14)$$

Since

$$2s_{ij} = b_{i|j} - b_{j|i}, \quad (2.15)$$

therefore putting the value from (2.11) in (2.15), we have

$$\begin{aligned} 2\left(1 + \frac{\beta}{L}\right)s_{ij} &= \overline{L}_{ir}D_j^r - \overline{L}_{jr}D_i^r + \frac{1}{L^2}(\beta L_i - Lb_i)(r_{j0} + s_{j0}) \\ &\quad - \frac{1}{L^2}(\beta L_j - Lb_j)(r_{i0} + s_{i0}). \end{aligned} \quad (2.16)$$

Subtract (2.8) from (2.16) and contract the resulting equation by  $y^j b^i$ , we have

$$\begin{aligned} \beta\{3\beta^2 - 2(1 + b^2)L^2\}L_r D^r - L\{3\beta^2 - 2(1 + b^2)L^2\}b_r D^r \\ = L\{2L^2(\beta + L)s_0 + r_{00}(L^2 b^2 - \beta^2)\}, \end{aligned} \quad (2.17)$$

Solution of algebraic equation (2.14) and (2.17) is given by

$$b_r D^r = \frac{2L^2(2L^2 - \beta^2)(\beta + L)s_0 + \{(L^2 b^2 - \beta^2)(\beta^2 + 2\beta L + 2L^2) + \beta(\beta + L)(2L^2 - \beta^2)\}r_{00}}{2(\beta^2 + 2\beta L + 2L^2)\{2(1 + b^2)L^2 - 3\beta^2\}}, \quad (2.18)$$

$$L_r D^r = \frac{L(\beta + L)\{(2L^2 - \beta^2)r_{00} - 4L^2(\beta + L)s_0\}}{2(\beta^2 + 2\beta L + 2L^2)(2(1 + b^2)L^2 - 3\beta^2)}. \quad (2.19)$$

Subtract (2.8) from (2.16) and contract the resulting equation by  $y^j$ , we have

$$\left(1 + \frac{\beta}{L}\right)s_{i0} + \frac{1}{2L^2}(Lb_i - \beta L_i)r_{00} = \overline{L}_{ir}D^r. \quad (2.20)$$

Putting the value from (2.2) in (2.20) using  $LL_{ir} = g_{ir} - L_i L_r$ ,  $L_i = l_i$  and contracting the resulting equation by  $g^{ij}$ , we have

$$D^j = \frac{2L^2(\beta + L)}{(2L^2 - \beta^2)}s_0^j + \frac{2L^2\{(2L^2 - \beta^2)r_{00} - 4L^2(\beta + L)s_0\}}{(2L^2 - \beta^2)\{2(1 + b^2)L^2 - 3\beta^2\}} \left[ \frac{(2L^3 - 3\beta^2 L - 2\beta^3)}{L^2(\beta^2 + 2\beta L + 2L^2)} y^j + b^j \right]. \quad (2.21)$$

**Proposition 2.1** *Difference tensor of first approximate exponential change of Finsler metric  $L$  is given by equations (2.21).*

### §3. Projective Change of Finsler Metric

**Definition 3.1**([4]) *A Finsler space  $\overline{F}^n$  is called projective to Finsler space  $F^n$  if there is geodesics correspond between  $\overline{F}^n$  and  $F^n$ . That is,  $L \rightarrow \overline{L}$  is projective if  $\overline{G}^i = \overline{G}^i + P(x, y) y^i$ , where  $P(x, y)$  is called projective factor, this is homogeneous scalar function of degree one in  $y^i$ .*

Putting  $D^j = Py^j$  in equation (2.21), where  $P$  is projective factor and contracting the resulting equation by  $y_j$ , we have

$$P = \frac{(\beta + L)\{(2L^2 - \beta^2)r_{00} - 4L^2(\beta + L)s_0\}}{(\beta^2 + 2\beta L + 2L^2)\{2(1 + b^2)L^2 - 3\beta^2\}}. \quad (3.1)$$

Putting  $D^j = Py^j$  in equation (2.21) the value from (3.1) in (2.21), we get

$$\frac{2\{(2L^2 - \beta^2)r_{00} - 4L^2(\beta + L)s_0\}}{(2L^2 - \beta^2)\{2(1 + b^2)L^2 - 3\beta^2\}} (\beta y^j - L^2 b^j) = \frac{2L^2(\beta + L)}{(2L^2 - \beta^2)} s_0^j. \quad (3.2)$$

Contracting (3.2) by  $b_j$ , we have

$$r_{00} = \frac{2L^2(\beta + L)}{(\beta^2 - L^2 b^2)} s_0. \quad (3.3)$$

Putting the value from (3.3) in (3.1), we have

$$P = \frac{2L^2(L + \beta)^2}{(\beta^2 - L^2 b^2)\{(\beta^2 + 2\beta L + 2L^2)\}} s_0. \quad (3.4)$$

Eliminating  $P$  and  $r_{00}$  from (3.4), (3.3) and (2.21), we have

$$s_0^j = \left\{ b^j - \left( \frac{\beta}{L^2} \right) y^j \right\} \frac{L^2 s_0}{(L^2 b^2 - \beta^2)}. \quad (3.5)$$

Equation (3.3) and (3.5) are necessary condition for first approximate exponential change of Finsler metric to be projective.

Conversely, if condition (3.3) and (3.5) are satisfied, then put these value in (2.21), we have

$$D^j = \frac{2L^2(L + \beta)^2}{(\beta^2 - L^2 b^2)(\beta^2 + 2\beta L + 2L^2)} s_0 y^j = Py^j.$$

That is  $\overline{F}^n$  is projective to  $F^n$ .

**Theorem 3.1** *The first approximate exponential change of Finsler space is projective iff equation (3.3) and (3.5) are satisfied and then projective factor  $P$  is given by  $P = \frac{2L^2(L + \beta)^2}{(\beta^2 - L^2 b^2)\{(\beta^2 + 2\beta L + 2L^2)\}} s_0$ .*

#### §4. Douglas Space

**Definition 4.1**([5]) *A Finsler space  $F^n$  is called Douglas space if  $G^i y^j - G^j y^i$  is homogeneous polynomial of degree three in  $y^i$ . In brief, homogeneous polynomial of degree  $r$  in  $y^i$  is denoted by  $hp(r)$ .*

If we denote

$$B^{ij} = D^i y^j - D^j y^i \quad (4.1)$$

from equation (2.21), we have

$$B^{ij} = \frac{2L^2\{(2L^2 - \beta^2)r_{00} - 4L^2(\beta + L)s_0\}}{(2L^2 - \beta^2)\{2(1 + b^2)L^2 - 3\beta^2\}} (b^i y^j - b^j y^i) + \frac{2L^2(\beta + L)}{(2L^2 - \beta^2)} (s_0^i y^j - s_0^j y^i). \quad (4.2)$$



From (4.2), we see that  $B^{ij}$  is  $hp(3)$ .

That is, if Douglas space is transformed to be Douglas space by first approximate exponential change of Finsler metric, then  $B^{ij}$  is  $hp(3)$  and if  $B^{ij}$  is  $hp(3)$  then Douglas space transformed by first approximate exponential change is Douglas space.

**Theorem 4.1** *The first approximate exponential change of Douglas space is Douglas space iff  $B^{ij}$  given by (4.2) is  $hp(3)$ .*

## References

- [1] H.S.Shukla, B.N.Prasad, O.P.Pandey, paper tittle? *Int. J. Contemp. Math. Sciences*, 7 (2012), no. 46, 2253 - 2263.
- [2] Ioan Bucataru, R.Miron, *Finsler-Lagrange geometry-Application to dynamical system*, Publisher? July 2007.
- [3] M.Matsumoto, *Foundation of Finsler Geometry and Special Finsler Space*, Publisher? 1982.
- [4] M.Matsumoto, Theory of Finsler space with  $(\alpha, \beta)$ -metric, *Rep. Math. Phy.*, 31 (1992), 43-83.
- [5] M.Matsumoto, Finsler space with  $(\alpha, \beta)$ -metric of Douglas type, *Tensor N. S.*, 60 (1998), 123-134.

## Difference Cordiality of Some Derived Graphs

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**Abstract:** Let  $G$  be a  $(p, q)$  graph. Let  $f : V(G) \rightarrow \{1, 2, \dots, p\}$  be a function. For each edge  $uv$ , assign the label  $|f(u) - f(v)|$ .  $f$  is called a difference cordial labeling if  $f$  is a one to one map and  $|e_f(0) - e_f(1)| \leq 1$  where  $e_f(1)$  and  $e_f(0)$  denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph. In this paper we investigate the difference cordial labeling behavior of Splitting, Degree splitting and Shadow graph of some standard graphs.

**Key Words:** Splitting graph, degree splitting graph, shadow graph, corona.

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### §1. Introduction

Let  $G = (V, E)$  be  $(p, q)$  graph. Throughout this paper we have considered only simple and undirected graphs. The number of vertices of  $G$  is called the order of  $G$  and the number of edges of  $G$  is called the size  $G$ . Graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling plays an important role of various fields of science and few of them are astronomy, coding theory, x-ray crystallography, radar, circuit design, communication network addressing, database management, secret sharing schemes, and models for constraint programming over finite domains [4]. The graph labeling problem was introduced by Rosa and he has introduced graceful labeling of graphs [21] in the year 1967. In 1980, Cahit [2] introduced the Cordial labeling of graphs. Kuo, Chang, and Kwong [8], Youssef [25], Liu and Zhu [10], Kirchherr [7], Ho, Lee, and Shee [6], Riskin [20], Seoud and Abdel Maqusoud [23], Diab [3], Lee and Liu [9], Andar, Boxwala, and Limaye [1], Vaidya, Ghodasara, Srivastav, and Kaneria [24] were worked in cordial labeling. Ponraj et al. introduced  $k$ -product cordial labeling [17],  $k$ -total product cordial labeling [18] recently. Inspiration of the above work, R. Ponraj, S. Sathish Narayanan and R. Kala introduced difference cordial labeling of graphs [11]. Let  $G$  be a  $(p, q)$  graph. Let  $f$  be a map from  $V(G)$  to  $\{1, 2, \dots, p\}$ . For each edge  $uv$ , assign the label  $|f(u) - f(v)|$ .  $f$  is called difference cordial labeling if  $f$  is 1 - 1 and  $|e_f(0) - e_f(1)| \leq 1$  where  $e_f(1)$  and  $e_f(0)$  denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a

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difference cordial graph. In [11]-[16] difference cordial labeling behavior of several graphs like path, cycle, complete graph, complete bipartite graph, bistar, wheel, web, grid, prism, book and some more standard graphs have been investigated. In this paper, we investigate the difference cordial labeling behavior of some derived graphs like splitting graph, Degree splitting graph and shadow graphs. Let  $x$  be any real number. Then the symbol  $\lfloor x \rfloor$  stands for the largest integer less than or equal to  $x$  and  $\lceil x \rceil$  stands for the smallest integer greater than or equal to  $x$ . Terms and definitions not defined here are used in the sense of Harary [5].

## §2. Splitting Graphs

A splitting graph of a graph was introduced by E.Sampath Kumar and H.B.Waliker [22]. For a graph  $G$ , the splitting graph of  $G$ ,  $S'(G)$ , is obtained from  $G$  by adding for each vertex  $v$  of  $G$  a new vertex  $v'$  so that  $v'$  is adjacent to every vertex that is adjacent to  $v$ . Note that if  $G$  is a  $(p, q)$  graph then  $S'(G)$  is a  $(2p, 3q)$  graph.

**Theorem 2.1**  $S'(P_n)$  is difference cordial.

*Proof* Let  $P_n : u_1 u_2 \dots u_n$  be the path. Let  $V(S'(P_n)) = \{v_i : 1 \leq i \leq n\} \cup V(P_n)$  and  $E(S'(P_n)) = E(P_n) \cup \{u_i v_{i+1}, v_i u_{i+1} : 1 \leq i \leq n-1\}$ .

**Case 1**  $n$  is odd.

Define a map  $f : V(S'(P_n)) \rightarrow \{1, 2 \dots 2n\}$  by

$$\begin{aligned} f(u_{2i-1}) &= 4i-1, & 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\ f(u_{2i}) &= 4i-2, & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ f(v_{2i-1}) &= 4i-3, & 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ f(v_{2i-1}) &= 4i, & 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\ f(u_n) &= 2n. \end{aligned}$$

Obviously, the above labeling is a difference cordial labeling of  $S'(P_n)$ .

**Case 2**  $n$  is even.

Assign the label to vertices  $u_{2i-1}, v_{2i}, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, u_{2i}, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, v_{2i-1}, 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$  as in Case 1 and then define  $f(u_{n-1}) = 2n$  and  $f(v_n) = 2n-1$ . Since  $e_f(0) = \frac{3n-4}{2}$  and  $e_f(1) = \frac{3n-2}{2}$ ,  $f$  is a difference cordial labeling of  $S'(P_n)$ .  $\square$

**Theorem 2.2**  $S'(C_n)$  is difference cordial.

*Proof* Let  $C_n : u_1 u_2 \dots u_n u_1$  be a cycle. Let  $V(S'(C_n)) = V(C_n) \cup \{v_i : 1 \leq i \leq n\}$  and

$$E\left(S'(C_n)\right) = \left\{u_i v_{i+1(\bmod n)}, v_i u_{i+1(\bmod n)} : 1 \leq i \leq n\right\} \cup E(C_n).$$

**Case 1**  $n$  is odd.

Let  $f$  be a difference cordial defined in Case 1 of Theorem 2.1. Define a map  $g : V\left(S'(C_n)\right) \rightarrow \{1, 2 \dots 2n\}$  by

$$\begin{aligned} g(u_i) &= f(u_i), & 1 \leq i \leq n-1, \\ g(v_i) &= f(v_i), & 1 \leq i \leq n-1, \\ g(v_n) &= f(u_n), \\ g(u_n) &= f(v_n). \end{aligned}$$

Since  $e_f(0) = \frac{3n+1}{2}$  and  $e_f(1) = \frac{3n-1}{2}$ ,  $g$  is a difference cordial labeling of  $S'(C_n)$ .

**Case 2**  $n$  is even.

Let  $f$  be a difference cordial labeling defined in Case 2 of Theorem 2.1. Define a map  $h : V\left(S'(C_n)\right) \rightarrow \{1, 2 \dots 2n\}$  by

$$\begin{aligned} h(u_i) &= f(u_i) & \forall i \neq n-1 \\ h(v_i) &= f(v_i) & \forall i \neq n-1 \end{aligned}$$

$h(u_{n-1}) = f(u_n)$ ,  $h(v_n) = f(u_{n-1})$ . Since  $e_f(0) = e_f(1) = \frac{3n}{2}$ ,  $h$  is a difference cordial labeling of  $S'(C_n)$ .  $\square$

The corona of  $G$  with  $H$ ,  $G \odot H$  is the graph obtained by taking one copy of  $G$  and  $p$  copies of  $H$  and joining the  $i^{th}$  vertex of  $G$  with an edge to every vertex in the  $i^{th}$  copy of  $H$ . The graph  $P_n \odot K_1$  is called a comb.

**Theorem 2.3**  $S'(P_n \odot K_1)$  is difference cordial.

*Proof* Let  $u'_i$  ( $1 \leq i \leq n$ ) be the vertex corresponding to  $u_i$  ( $1 \leq i \leq n$ ) and  $v'_i$  ( $1 \leq i \leq n$ ) be the vertex corresponding to  $v_i$  ( $1 \leq i \leq n$ ). Define a map  $f : V\left(S'(P_n \odot K_1)\right) \rightarrow \{1, 2 \dots 4n\}$  by  $f(u_1) = 3$ ,  $f(u'_1) = 1$ ,

$$\begin{aligned} f(u_{i+1}) &= 3i+1 & 1 \leq i \leq n-1 \\ f(u'_{i+1}) &= 3i+3 & 1 \leq i \leq n-1 \\ f(v_i) &= 3i-1 & 1 \leq i \leq n \\ f(v'_i) &= 3n+1 & 1 \leq i \leq n \end{aligned}$$

Since  $e_f(0) = 3n-2$  and  $e_f(1) = 3n-1$ ,  $f$  is a difference cordial labeling of  $S'(P_n \odot K_1)$ .  $\square$

**Theorem 2.4** ([11]) If  $G$  is a  $(p, q)$  difference cordial graph, then  $q \leq 2p-1$ .

The graph  $W_n = C_n + K_1$  is called a wheel.

**Theorem 2.5**  $S'(W_n)$  is not difference cordial.

*Proof* Clearly, the number of vertices and edges in  $S'(W_n)$  are  $2n + 2$  and  $6n$  respectively. By Theorem 2.4,  $6n \leq 2(2n + 2) - 1 \leq 4n - 3$ . This is impossible.  $\square$

**Theorem 2.6**([11]) *Any Path is a difference cordial graph.*

**Theorem 2.7**  $S'(K_{1,n})$  is difference cordial iff  $n \leq 3$ .

*Proof* Since  $S'(K_{1,1}) \cong P_4$ , by theorem 2.6,  $S'(K_{1,1})$  is difference cordial. The difference cordial labeling of  $S'(K_{1,2})$  and  $S'(K_{1,3})$  is shown in Figure 1.

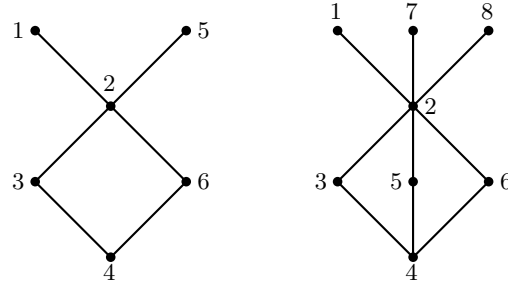


Figure 1

Suppose  $f$  is a difference cordial labeling of  $K_{1,n}$  with  $n > 3$ . Clearly,  $e_f(1) \leq 4$ . Then  $e_f(0) \geq q - 4 \geq 3n - 4$ . This implies,  $e_f(0) - e_f(1) \geq 3n - 8 > 1$ , a contradiction.  $\square$

**Theorem 2.8**  $S'(K_n)$  is not difference cordial.

*Proof* The order and size of  $S'(K_n)$  are  $2n$  and  $\frac{3n(n-1)}{2}$  respectively. By Theorem 2.4,  $\frac{3n(n-1)}{2} \leq 2(2n) - 1$ . This implies  $2 \leq 5n^2 + 3n$ . Hence,  $S'(K_n)$  is not difference cordial.  $\square$

The graph  $C_n \times P_2$  is called prism.

**Theorem 2.9**  $S'(C_n \times P_2)$  is not difference cordial.

*Proof* The order and size of  $S'(C_n \times P_2)$  are  $4n$  and  $9n$  respectively. By Theorem 2.4,  $9n \leq 2(4n) - 1 \Rightarrow n \leq -1$ . This is impossible.  $\square$

The helm  $H_n$  is the graph obtained from a wheel by attaching a pendant edge at each vertex of the  $n$ -cycle. A flower  $Fl_n$  is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.

**Theorem 2.10** A splitting graph of a flower graph is not difference cordial.

*Proof* The number of vertices and edges in the splitting graph of a flower graph are  $4n + 2$  and  $12n$  respectively. By theorem 2.4,  $12n \leq 2(4n + 2) - 1$ . This implies  $4n \leq 3$ . This is impossible.  $\square$

### §3. Degree Splitting Graphs

The concept of Degree splitting graph was introduced by R.Ponraj and S.Somasundaram in [19]. Let  $G = (V, E)$  be a graph with  $V = S_1 \cup S_2 \cup \dots \cup S_t \cup T$  where each  $S_i$  is a set of vertices having at least two vertices and having the same degree and  $T = V - \bigcup_{i=1}^t S_i$ . The Degree Splitting graph of  $G$  denoted by  $DS(G)$  is obtained from  $G$  by adding vertices  $w_1, w_2, \dots, w_t$  and joining  $w_i$  to each vertex of  $S_i$  ( $1 \leq i \leq t$ ).

**Theorem 3.1**  $DS(P_n)$  is difference cordial.

*Proof* Let  $P_n$  be the path  $u_1 u_2 \dots u_n$ . Let  $V(DS(P_n)) = V(P_n) \cup \{u, v\}$  and  $E(DS(P_n)) = \{uu_i : 2 \leq i \leq n-1\} \cup \{uu_1, vu_n\}$ . Define  $f : V(DS(P_n)) \rightarrow \{1, 2, \dots, n+2\}$  by  $f(u_i) = i$ ,  $1 \leq i \leq n$ ,  $f(v) = n+1$ ,  $f(u) = n+2$ . Since  $e_f(1) = n$ ,  $e_f(0) = n-1$ ,  $f$  is a difference cordial labeling of  $DS(P_n)$ .  $\square$

**Theorem 3.2**([11]) The wheel  $W_n$  is difference cordial.

**Theorem 3.3**  $DS(C_n)$  is difference cordial.

*Proof* Since  $DS(C_n) \cong W_n$ , the proof follows from Theorem 3.2.  $\square$

**Theorem 3.4**([11])  $K_n$  is difference cordial iff  $n \leq 4$ .

**Theorem 3.5**  $DS(K_n)$  is difference cordial iff  $n \leq 3$ .

*Proof* Since  $DS(K_n) \cong K_{n+1}$ , the proof follows from Theorem 3.4.  $\square$

**Theorem 3.6**([11])  $K_{2,n}$  is difference cordial iff  $n \leq 4$ .

**Theorem 3.7**  $DS(K_{1,n})$  is difference cordial iff  $n \leq 4$ .

*Proof* Since  $DS(K_{1,n}) \cong K_{2,n}$ , the proof follows from Theorem 3.6.  $\square$

**Theorem 3.8**  $DS(W_n)$  is difference cordial iff  $n = 3$ .

*Proof* The difference cordial labeling of  $DS(W_3)$  is given in figure 2.

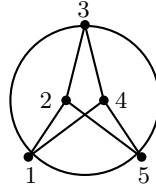


Figure 2

Suppose  $DS(W_n)$  is difference cordial, then by Theorem 2.4,  $3n \leq 2(n+2) - 1$ . This implies  $n = 3$ .  $\square$

**Theorem 3.9**  $DS(K_n^c + 2K_2)$  is difference cordial iff  $n = 1$ .

*Proof* The order and size of  $DS(K_n^c + 2K_2)$  are  $n + 6$  and  $5n + 6$  respectively. Suppose  $DS(K_n^c + 2K_2)$  is difference cordial, then by theorem 2.4,  $5n + 6 \leq 2(n + 6) - 1$ . This is true when  $n = 1$ . The difference cordial labeling of  $DS(K_1^c + 2K_2)$  is given in Figure 3.

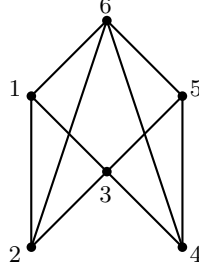


Figure 3

This completes the proof.  $\square$

**Theorem 3.10**  $DS(K_2 + mK_1)$  is difference cordial iff  $n \leq 3$ .

*Proof* The graph  $DS(K_2 + mK_1)$  consists of  $m + 4$  vertices and  $3m + 3$  edges. Since  $DS(K_2 + K_1) \cong W_3$ , using Theorem 3.2,  $DS(K_2 + K_1)$  is difference cordial. the difference cordial labeling of  $DS(K_2 + 2K_1)$  and  $DS(K_2 + 3K_1)$  are given in Figure 4.

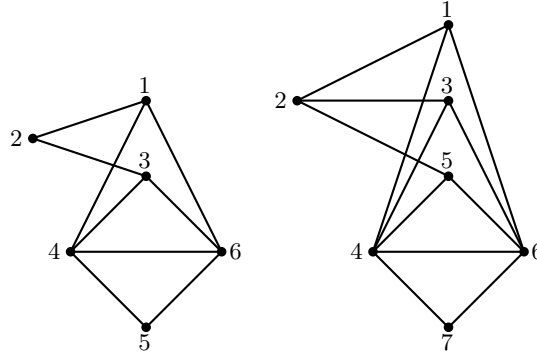


Figure 4

Suppose  $DS(K_2 + mK_1)$  is difference cordial, then by theorem 2.4,  $3m + 3 \leq 2(m + 4) - 1$ .  $\Rightarrow m \leq 4$ . When  $m = 4$ ,  $e_f(0) \geq 4 + 3 + 2 \geq 9$ . Obviously,  $e_f(1) \leq 7$ . Hence  $e_f(0) - e_f(1) \geq 2$ . This implies  $DS(K_2 + mK_1)$  is not difference cordial.  $\square$

**Theorem 3.11**  $DS(K_{n,n})$  is difference cordial iff  $n \leq 2$ .

*Proof* The order and size of  $DS(K_{n,n})$  are  $2n + 1$  and  $n^2 + 2n$  respectively. Suppose  $DS(K_{n,n})$  is difference cordial, then by theorem 2.4,  $n^2 + 2n \leq 2(2n + 1) - 1$ ,  $\Rightarrow n^2 - 2n - 1 \leq 0$ .  $\Rightarrow n \leq 2$ . Since  $DS(K_{1,1}) \cong K_3$ ,  $DS(K_{2,2}) \cong W_4$ , using Theorems 3.2 and 3.4,  $DS(K_{1,1})$  and  $DS(K_{2,2})$  are difference cordial.  $\square$

The triangular snake  $T_n$  is obtained from the path  $P_n$  by replacing each edge of the path by a triangle  $C_3$ .

**Theorem 3.12**  $DS(T_n)$  is difference cordial iff  $n \leq 5$ .

*Proof* Clearly, the order and size of  $DS(T_n)$  ( $n > 3$ ) are  $2n + 1$  and  $5n - 4$  respectively. By Theorem 2.4,  $5n - 4 \leq 2(2n + 1) - 1$ . This implies  $n \leq 5$ . Since  $DS(T_2) \cong W_3$ , using theorem 3.2,  $DS(T_2)$  is difference cordial. The difference cordial labeling of  $DS(T_3)$ ,  $DS(T_4)$  and  $DS(T_5)$  are given in Figure 5.

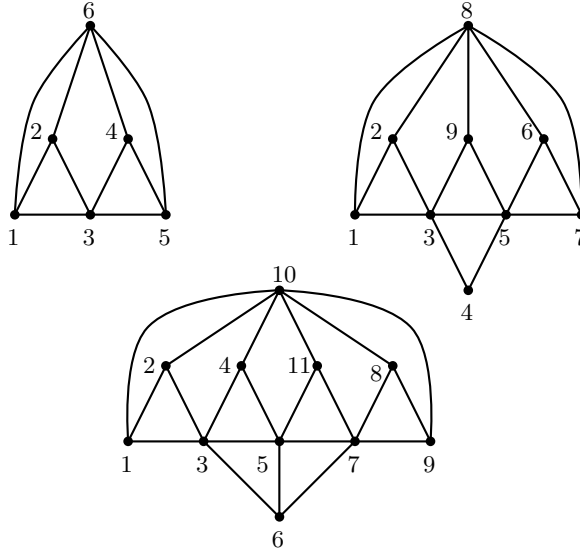


Figure 5

This completes the proof.  $\square$

The Quadrilateral snake  $Q_n$  is obtained from the path  $P_n$  by replacing each edge of the path by a cycle  $C_4$ .

**Theorem 3.13**  $DS(Q_n)$  is difference cordial iff  $n \leq 5$ .

*Proof* The difference cordial labeling of  $DS(Q_n)$  ( $n \leq 5$ ) is given in Figure 6.

The number of vertices and edges in  $DS(Q_n)$  are  $3n$  and  $7n - 6$  respectively. Suppose  $DS(Q_n)$  is difference cordial, then by Theorem 2.4,  $7n - 6 \leq 2(3n) - 1$ . This implies  $n \leq 5$ .  $\square$

The sunflower graph  $S_n$  is obtained by taking a wheel with central vertex  $v_0$  and the cycle  $C_n : v_1v_2 \dots v_nv_1$  and new vertices  $w_1w_2 \dots w_n$  where  $w_i$  is joined by vertices  $v_i, v_{i+1(\text{mod } n)}$ .

The Lotus inside a circle  $LC_n$  is a graph obtained from the cycle  $C_n : u_1u_2 \dots u_nu_1$  and a star  $K_{1,n}$  with central vertex  $v_0$  and the end vertices  $v_1v_2 \dots v_n$  by joining each  $v_i$  to  $u_i$  and  $u_{i+1(\text{mod } n)}$ .



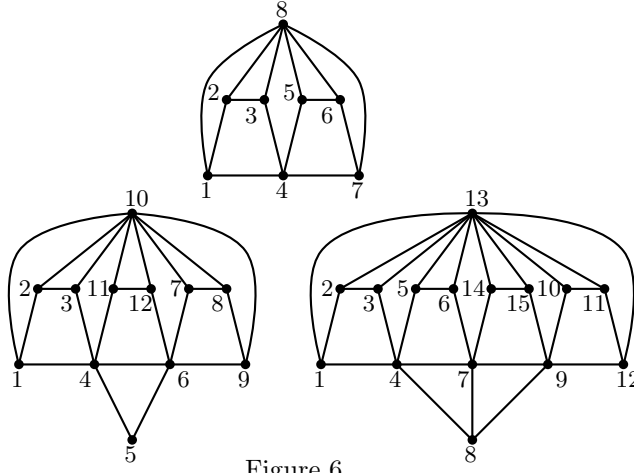


Figure 6

**Theorem 3.14** *The following are not difference cordial:  $DS(S_n)$ ,  $DS(LC_n)$  and  $DS(Fl_n)$ .*

*Proof* Since the order and size of the graphs given above are  $2n + 3$  and  $6n$  respectively. Suppose the graphs given above are difference cordial, then using theorem 2.4,  $6n \leq 2(2n + 3) - 1$ . This implies  $2n \leq 5$ , a contradiction.  $\square$

The graph  $L_n = P_n \times P_2$  is called ladder.

**Theorem 3.15**  *$DS(L_n)$  is difference cordial iff  $n \leq 5$ .*

*Proof* Since  $DS(L_2) \cong W_4$ , by theorem 3.2,  $DS(L_2)$  is difference cordial. For  $n \geq 3$ ,  $V(DS(L_n)) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{u, v\}$  and  $E(DS(L_n)) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{uu_i, uv_i : 2 \leq i \leq n-1\} \cup \{vu_1, vv_1, vu_n, vv_n\}$ . The difference cordial labeling of  $DS(L_n)$  is given in Table 1.

$n$	$u$	$v$	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$
3	4	8	1	5	6			2	3	7		
4	5	10	1	4	7	8		2	3	6	9	
5	7	12	1	4	5	8	11	2	3	6	9	10

Table 1:

Conversely, Suppose  $f$  is a difference cordial labeling, then by Theorem 2.4,  $5n - 2 \leq 2(2n + 2) - 1$ ,  $\Rightarrow n \leq 5$ .  $\square$

**Theorem 3.16** *If  $m + n > 8$  then  $DS(B_{m,n})$  ( $m \neq n$ ) is not difference cordial.*

*Proof* Clearly, the order and size of  $DS(B_{m,n})$  are  $m + n + 3$  and  $2m + 2n + 1$  respectively. Obviously,  $e_f(1) \leq m + m + 2$ . Also we observe that  $e_f(0) \geq m + n - 2 + m - 1 + n - 2 \geq 2m + 2n - 5$ .

$\Rightarrow e_f(0) - e_f(1) \geq m + n - 7 \longrightarrow (1)$ . Suppose  $DS(B_{m,n})$  is difference cordial with  $m + n > 8$ , then a contradiction arises to (1).  $\square$

**Theorem 3.17**  $DS(B_{n,n})$  is difference cordial iff  $n \leq 2$ .

*Proof* The order and size of  $DS(B_{n,n})$  are  $2n + 4$  and  $4n + 3$  respectively. Clearly  $e_f(0) \geq 2n - 2 + n + n - 1 \geq 4n - 3$ . Also  $e_f(1) \leq 2n + 3$ . Then  $e_f(0) - e_f(1) \geq 2n - 6$ . It follows that  $n \leq 3$ . when  $n = 3$ ,  $e_f(1) \leq 6$ . This implies  $DS(B_{3,3})$  is not difference cordial.  $DS(B_{1,1}) \cong DS(P_4)$ , using Theorem 3.1,  $DS(B_{1,1})$  is difference cordial. The difference cordial labeling of  $DS(B_{2,2})$  is given in Figure 7.

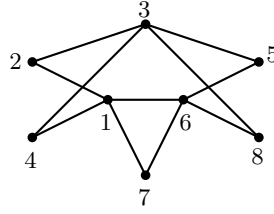


Figure 7

This completes the proof.  $\square$

**Theorem 3.18**  $DS(B_{1,n})$  is difference cordial iff  $n \leq 4$ .

*Proof*  $DS(B_{1,n})$  consists of  $n + 4$  vertices and  $2n + 3$  edges. Note that  $e_f(1) \leq 5$ . Then  $e_f(0) \geq q - 5 \geq 2n - 2$ .  $\Rightarrow e_f(0) - e_f(1) \geq 2n - 7$ . It follows that  $n \leq 4$ . By Theorem 3.17,  $DS(B_{1,1})$  is difference cordial. The difference cordial labeling of  $DS(B_{1,2})$ ,  $DS(B_{1,3})$  and  $DS(B_{1,4})$  are given in Figure 8.

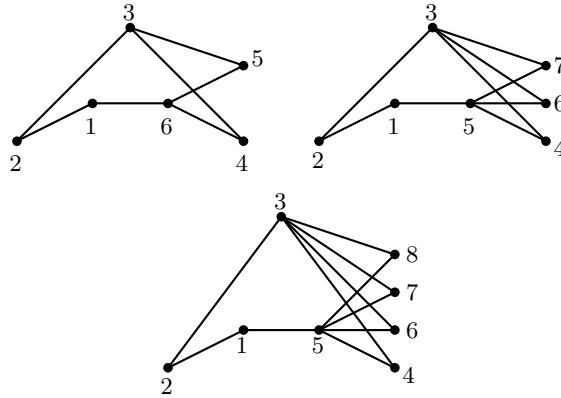


Figure 8

This completes the proof.  $\square$

**Theorem 3.19**  $DS(B_{2,n})$  is difference cordial iff  $n \leq 4$ .

*Proof* The order and size of  $DS(B_{2,n})$  are  $n + 5$  and  $2n + 5$  respectively. It is clear that  $e_f(1) \leq 6$ . Then  $e_f(0) \geq q - 6 \geq 2n - 1$ . Hence  $e_f(0) - e_f(1) \geq 2n - 7$ . This implies  $n \leq 4$ .

Using theorems 3.18, 3.17,  $DS(B_{2,1})$  and  $DS(B_{2,2})$  are difference cordial. The difference cordial labeling of  $DS(B_{2,3})$  and  $DS(B_{2,4})$  are given in Figure 9.

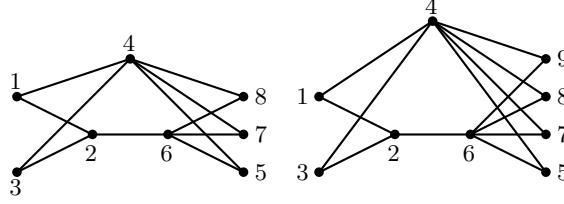


Figure 9

This completes the proof.  $\square$

**Theorem 3.20**  $DS(B_{3,n})$  is difference cordial iff  $n \leq 2$ .

*Proof* The number of vertices and edges in  $DS(B_{3,n})$  are  $n + 6$  and  $2n + 7$  respectively. obviously  $e_f(1) \leq 6 \Rightarrow e_f(0) \geq q - 6 \geq 2n + 1$ . Therefore  $e_f(0) - e_f(1) \geq 2n - 5 \rightarrow (1)$ . Suppose  $n > 3$  then a contradiction arises to (1). By Theorem 3.17,  $DS(B_{3,3})$  is not difference cordial. Using theorems 3.18, 3.19,  $DS(B_{3,1})$  and  $DS(B_{3,2})$  are difference cordial.  $\square$

#### §4. Shadow Graphs

The shadow graph  $D_2(G)$  of a connected graph  $G$  is constructed by taking two copies of  $G$ ,  $G'$  and  $G''$  and joining each vertex  $u'$  in  $G'$  to the neighbors of the corresponding vertex  $u''$  in  $G''$ .

**Theorem 4.1** Let  $G$  be a  $(p, q)$  graph with  $q \geq p$ . Then  $D_2(G)$  is not difference cordial.

*Proof* Suppose  $G$  is a difference cordial graph with  $q \geq p$ . Clearly,  $D_2(G)$  consists of  $2p$  vertices and  $4q$  edges. By Theorem 2.4,  $4q \leq 2(2p) - 1$ . This implies  $4q \leq 4p - 1$ , a contradiction.  $\square$

**Theorem 4.2**  $D_2(P_n)$  is difference cordial.

*Proof* Let  $V(D_2(P_n)) = \{u_i, v_i : 1 \leq i \leq n\}$  and  $E(D_2(P_n)) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_{i+1}, v_i u_{i+1} : 1 \leq i \leq n-1\}$ . Define a map  $f : V(D_2(P_n)) \rightarrow \{1, 2, \dots, 2n\}$  by

$$\begin{aligned} f(u_i) &= i & 1 \leq i \leq n \\ f(v_i) &= n + i & 1 \leq i \leq n. \end{aligned}$$

Since  $e_f(0) = e_f(1) = 2n - 2$ ,  $f$  is a difference cordial labeling of  $D_2(P_n)$ .  $\square$

**Theorem 4.3**([11]) Any Cycle is a difference cordial graph.

**Theorem 4.4**  $D_2(K_n)$  is difference cordial iff  $n \leq 2$ .

*Proof* The order and size of  $D_2(K_n)$  are  $2n$  and  $2\binom{n}{2} + n(n-1)$  respectively. Suppose  $D_2(K_n)$  is difference cordial. By Theorem 2.4,  $2\binom{n}{2} + n(n-1) \leq 2(2n) - 1$ . This implies,

$2n^2 - 6n + 1 \leq 0$ . It follows that,  $n \leq 2$ . Since  $D_2(K_2) \cong C_4$ , using Theorem 4.3,  $D_2(K_2)$  is difference cordial.  $\square$

**Theorem 4.5**  $D_2(K_{1,m})$  is difference cordial iff  $m \leq 2$ .

*Proof* The order and size of  $D_2(K_{1,m})$  are  $2m + 2$  and  $4m$  respectively. Clearly,  $e_f(1) \leq 2m + 1$ . let  $v$  be the central vertex of  $K_{1,m}$  and  $v'$  be the corresponding shadow vertex. Note that the degree of  $v$  and  $v'$  in  $D_2(K_{1,m})$  are  $2m$ . Therefore,  $e_f(0) \geq (2m - 2) + (2m - 2) \geq 4m - 4$ . Hence,  $e_f(0) - e_f(1) \geq 2m - 3$ . This implies,  $m \leq 2$ . Since  $D_2(K_{1,1}) \cong C_4$ , by Theorem 4.3,  $D_2(K_{1,1})$  is difference cordial. A difference cordial labeling of  $D_2(K_{1,2})$  is given in Figure 10.

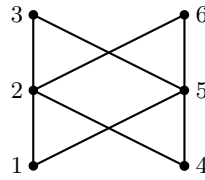


Figure 10

This completes the proof.  $\square$

## References

- [1] M.Andar, S.Boxwala, and N.Limaye, Cordial labelings of some wheel related graphs, *J. Combin. Math. Combin. Comput.*, **41** (2002) 203-208.
- [2] I.Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars combin.*, **23** (1987) 201-207.
- [3] A.Diab, Study of some problems of cordial graphs, *Ars Combin.*, **92** (2009) 255-261.
- [4] J.A.Gallian, *A Dynamic survey of graph labeling*, *The Electronic Journal of Combinatorics*, **18** (2012) Ds6.
- [5] F.Harary, *Graph theory*, Addison wesley, New Delhi (1969).
- [6] Y.S.Ho, S.M.Lee and S.C.Shee, Cordial labellings of the cartesian product and composition of graphs, *Ars Combin.*, **29** (1990) 169-180.
- [7] W.W.Kirchherr, On the cordiality of some specific graphs, *Ars Combin.*, **31** (1991) 127-138.
- [8] D.Kuo, G.Chang, and Y.-H.Kwong, Cordial labeling of  $mK_n$ , *Discrete Math.*, **169** (1997) 121-131.
- [9] S.M.Lee and A.Liu, A construction of cordial graphs from smaller cordial graphs, *Ars Combin.*, **32** (1991) 209-214.
- [10] Z.Liu and B.Zhu, A necessary and sufficient condition for a 3-regular graph to be cordial, *Ars Combin.*, **84** (2007) 225-230.
- [11] R.Ponraj, S.Sathish Narayanan and R.Kala, Difference cordial labeling of graphs, *Global Journal of Mathematical Sciences: Theory and Practical*, **5**(2013), 185-196.
- [12] R.Ponraj, S.Sathish Narayanan, R.Kala, Difference cordial labeling of graphs obtained from Double snakes, *International Journal of Mathematics Research*, **5**(2013), 317-322.

- [13] R.Ponraj, S.Sathish Narayanan, Difference cordiality of some graphs obtained from Double Alternate Snake Graphs, *Global Journal of Mathematical Sciences: Theory and Practical*, **5**(2013), 167-175.
- [14] R.Ponraj, S.Sathish Narayanan and R.Kala, Difference cordial labeling of corona graphs, *J. Math. Comput. Sci.*, **3**(2013), 1237-1251.
- [15] R.Ponraj and S.Sathish Narayanan, Further results on difference cordial labeling of corona graphs, *The Journal of the Indian Academy of Mathematics*, **35**(2013).
- [16] R.Ponraj, S.Sathish Narayanan and R.Kala, Difference cordiality of Some Product related graphs (communicated).
- [17] R.Ponraj, M.Sivakumar and M.Sundaram, k-Product cordial labeling of graphs, *Int. J. Contemp. Math. Sciences*, Vol. 7, 2012, no. 15, 733 - 742.
- [18] R.Ponraj, M.Sivakumar, M.Sundaram, k- Total product cordial labelings of graphs, *Applications and Applied Mathematics*, Vol. 7, Issue 2 (December 2012), pp. 708- 716
- [19] R.Ponraj and S.Somasundaram, On the Degree Splitting graph of a graph, *National Academy Science Letter*, **27**(2004) 275-278.
- [20] A.Riskin, Cordial deficiency, *Bull. Malays. Math. Sci. Soc.*, **30** (2007) 201-204. arXiv:math/0610760v1 [math.CO] 25 Oct 2006.
- [21] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966)*, Gordon and Breach, N. Y. and Dunod Paris (1967) 349-355.
- [22] E.Sampathkumar and H.B.Waliker, On the Splitting graph of a graph, *J. Karnatak Uni., Sci*, **25** (1980)13-16.
- [23] M.A.Seoud and A.E.I.Abdel Maqsood, On cordial and balanced labelings of graphs, *J. Egyptian Math. Soc.*, **7**(1999) 127-135.
- [24] S.K.Vaidya, G.Ghodesara, S.Srivastav, and V.Kaneria, Cordial labeling for two cycle related graphs, *Math. Student*, **76** (2007) 237-246.
- [25] M.Z.Youssef, On Skolem-graceful and cordial graphs, *Ars Combin.*, **78** (2006) 167-177.

## Computation of Four Orthogonal Polynomials Connected to Euler's Generating Function of Factorials

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**Abstract:** In the present paper, first we describe the orthogonality relations between denominator polynomials of  $[n-1/n]$  Pade approximants and related power series expansion; next we derive a continued fraction expansion called regular C-fraction for Euler's generating function of factorials and finally four orthogonal polynomials are extracted from numerator as well as denominator polynomials of both even and odd order convergents of the regular C-fraction connected to Pade approximants.

**Key Words:** Euler's generating function for factorials, regular C-fraction, orthogonal polynomials, Pade approximants.

**AMS(2010):** 05A15, 11J70, 30B70, 33C45, 41A21

### §1. Introduction

There is a very interesting literature [2-3] which interprets that  $[n-1]/n$  order Pade approximants provides an orthogonality relation between its denominator polynomials and the power series expansion. They are nothing but even order convergents [2-3,7,9] of a regular C-fraction expansion of the power series expansion. The denominator polynomials transformed to monic form are orthogonal polynomials with respect to a linear moment functional  $L : \mathbb{P} \longrightarrow \mathbb{R}$  from the space of all polynomials over  $\mathbb{R}$  into  $\mathbb{R}$  which has  $n^{th}$  moment same as the coefficient of  $x^n$  in a known power series. According to Favard's theorem [4-6,8] the necessary and sufficient condition for orthogonality of  $P_n(x)$  is to satisfy the following three term recurrence relation:

$$\begin{aligned} P_{-1}(x) &:= 0, & P_0(x) &:= 1, \\ P_n(x) &:= (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), & n &= 1, 2, 3, 4, \dots, \end{aligned} \quad (1)$$

where  $c_n$ 's are real and  $\lambda_n$ 's are non-zero numbers. The orthogonality relation [5-6,8] is given by

$$L\{P_m(x)P_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1 \lambda_2 \cdots \lambda_{n+1}, & m = n. \end{cases} \quad (2)$$

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Motivated strongly by the above works, in the present paper, four orthogonal polynomials are extracted from numerator as well as denominator polynomials of both even and odd order convergence of a regular C-fraction connected to Pade approximants for the Euler's generating function of factorials. In Section two, main results of Pade approximation, related continued fractions and orthogonal polynomials are reviewed which will be useful in the next Sections. In Section three, we derive a continued fraction expansion called regular C-fraction for Euler's generating function of factorials. In the last Section, we take the help of the regular C-fraction expansion derived in the previous Section to compute four orthogonal polynomials.

## §2. Main Results of Pade Approximation, Related Continued Fractions and Orthogonal Polynomials

In this section, we review some results which are used for next Sections. We begin with the definition of Pade approximants and a standard result.

### 1. A rational function

$$[m/n]_f(x) = \frac{p_0^{(m,n)} + p_1^{(m,n)} x + \cdots + p_m^{(m,n)} x^m}{1 + q_1^{(m,n)} x + \cdots + q_n^{(m,n)} x^n} = \frac{P_m^{(m,n)}(x)}{Q_n^{(m,n)}(x)}$$

is said to be  $(m, n)$  order Pade approximants [2-3] for a formal power series

$$f(x) := a_0 + a_1 x + \cdots + a_N x^N + \cdots,$$

if

$$(1 + q_1^{(m,n)} x + \cdots + q_n^{(m,n)} x^n) \times (a_0 + a_1 x + \cdots + a_{m+n} x^{m+n}) - (p_0^{(m,n)} + p_1^{(m,n)} x + \cdots + p_m^{(m,n)} x^m) = \mathbf{O}(x^{m+n+1}),$$

where  $q_i^{(m,n)}, p_j^{(m,n)}$  may or may not be zero  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, m$ .

The standard result [2-3] states that  $[m/n]_f(x)$  exists and unique if and only if the  $n \times n$  Hankel determinant:

$$H_{m,n} = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix} \neq 0,$$

where  $a_i = 0$ , if  $i < 0$ .

### 2. Let

$$[n-1/n]_f(x) = \frac{P_{n-1}^{(n-1,n)}(x)}{Q_n^{(n-1,n)}(x)} \text{ be } (n-1, n) \text{ Pade approximants for } f(x).$$

Then  $Q_n^{(n-1,n)}(x)$  satisfies

$$a_{n+k} + a_{n+k-1} q_1^{(n-1,n)} + \cdots + a_{k+1} q_{n-1}^{(n-1,n)} + a_k q_n^{(n-1,n)} = 0 \quad (3)$$

for  $k = 0, 1, 2, \dots, n-1$ . Put

$$\tilde{q}_n^{(n-1,n)}(x) = x^n Q_n^{(n-1,n)}\left(\frac{1}{x}\right) = x^n + q_1^{(n-1,n)}x^{n-1} + \dots + q_n^{(n-1,n)}.$$

For the power series

$$f(x) := a_0 + a_1 x + \dots + a_N x^N + \dots,$$

define a linear moment generating function by

$$L_f(x^n) = a_n.$$

Then (3) simply states that

$$L_f(x^k \tilde{q}_n^{(n-1,n)}(x)) = 0, \quad n = 0, 1, 2, 3, \dots, n-1. \quad (4)$$

Hence

$$L_f(\tilde{q}_m^{(n-1,n)}(x) \tilde{q}_n^{(n-1,n)}(x)) = 0, \quad \text{if } m \neq n.$$

Therefore  $\{\tilde{q}_n^{(n-1,n)}(x)\}$  forms monic orthogonal polynomial with respect to  $L_f$ .

3. Let

$$[n/n]_f(x) = \frac{P_n^{(n,n)}(x)}{Q_n^{(n,n)}(x)}$$

be  $(n, n)$  Pade approximants for  $f(x)$ . Then, we have

$$\frac{\tilde{P}_{n-1}^{(n,n)}(x)}{Q_n^{(n,n)}(x)} = [n-1/n]_{f_1}(x),$$

where  $f_1(x) = a_1 + a_2x + \dots + a_{n+1}x^n + \dots = \frac{f(x) - a_0}{x}$ . Using item 2,  $\{\tilde{q}_n^{(n,n)}(x)\}$  forms monic orthogonal polynomial with respect to  $L_{f_1}$ .

4. Let

$$[n+1/n]_f(x) = \frac{P_{n+1}^{(n+1,n)}(x)}{Q_n^{(n+1,n)}(x)}$$

be  $(n+1, n)$  Pade approximants for  $f(x)$ . Then, we have

$$\frac{\tilde{P}_{n-1}^{(n+1,n)}(x)}{Q_n^{(n+1,n)}(x)} = [n-1/n]_{f_2}(x),$$

where  $f_2(x) = a_2 + a_3x + \dots + a_{n+1}x^n + \dots = \frac{f(x) - a_0 - a_1x}{x^2}$ . Using item 2,  $\{\tilde{q}_n^{(n+1,n)}(x)\}$  forms monic orthogonal polynomial with respect to  $L_{f_2}$ .

5. Let  $[m/n]_f(x) = \frac{P_m^{(m,n)}(x)}{Q_n^{(m,n)}(x)}$  be  $(m, n)$  order Pade approximants for  $f(x)$ . Then

$$\begin{aligned} \frac{1}{f(x)} - \frac{Q_n^{(m,n)}(x)}{P_m^{(m,n)}(x)} &= - \left[ f(x) - \frac{P_m^{(m,n)}(x)}{Q_n^{(m,n)}(x)} \right] \left[ \frac{f(x)P_m^{(m,n)}(x)}{Q_n^{(m,n)}(x)} \right]^{-1} \\ &= \mathbf{O}(x^{m+n+1}). \end{aligned}$$



Hence

$$\frac{Q_n^{(m,n)}(x)}{P_m^{(m,n)}(x)} = [n/m]_{\frac{1}{f}}(x).$$

6. Suppose

$$f(x) = \frac{c_0}{1} + \frac{c_1 x}{1} + \dots + \frac{c_n x}{1} + \dots$$

Then [2-3]

$$\begin{aligned} \frac{P_1}{Q_1} &= \frac{c_0}{1}, \quad \frac{P_2}{Q_2} = \frac{c_0}{1+c_1 x}, \dots, \quad \frac{P_{2n}}{Q_{2n}} = \frac{P_{2n-1} + c_{2n} P_{2n-2}}{Q_{2n-1} + c_{2n} Q_{2n-2}} = [n-1/n]_f(x), \\ \frac{P_{2n+1}}{Q_{2n+1}} &= \frac{P_{2n} + c_{2n+1} P_{2n-1}}{Q_{2n} + c_{2n+1} Q_{2n-1}} = [n/n]_f(x). \end{aligned}$$

### §3. Derivation of Regular C-Fraction for Euler's Generating Function of Factorials

The generating function for factorial numbers with its asymptotic relation

$$E(x) = \int_0^\infty \frac{e^{-t}}{1+xt} dt \sim \sum_{n=0}^\infty (-1)^n n! x^n, \quad \text{as } x \longrightarrow 0,$$

was first studied by Euler [2]. It has a remarkable regular C-fraction expansion:

$$\int_0^\infty \frac{e^{-t}}{1+xt} dt = \frac{1}{1} + \frac{x}{1} + \frac{x}{1} + \dots + \frac{nx}{1} + \frac{nx}{1} + \dots \quad (5)$$

**Theorem 3.1** Let  $I_{2n-1} = \int_0^\infty \frac{t^{n-1} e^{-t}}{(1+xt)^n} dt$  and  $I_{2n} = \int_0^\infty \frac{t^n e^{-t}}{(1+xt)^n} dt$ ,  $n = 0, 1, 2, 3, \dots$

Then

$$\begin{aligned} I_{2n-2} - I_{2n-1} &= x I_{2n}, \\ n I_{2n-1} - I_{2n} &= n x I_{2n+1} \end{aligned}$$

and as a result

$$\frac{I_{2n-1}}{I_{2n-2}} = \frac{1}{1} + \frac{n x}{1 + n x \frac{I_{2n+1}}{I_{2n}}}, \quad n = 1, 2, 3, \dots,$$

which readily gives

$$\frac{I_1}{I_0} = \int_0^\infty \frac{e^{-t}}{1+xt} dt = \frac{1}{1} + \frac{x}{1} + \frac{x}{1} + \dots + \frac{nx}{1} + \frac{nx}{1} + \dots$$

*Proof* We can show that

$$\begin{aligned} I_{2n-2} - I_{2n-1} &= \int_0^\infty \frac{[(1+xt)t^{n-1} - t^{n-1}]e^{-t}}{(1+xt)^n} dt \\ &= x I_{2n} \\ \frac{I_{2n-1}}{I_{2n-2}} &= \frac{1}{1 + n x \frac{I_{2n}}{I_{2n-1}}} \end{aligned}$$

and

$$\begin{aligned} nI_{2n-1} - I_{2n} &= \int_0^\infty \frac{d(t^n e^{-t})}{(1+xt)^n} \\ &= n x I_{2n+1}. \\ \frac{I_{2n}}{n I_{2n-1}} &= \frac{1}{1 + n x \frac{I_{2n+1}}{I_{2n}}}. \end{aligned}$$

For  $n = 1$ , also the identities hold:

$$I_1 = \frac{1}{1 + x \frac{I_2}{I_1}}, \quad \frac{I_2}{I_1} = \frac{1}{1 + x \frac{I_2}{I_1}}.$$

Therefore

$$\frac{I_{2n-1}}{I_{2n-2}} = \frac{1}{1 + \frac{n x}{1 + n x \frac{I_{2n+1}}{I_{2n}}}}, \quad n = 1, 2, 3, \dots$$

Hence

$$\frac{I_1}{I_0} = I_1 = \int_0^\infty \frac{e^{-t}}{1+xt} dt = \frac{1}{1} + \frac{x}{1} + \frac{x}{1} + \dots + \frac{nx}{1} + \frac{nx}{1} + \dots \quad \square$$

#### §4. Computation of Four Orthogonal Polynomials

Let us consider (5),

$$E(x) = \frac{1}{1} + \frac{x}{1} + \frac{x}{1} + \dots + \frac{nx}{1} + \frac{nx}{1} + \dots$$

In the language of Pade approximants, the continued fraction provides diagonal Pade approximants [2-3] which are given by (please see item 6 of Section two)

$$\frac{A_1}{B_1} = \frac{1}{1} = \frac{P_0^{(0,0)}}{Q_0^{(0,0)}}, \quad \frac{A_3}{B_3} = \frac{1+x}{1+2x} = \frac{P_1^{(1,1)}}{Q_1^{(1,1)}}, \quad \dots, \quad \frac{A_{2n+1}}{B_{2n+1}} = \frac{P_n^{(n,n)}}{Q_n^{(n,n)}}$$

and lower diagonal Pade approximants are given by (please see item 6 of Section 2)

$$\frac{A_2}{B_2} = \frac{1}{1+x} = \frac{P_0^{(0,1)}}{Q_0^{(0,1)}}, \quad \frac{A_4}{B_4} = \frac{1+3x}{1+4x+2x^2} = \frac{P_1^{(1,2)}}{Q_1^{(1,2)}}, \quad \dots, \quad \frac{A_{2n+2}}{B_{2n+2}} = \frac{P_n^{(n-1,n)}}{Q_n^{(n-1,n)}}.$$

Let us make use of definitions of even and odd parts of continued fraction as given in [9].

$[n - 1/n]$  Pade approximants can be computed using the even part of continued fraction (5):

$$\frac{1}{1+x} - \frac{1^2 x^2}{1+3x} - \dots - \frac{n^2 x^2}{1+(2n+1)x} - \dots,$$

we obtain even order convergents:

$$\frac{A_{2n+2}(x)}{B_{2n+2}(x)} = \frac{(1+(2n+1)x)A_{2n}(x) - n^2 x^2 A_{2n-2}(x)}{(1+(2n+1)x)B_{2n}(x) - n^2 x^2 B_{2n-2}(x)}$$

with

$$\frac{A_2}{B_2} = \frac{1}{1+x}, \quad \frac{A_4}{B_4} = \frac{1+3x}{1+4x+2x^2}, \quad n = 2, 3, 4, \dots$$

Similarly,  $[n/n]$  Pade approximants can be computed using the odd part of continued fraction (5):

$$1 - \frac{x}{1+2x} - \frac{1.2x^2}{1+4x} - \dots - \frac{n(n+1)x^2}{1+(2n+2)x} - \dots,$$

we obtain odd order convergents:

$$\frac{A_{2n+3}(x)}{B_{2n+3}(x)} = \frac{(1+(2n+2)x)A_{2n+1}(x) - n(n+1)x^2A_{2n-1}(x)}{(1+(2n+2)x)B_{2n+1}(x) - n(n+1)x^2B_{2n-1}(x)}$$

with

$$\frac{A_1}{B_1} = \frac{1}{1}, \quad \frac{A_3}{B_3} = \frac{1+x}{1+2x}, \quad n = 2, 3, 4, \dots$$

The desired orthogonal polynomials are nothing but

$$\begin{aligned} p_n(x) &= x^n A_{2n+2} \left( \frac{1}{x} \right), \quad n = 0, 1, 2, \dots, \\ q_n(x) &= x^n B_{2n} \left( \frac{1}{x} \right), \quad n = 0, 1, 2, \dots, \\ r_n(x) &= x^n A_{2n+1} \left( \frac{1}{x} \right), \quad n = 0, 1, 2, \dots, \\ s_n(x) &= x^n B_{2n+1} \left( \frac{1}{x} \right), \quad n = 0, 1, 2, \dots, \end{aligned}$$

where we can select  $B_0 \left( \frac{1}{x} \right) := 1$ . Now we can describe orthogonality of  $q_n(x)$ ,  $s_n(x)$ ,  $p_n(x)$  and  $r_n(x)$  as follows:

(1) Consider the series

$$E(x) = 0! - 1!x + 2!x^2 - 3!x^3 + \dots + (-1)^n n!x^n + \dots$$

The linear moment generating function with respect to  $E(x)$  denoted by  $L_E$  has  $n^{th}$  moment,

$$L_E\{x^n\} = (-1)^n n!.$$

The three term recurrence relation of  $q_n(x)$  is

$$\begin{aligned} q_{n+1}(x) &= (x+2n+1)q_n(x) - n^2q_{n-1}(x), \\ q_0(x) &= 1, \quad q_1(x) = x+1, \quad n = 1, 2, 3, \dots \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of  $q_n(x)$  is

$$L_E\{q_m(x)q_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2 \cdots \lambda_{n+1}, & m = n, \end{cases}$$

where  $\lambda_1 = 1$  and  $\lambda_k = (k-1)^2$ ,  $k = 2, 3, \dots, n+1$ . Following [1], it is well known fact in the literature that

$$q_n(x) = n!L_n(-x) = \sum_{r=0}^n r! \left[ \binom{n}{r} \right]^2 x^{n-r},$$

where  $L_n(x)$  is Laguerre polynomials of order 0.

(2) Using item 3 of Section two, we obtain the series

$$E_1(x) = \frac{E(x) - 1}{x} = 1! - 2!x + \dots + (-1)^n(n+1)!x^n + \dots.$$

The linear moment generating function with respect to  $E_1(x)$  denoted by  $L_{E_1}$  has  $n^{th}$  moment

$$L_{E_1}\{x^n\} = (-1)^n(n+1)!.$$

The three term recurrence relation of  $s_n(x)$  is

$$\begin{aligned} s_{n+1}(x) &= (x + 2n + 2)s_n(x) - n(n+1)s_{n-1}(x), \\ s_0(x) &= 1, \quad s_1(x) = x + 2, \quad n = 1, 2, 3, \dots \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of  $s_n(x)$  is

$$L_{E_1}\{s_m(x)s_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2 \cdots \lambda_{n+1}, & m = n, \end{cases}$$

where  $\lambda_1 = 1$  and  $\lambda_k = (k-1)k$ ,  $k = 2, 3, \dots, n+1$ . Following [1], it is well known fact in the literature that

$$s_n(x) = n!L_n^1(-x) = \sum_{r=0}^n r! \binom{n}{r} \binom{n+1}{r} x^{n-r},$$

where  $L_n^1(x)$  is Laguerre polynomials of order 1.

(3) Using items 4 and 5 of Section 2, we obtain the series

$$\frac{1}{E(x)} = 1 + x - d_1x^2 + d_2x^3 - d_3x^4 + \dots + (-1)^n d_n x^{n+1} + \dots.$$

$$E_2(x) = \frac{\frac{-1}{E(x)} + 1 + x}{x^2} = d_1 - d_2x + d_3x^2 - d_4x^3 + \dots + (-1)^n d_{n+1}x^n + \dots.$$

The linear moment generating function with respect to  $E_2(x)$  denoted by  $L_{E_2}$  has  $n^{th}$  moment

$$L_{E_2}\{x^n\} = (-1)^n d_{n+1}.$$

The three term recurrence relation of  $p_n(x)$  is

$$\begin{aligned} p_{n+1}(x) &= (x + 2n + 3)p_n(x) - (n+1)^2 p_{n-1}(x), \\ p_0(x) &= 1, \quad p_1(x) = x + 3, \quad n = 1, 2, 3, \dots \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of  $p_n(x)$  is

$$L_{E_2}\{p_m(x)p_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2 \cdots \lambda_{n+1}, & m = n, \end{cases}$$

where  $\lambda_1 = 1$  and  $\lambda_k = k^2$ ,  $k = 2, 3, \dots, n+1$ .

(4) Using item 3 and item 5 of Section 2, we obtain the series

$$E_3(x) = \frac{\frac{1}{x}(1 - E(x))}{E(x)} = 1 - d_1x + d_2x^2 - d_3x^3 + \dots + (-1)^n d_n x^n + \dots$$

The linear moment generating function with respect to  $E_3(x)$  denoted by  $L_{E_3}$  has  $n^{th}$  moment

$$L_{E_3}\{x^n\} = (-1)^n d_n.$$

The three term recurrence relation of  $r_n(x)$  is

$$\begin{aligned} r_{n+1}(x) &= (x + 2n + 2)r_n(x) - n(n+1)r_{n-1}(x), \\ r_0(x) &= 1, \quad r_1(x) = x + 1, \quad n = 1, 2, 3, \dots \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of  $r_n(x)$  is

$$L_{E_3}\{r_m(x)r_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2 \dots \lambda_{n+1}, & m = n, \end{cases}$$

where  $\lambda_1 = 1$  and  $\lambda_k = k(k-1)$ ,  $k = 2, 3, \dots, n+1$ .

Suppose  $r_n(x) = x^n + r_{n-1}x^{n-1} + \dots + r_1x + r_0$ . Since  $L_{E_3}\{r_0(x)r_n(x)\} = 0$ , we can compute  $d_n$  using

$$d_n = -[r_{n-1}d_{n-1} + \dots + r_1d_1 + r_0], \quad d_0 = 1, \quad n = 1, 2, \dots$$

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### References

- [1] G.B.Arften and H.J.Weber, *Mathematical Methods for Physicists*, Elsevier Academic Press, New Delhi, 2005.
- [2] G.A.Baker Jr., *Essentials of Pade Approximants*, Academic Press, New York, 1975.
- [3] G.A.Baker Jr. and P. Graves-Morris, *Pade Approximants*, Second Edition, Cambridge University Press, New York, 1996.
- [4] A.Branquinhio, A note on semi-classical orthogonal polynomials, *Bull. Belg. Math. Soc.*, Vol.3, 1-12(1996).
- [5] T.S.Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [6] W.Gautchi, *Orthogonal Polynomials: Computation and Approximation*, Oxford University Press, New York, 2004.

- [7] W.B.Jones and W.J.Thron, *Analytic Theory of Continued Fractions*, Cambridge University Press, New York, 1984.
- [8] W.Hahn, On differential equations for orthogonal polynomials, *Funk. Ekv.* Vol.21, 1-9(1978).
- [9] H.S.Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand Company, New York, 1948.

## On Odd Sum Graphs

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**Abstract:** An injective function  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is an odd sum labeling if the induced edge labeling  $f^*$  defined by  $f^*(uv) = f(u) + f(v)$ , for all  $uv \in E(G)$ , is bijective and  $f^*(E(G)) = \{1, 3, 5, \dots, 2q - 1\}$ . A graph is said to be an odd sum graph if it admits an odd sum labeling. In this paper, we have studied the odd sum property for the graphs paths  $P_p$ , cycles  $C_p$ ,  $C_p \odot K_1$ , the ladder  $P_2 \times P_p$ ,  $P_m \odot nK_1$ , the balloon graph  $P_n(C_p)$ , quadrilateral snake  $Q_n$ ,  $[P_m; C_n]$ ,  $(P_m; Q_3)$ ,  $T_p^{(n)}$ ,  $H_n \odot mK_1$ , bistar graph and cyclic ladder  $P_2 \times C_p$ .

**Key Words:** Labeling, odd sum labeling, odd sum graph.

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### §1. Introduction

Throughout this paper, by a graph we mean a finite, undirected simple graph. Let  $G(V, E)$  be a graph with  $p$  vertices and  $q$  edges. For notations and terminology we follow [1].

Path on  $p$  vertices is denoted by  $P_p$  and a cycle on  $p$  vertices is denoted by  $C_p$  whose length is  $p$ . If  $m$  number of pendant vertices are attached at each vertex of  $G$ , then the resultant graph obtained from  $G$  is the graph  $G \odot mK_1$ . When  $m = 1$ ,  $G \odot K_1$  is the corona of  $G$ . The bistar graph  $B_{m,n}$  is the graph obtained from  $K_2$  by identifying the central vertices of  $K_{1,m}$  and  $K_{1,n}$  at the end vertices of  $K_2$  respectively. The graph  $P_2 \times P_p$  is the ladder and  $P_2 \times C_p$  is the cyclic ladder. The balloon of a graph  $G$ ,  $P_n(G)$  is the graph obtained from  $G$  by identifying an end vertex of  $P_n$  at a vertex of  $G$ . Let  $v$  be a fixed vertex of  $G$ . The graph  $[P_m; G]$  is obtained from  $m$  copies of  $G$  and the path  $P_m : u_1 u_2 \dots u_m$  by identifying  $u_i$  with the vertex  $v$  of the  $i^{th}$  copy of  $G$ , for  $1 \leq i \leq m$ . The graph  $(P_m; G)$  is obtained from  $m$  copies of  $G$  and the path  $P_m : u_1 u_2 \dots u_m$  by joining  $u_i$  with the vertex  $v$  of the  $i^{th}$  copy of  $G$  by means of an edge, for  $1 \leq i \leq m$  [7]. The cube graph  $Q_3$  is  $P_2 \times C_4$ . A quadrilateral snake is obtained from a path by

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identifying each edge of the path with an edge of the cycle  $C_4$ . The graph  $T_p^{(n)}$  is a tree formed from  $n$  copies of path on  $p$  vertices by joining an edge  $uu'$  between every pair of consecutive paths where  $u$  is a vertex in  $i^{th}$  copy of the path and  $u'$  is the corresponding vertex in the  $(i+1)^{th}$  copy of the path.

In [2], an odd edge labeling of a graph is defined as follows: A labeling  $f : V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$  is called an odd edge labeling of  $G$  if for the edge labeling  $f^+$  on  $E(G)$  defined by  $f^+(uv) = f(u) + f(v)$  for any edge  $uv \in E(G)$ , for a connected graph  $G$ , the edge labeling is not necessarily injective. In [5], the concept of pair sum labeling was introduced. An injective function  $f : V(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm p\}$  is said to be a pair sum labeling if the induced edge function  $f_e : E(G) \rightarrow \mathbb{Z} - \{0\}$  defined by  $f_e(uv) = f(u) + f(v)$  is one-one and  $f_e(E(G))$  is either of the form  $\{\pm k_1, \pm k_2, \dots, \pm \frac{k_q}{2}\}$  or  $\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{q-1}{2}}\} \cup \{\frac{k_{q+1}}{2}\}$  according as  $q$  is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph. In [6], the concept of mean labeling was introduced. An injective function  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is said to be a mean labeling if the induced edge labeling  $f^*$  defined by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even,} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

is injective and  $f^*(E(G)) = \{1, 2, \dots, q\}$ . A graph  $G$  is said to be odd mean if there exists an injective function  $f$  from  $V(G)$  to  $\{0, 1, 2, 3, \dots, 2q-1\}$  such that the induced map  $f^*$  from  $E(G) \rightarrow \{1, 3, 5, \dots, 2q-1\}$  defined by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even,} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

is a bijection [6].

Motivated by these, we introduce a new concept called odd sum labeling. An injective function  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is an odd sum labeling if the induced edge labeling  $f^*$  defined by  $f^*(uv) = f(u) + f(v)$ , for all  $uv \in E(G)$ , is bijective and  $f^*(E(G)) = \{1, 3, 5, \dots, 2q-1\}$ . A graph is said to be an odd sum graph if it admits an odd sum labeling. In this paper, we have studied the odd sum property for the graphs paths  $P_p$ , cycles  $C_p$ ,  $C_p \odot K_1$ , the ladder  $P_2 \times P_p$ ,  $P_m \odot nK_1$ , the balloon graph  $P_n(C_p)$ , quadrilateral snake  $Q_n$ ,  $[P_m; C_n]$ ,  $(P_m; Q_3)$ ,  $T_p^{(n)}$ ,  $H_n \odot mK_1$ , bistar graph and cyclic ladder  $P_2 \times C_p$ .

## §2. Main Results

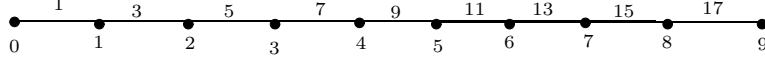
**Observation 2.1** *Every graph having an odd cycle is not an odd sum graph.*

*Proof* If a graph has a cycle of odd length, then at least one edge  $uv$  on the cycle such that  $f(u)$  and  $f(v)$  are of same suit and hence its induced edge label  $f^*(uv)$  is even.  $\square$

**Proposition 2.2** *Every path  $P_p$ ,  $p \geq 2$  is an odd sum graph.*

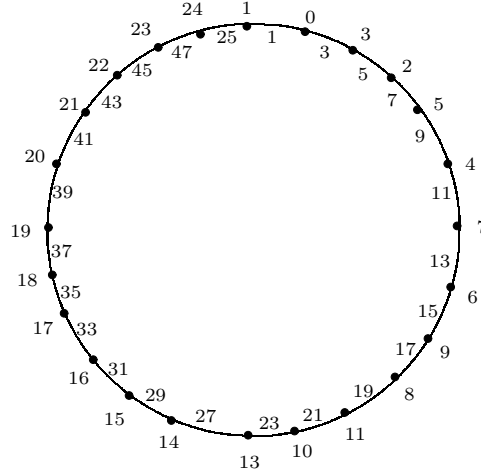


*Proof* Let  $v_1, v_2, \dots, v_p$  be the vertices of the path  $P_p$ . The labeling  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is defined as  $f(v_i) = i - 1$  for  $1 \leq i \leq p$  and the induced edge label is  $f^*(v_i v_{i+1}) = 2i - 1$ , for  $1 \leq i \leq p - 1$ . Then  $f$  is an odd sum labeling and hence  $P_p$  is an odd sum graph.  $\square$



**Figure 1:** An odd sum labeling of  $P_{10}$ .

**Proposition 2.3** Cycle  $C_p$  is an odd sum graph only when  $p \equiv 0 \pmod{4}$ .



**Figure 2:** An odd sum labeling of  $C_{24}$ .

*Proof* By Observation 2.1,  $C_p$  is not an odd sum graph when  $p$  is odd. Suppose  $p = 2m, m \geq 2$  and  $C_p$  admits an odd sum labeling. Then  $\sum_{uv \in E(G)} f^*(uv) = \sum_{uv \in E(G)} (f(u) + f(v))$ . This implies that  $1 + 3 + \dots + (4m - 1) = 2(0 + 1 + 2 + \dots + 2m) - 2i$  where  $i$  is not a vertex label of  $C_p$ . From this we have,  $i = m$ . If  $m$  is odd, then the number of even values is in excess of 2 that of the number of odd values and they are to be assigned as vertex labels in  $C_p$ . Thus if  $C_p$  admits an odd sum labeling, then  $m$  should be even and hence  $p$  is a multiple of 4.

Suppose  $p = 4m, m \geq 1$ . Let  $v_1, v_2, \dots, v_p$  be the vertices of the cycle  $C_p$ . The labeling  $f : V(G) \rightarrow \{0, 1, 2, \dots, 4m\}$  is defined as follows.

$$f(v_i) = \begin{cases} i, & 1 \leq i \leq 2m - 1 \text{ and } i \text{ is odd,} \\ i - 2, & 1 \leq i \leq 2m \text{ and } i \text{ is even,} \\ i, & 2m + 1 \leq i \leq 4m. \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(v_i v_{i+1}) = \begin{cases} 2i - 1, & 1 \leq i \leq 2m, \\ 2i + 1, & 2m + 1 \leq i \leq 4m - 1 \text{ and} \\ f^*(v_{4m} v_1) = 4m + 1. \end{cases}$$

Hence  $f$  is an odd sum labeling of  $C_p$  only when  $p \equiv 0(\text{mod } 4)$ .  $\square$

**Proposition 2.4** *For each even integer  $p \geq 4$ ,  $C_p \odot K_1$  is an odd sum graph.*

*Proof* In  $C_p \odot K_1$ , let  $v_1, v_2, \dots, v_p$  be the vertices on the cycle and let  $u_i$  be the pendant vertex of  $v_i$  at each  $i$ ,  $1 \leq i \leq p$ .

**Case 1**  $p = 4m$ , for  $m \geq 1$ .

The labeling  $f : V(C_p \odot K_1) \rightarrow \{0, 1, 2, \dots, 8m\}$  is defined as follows.

$$f(v_i) = \begin{cases} 2i - 2, & 1 \leq i \leq 2m - 1 \text{ and } i \text{ is odd,} \\ 2i, & 2m + 1 \leq i \leq 4m - 1 \text{ and } i \text{ is odd,} \\ 2i - 1, & 2 \leq i \leq 4m \text{ and } i \text{ is even and} \end{cases}$$

$$f(u_i) = \begin{cases} 2i - 1, & 1 \leq i \leq 4m - 1 \text{ and } i \text{ is odd,} \\ 2i - 2, & 2 \leq i \leq 2m \text{ and } i \text{ is even,} \\ 2i, & 2m + 2 \leq i \leq 4m \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(v_i v_{i+1}) = \begin{cases} 4i - 1, & 1 \leq i \leq 2m - 1, \\ 4i + 1, & 2m \leq i \leq 4m - 1, \end{cases}$$

$$f^*(v_{4m} v_1) = 8m - 1 \text{ and}$$

$$f^*(u_i v_i) = \begin{cases} 4i - 3, & 1 \leq i \leq 2m, \\ 4i - 1, & 2m + 1 \leq i \leq 4m. \end{cases}$$

Thus  $f$  is an odd sum labeling of  $C_p \odot K_1$ . Hence  $C_p \odot K_1$  is an odd sum graph when  $p = 4m$ .

**Case 2**  $p = 4m + 2$ , for  $m \geq 1$ .

The labeling  $f : V(C_p \odot K_1) \rightarrow \{0, 1, 2, \dots, 8m + 4\}$  is defined as follows.

$$f(v_i) = \begin{cases} 2i - 2, & 1 \leq i \leq 2m + 1 \text{ and } i \text{ is odd,} \\ 2i, & 2m + 3 \leq i \leq 4m + 1 \text{ and } i \text{ is odd,} \\ 2i - 1, & 2 \leq i \leq 2m \text{ and } i \text{ is even,} \\ 2i + 1, & i = 2m + 2, \\ 2i - 1, & 2m + 4 \leq i \leq 4m + 2 \text{ and } i \text{ is even and} \end{cases}$$

$$f(u_i) = \begin{cases} 2i - 1, & 1 \leq i \leq 2m + 1 \text{ and } i \text{ is odd,} \\ 2i - 3, & i = 2m + 3, \\ 2i - 1, & 2m + 5 \leq i \leq 4m + 1 \text{ and } i \text{ is odd,} \\ 2i - 2, & 2 \leq i \leq 2m + 2 \text{ and } i \text{ is even,} \\ 2i, & 2m + 4 \leq i \leq 4m + 2 \text{ and } i \text{ is even.} \end{cases}$$

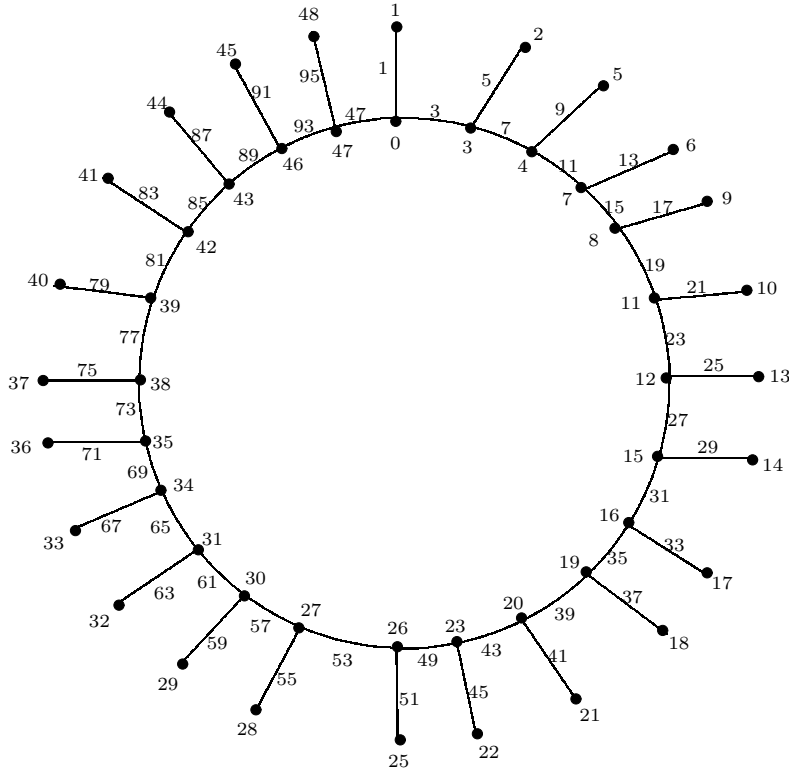
The induced edge labels are obtained as follows.

$$f^*(v_i v_{i+1}) = \begin{cases} 4i - 1, & 1 \leq i \leq 2m, \\ 4i + 1, & i = 2m + 1, \\ 4i + 3, & i = 2m + 2, \\ 4i + 1, & 2m + 3 \leq i \leq 4m + 1, \end{cases}$$

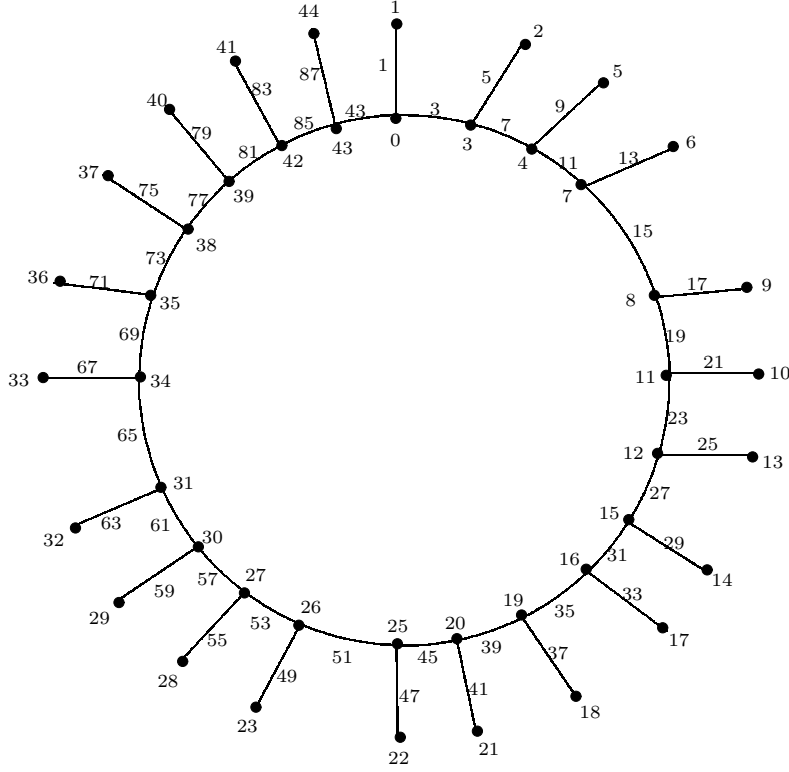
$$f^*(v_{4m+2} v_1) = 8m + 3 \text{ and}$$

$$f^*(u_i v_i) = \begin{cases} 4i - 3, & 1 \leq i \leq 2m + 1, \\ 4i - 1, & i = 2m + 2, \\ 4i - 3, & i = 2m + 3 \\ 4i - 1, & 2m + 4 \leq i \leq 4m + 2. \end{cases}$$

Thus  $f$  is an odd sum labeling of  $C_p \odot K_1$ . Hence  $C_p \odot K_1$  is an odd sum graph.  $\square$



**Figure 3:** An odd sum labeling of  $C_{24} \odot K_1$ .



**Figure 4:** An odd sum labeling of  $C_{22} \odot K_1$ .

**Proposition 2.5** For every positive integer  $p \geq 2$ , the ladder  $P_2 \times P_p$  is an odd sum graph.

*Proof* Let  $u_1, u_2, \dots, u_p$  and  $v_1, v_2, \dots, v_p$  be the vertices of the two copies of  $P_p$ . The labeling  $f : V(P_2 \times P_p) \rightarrow \{0, 1, 2, \dots, 3p - 2\}$  is defined as follows.

$$f(u_i) = 3i - 3, \text{ for } 1 \leq i \leq p \text{ and}$$

$$f(v_i) = 3i - 2, \text{ for } 1 \leq i \leq p.$$

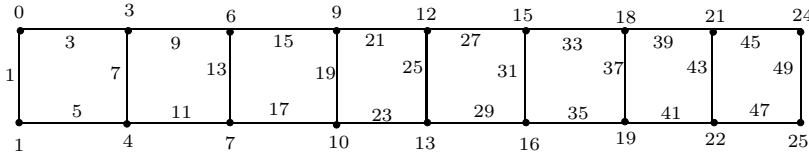
The induced edge labels are obtained as follows.

$$f^*(u_i u_{i+1}) = 6i - 3, \text{ for } 1 \leq i \leq p - 1,$$

$$f^*(v_i v_{i+1}) = 6i - 1, \text{ for } 1 \leq i \leq p - 1 \text{ and}$$

$$f^*(u_i v_i) = 6i - 5, \text{ for } 1 \leq i \leq p.$$

Thus  $f$  is an odd sum labeling of  $P_2 \times P_p$ . Hence  $P_2 \times P_p$  is an odd sum graph.  $\square$



**Figure 5:** An odd sum labeling of  $P_2 \times P_9$ .

**Proposition 2.6** *The graph  $P_m \odot nK_1$  is an odd sum graph if either  $m$  is an even positive integer and  $n$  is any positive integer or  $m$  is an odd positive integer and  $n = 1, 2$ .*

*Proof* In  $P_m \odot nK_1$ , let  $u_1, u_2, \dots, u_m$  be the vertices on the path and  $\{u_{i,j} : 1 \leq j \leq n\}$  be the pendant vertices attached at  $u_i, 1 \leq i \leq m$ .

**Case 1**  $m$  is even.

The labeling  $f : V(P_m \odot nK_1) \rightarrow \{0, 1, 2, \dots, m(n+1) - 1\}$  is defined as follows.

For  $1 \leq i \leq m$ ,

$$f(u_i) = \begin{cases} (n+1)(i-1), & i \text{ is odd,} \\ (n+1)i - 1, & i \text{ is even.} \end{cases}$$

For  $1 \leq i \leq m$  and  $1 \leq j \leq n$ ,

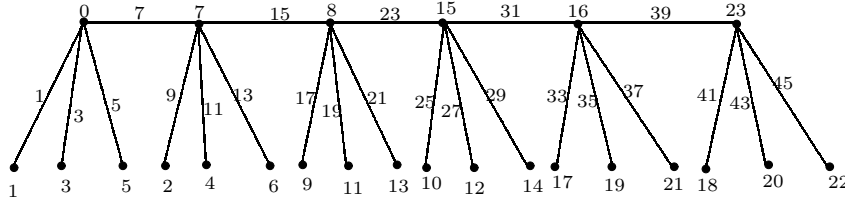
$$f(u_{i,j}) = \begin{cases} (n+1)(i-1) + 2j - 1, & i \text{ is odd,} \\ (n+1)(i-2) + 2j, & i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(u_i u_{i,j}) = 2(n+1)(i-1) + 2j - 1, \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n \text{ and}$$

$$f^*(u_i u_{i+1}) = 2(n+1)i - 1, \text{ for } 1 \leq i \leq m - 1.$$

Thus  $f$  is an odd sum labeling of  $P_m \odot nK_1$ .



**Figure 6:** An odd sum labeling of  $P_6 \odot 3K_1$ .

**Case 2**  $m$  is odd.

If  $P_m \odot nK_1$  has an odd sum labeling  $f$  when  $m$  is odd, then  $f$  is a bijection from  $V(P_m \odot nK_1)$  to the set  $\{0, 1, 2, \dots, m(n+1) - 1\}$ . Since the number of even integers in this set is either equal to or one excess to the number of odd integers in this set,  $n$  should be less than or equal to 2.

In case of  $m$  is odd and  $n = 1, 2$ , the labeling  $f : V(P_m \odot nK_1) \rightarrow \{0, 1, 2, \dots, m(n+1) - 1\}$  is defined as follows.

For  $1 \leq i \leq m$ ,

$$f(u_i) = \begin{cases} (n+1)(i-1) + 1, & i \text{ is odd,} \\ (n+1)i - 2, & i \text{ is even.} \end{cases}$$

For  $1 \leq i \leq m$  and  $1 \leq j \leq n$ ,

$$f(u_{i,j}) = \begin{cases} (n+1)(i-1) + 2(j-1), & i \text{ is odd,} \\ (n+1)(i-2) + 2j + 1, & i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows.

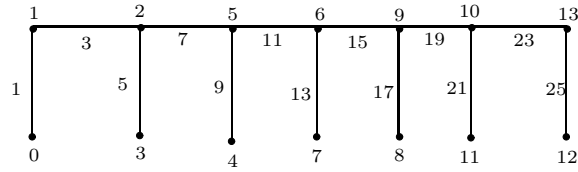
For  $1 \leq i \leq m$  and  $1 \leq j \leq n$ ,

$$f^*(u_i u_{i,j}) = 2(n+1)(i-1) + 2j - 1.$$

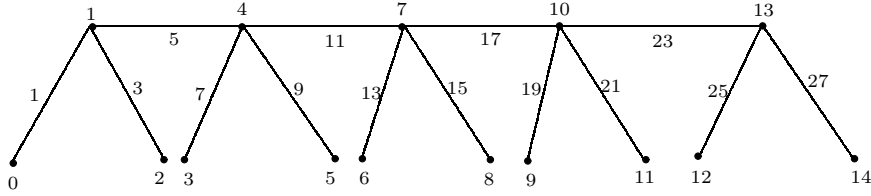
For  $1 \leq i \leq m-1$ ,

$$f^*(u_i u_{i+1}) = 2(n+1)i - 1.$$

Thus  $f$  is an odd sum labeling of  $P_m \odot nK_1$ .  $\square$



**Figure 7:** An odd sum labeling of  $P_7 \odot K_1$ .



**Figure 8:** An odd sum labeling of  $P_5 \odot 2K_1$ .

**Proposition 2.7** *The graph  $P_n(C_p)$  is an odd sum graph if either  $p \equiv 0 \pmod{4}$  or  $p \equiv 2 \pmod{4}$  and  $n \not\equiv 1 \pmod{3}$ .*

*Proof* Let  $u_1, u_2, \dots, u_p$  be the vertices of  $C_p$  and  $v_1, v_2, \dots, v_n$  be the vertices of the path  $P_n$  and  $u_p$  be identified with  $v_1$  in  $P_n(C_p)$ .

**Case 1**  $p \equiv 0 \pmod{4}$ .

Let  $p = 4m, m \geq 1$ . The labeling  $f : V(P_n(C_p)) \rightarrow \{0, 1, 2, \dots, 4m + n - 1\}$  is defined as follows.

$$f(u_i) = \begin{cases} i, & 1 \leq i \leq 4m \text{ and } i \text{ is odd,} \\ i - 2, & 1 \leq i \leq 2m \text{ and } i \text{ is even,} \\ i, & 2m + 1 \leq i \leq 4m \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_i) = 4m + i - 1, 2 \leq i \leq n.$$

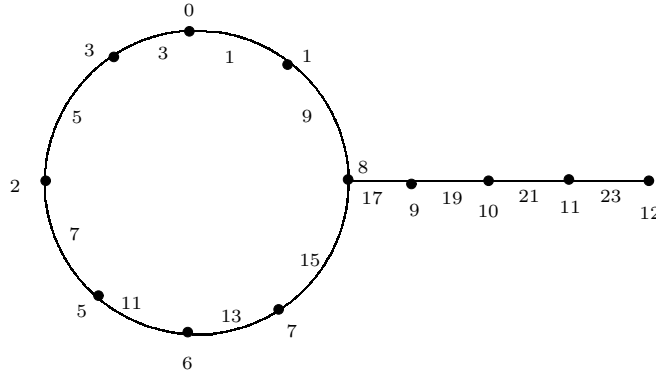
The induced edge labels are obtained as follows.

$$f^*(u_i u_{i+1}) = \begin{cases} 2i - 1, & 1 \leq i \leq 2m, \\ 2i + 1, & 2m + 1 \leq i \leq 4m - 1. \end{cases}$$

$$f^*(u_1 u_{4m}) = 4m + 1 \text{ and}$$

$$f^*(v_i v_{i+1}) = 8m + 2i - 1, \quad 1 \leq i \leq n - 1.$$

Thus  $f$  is an odd sum labeling of  $P_n(C_p)$ .



**Figure 9:** An odd sum labeling of  $P_5(C_8)$ .

**Case 2**  $p \equiv 2(\text{mod } 4)$ .

Let  $p = 4m + 2, m \geq 1$ .

**Subcase 2.1**  $n \equiv 0(\text{mod } 3)$ .

The labeling  $f : V(P_n(C_p)) \rightarrow \{0, 1, 2, \dots, 4m + n + 1\}$  is defined as follows.

$$f(u_1) = 4m + 3,$$

$$f(u_i) = \begin{cases} i - 2, & 1 \leq i \leq 2m + 3, \\ i, & 2m + 4 \leq i \leq 4m + 2 \text{ and } i \text{ is even,} \\ i - 2, & 2m + 4 \leq i \leq 4m + 2 \text{ and } i \text{ is odd and} \end{cases}$$

$$f(v_i) = \begin{cases} 4m + i + 1, & 1 \leq i \leq n - 3 \text{ and } i \equiv 1(\text{mod } 3), \\ 4m + i - 1, & 1 \leq i \leq n - 1 \text{ and } i \equiv 2(\text{mod } 3), \\ 4m + i + 3, & 1 \leq i \leq n - 1 \text{ and } i \equiv 0(\text{mod } 3), \\ 4m + n + 1, & i = n - 2, \\ 4m + n - 1, & i = n. \end{cases}$$

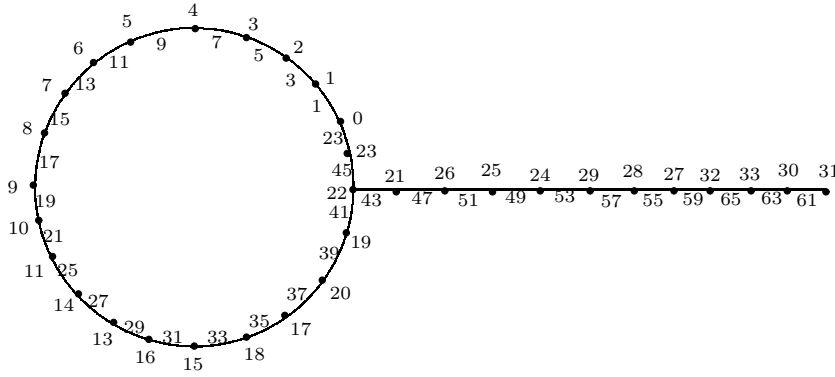
The induced edge labels are obtained as follows.

$$f^*(u_i u_{i+1}) = \begin{cases} 4m+3, & i=1, \\ 2i-3, & 2 \leq i \leq 2m+2, \\ 2i-1, & 2m+3 \leq i \leq 4m+1. \end{cases}$$

$$f^*(u_{4m+2} u_1) = 8m+5 \text{ and}$$

$$f^*(v_i v_{i+1}) = \begin{cases} 8m+2i+3, & 2 \leq i \leq n-4 \text{ and } i \equiv 2(\text{mod } 3), \\ 8m+2i+5, & 2 \leq i \leq n-4 \text{ and } i \equiv 0(\text{mod } 3), \\ 8m+2i+1, & 2 \leq i \leq n-4 \text{ and } i \equiv 1(\text{mod } 3), \\ 8m+4n-2i-5, & n-3 \leq i \leq n-1. \end{cases}$$

Thus  $f$  is an odd sum labeling of  $P_n(C_p)$ .



**Figure 10:** An odd sum labeling of  $P_{12}(C_{22})$ .

**Subcase 2.2**  $n \equiv 2(\text{mod } 3)$ .

The labeling  $f : V(P_n(C_p)) \rightarrow \{0, 1, 2, \dots, 4m+n+1\}$  is defined as follows.

$$f(u_1) = 4m+3,$$

$$f(u_i) = \begin{cases} i-2, & 1 \leq i \leq 2m+3, \\ i, & 2m+4 \leq i \leq 4m+2 \text{ and } i \text{ is even,} \\ i-2, & 2m+4 \leq i \leq 4m+2 \text{ and } i \text{ is odd,} \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 4m+i+1, & 1 \leq i \leq n \text{ and } i \equiv 1(\text{mod } 3), \\ 4m+i-1, & 1 \leq i \leq n \text{ and } i \equiv 2(\text{mod } 3), \\ 4m+i+3, & 1 \leq i \leq n \text{ and } i \equiv 0(\text{mod } 3). \end{cases}$$



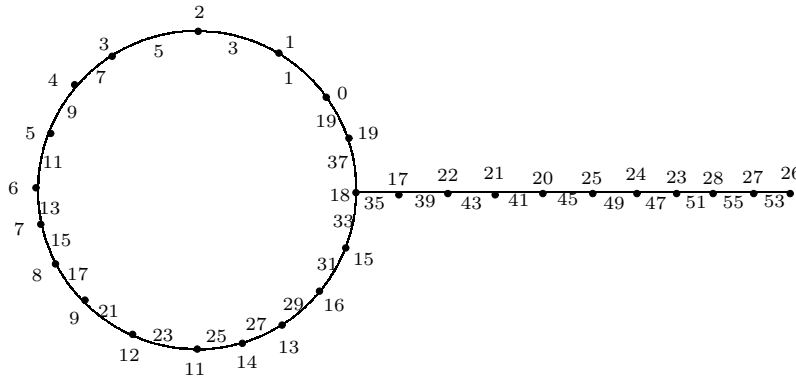
The induced edge labels are obtained as follows.

$$f^*(u_i u_{i+1}) = \begin{cases} 4m+3, & i=1, \\ 2i-3, & 2 \leq i \leq 2m+2, \\ 2i-1, & 2m+3 \leq i \leq 4m+1, \end{cases}$$

$$f^*(u_{4m+2} u_1) = 8m+5 \text{ and}$$

$$f^*(v_i v_{i+1}) = \begin{cases} 8m+2i+1, & 1 \leq i \leq n \text{ and } i \equiv 1(\text{mod } 3), \\ 8m+2i+3, & 1 \leq i \leq n \text{ and } i \equiv 2(\text{mod } 3), \\ 8m+2i+5, & 1 \leq i \leq n \text{ and } i \equiv 0(\text{mod } 3). \end{cases}$$

Thus  $f$  is an odd sum labeling of  $P_n(C_p)$ . Hence  $P_n(C_p)$  is an odd sum graph.  $\square$



**Figure 11:** An odd sum labeling of  $P_{11}(C_{18})$ .

**Proposition 2.8**  $[P_m; C_n]$  is an odd sum graph for  $n \equiv 0(\text{mod } 4)$  and any  $m \geq 2$ .

*Proof* In  $[P_m; C_n]$ , let  $v_1, v_2, \dots, v_m$  be the vertices on the path  $P_m$ ,  $v_{i,1}, v_{i,2}, \dots, v_{i,n}$  be the vertices of the  $i^{\text{th}}$  cycle  $C_n$ , for  $1 \leq i \leq m$  and each vertex  $v_{i,1}$  of the  $i^{\text{th}}$  cycle  $C_n$  is identified with the vertex  $v_i$  of the path  $P_m$ ,  $1 \leq i \leq m$ .

Suppose  $n = 4t, t \geq 1$ . The labeling  $f : V([P_m; C_n]) \rightarrow \{0, 1, 2, 3, \dots, m(n+1) - 1\}$  is defined as follows.

For  $1 \leq i \leq m$ ,

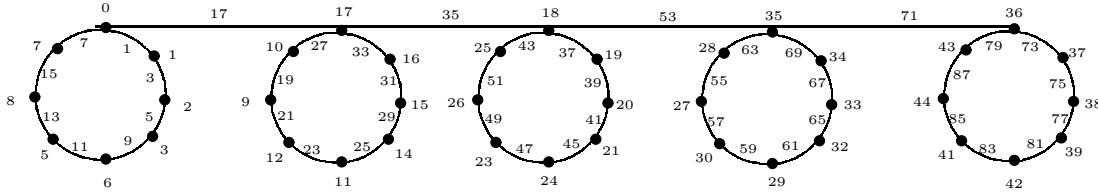
$$f(v_{i,j}) = \begin{cases} (n+1)(i-1) + j - 1, & 1 \leq j \leq 2t, i \text{ and } j \text{ are odd,} \\ (n+1)(i-1) + j + 1, & 2t+1 \leq j \leq 4t, i \text{ and } j \text{ are odd,} \\ (n+1)(i-1) + j - 1, & 1 \leq j \leq 4t, i \text{ is odd and } j \text{ are even,} \\ (n+1)i - j, & 1 \leq j \leq 2t, i \text{ is even and } j \text{ is odd,} \\ (n+1)i - j - 2, & 2t+1 \leq j \leq 4t, i \text{ is even and } j \text{ is odd,} \\ (n+1)i - j, & 1 \leq j \leq 4t, i \text{ is even and } j \text{ is even.} \end{cases}$$

For  $1 \leq i \leq m$ , the induced edge label is obtained as follows.

$$f^*(v_{i,j}v_{i,j+1}) = \begin{cases} 2(n+1)(i-1) + 2j - 1, & 1 \leq j \leq 2t - 1 \text{ and } i \text{ is odd,} \\ 2(n+1)(i-1) + 2j + 1, & 2t \leq j \leq 4t - 1 \text{ and } i \text{ is odd,} \\ 2(n+1)(i-1) + 9, & j = 1 \text{ and } i \text{ is even,} \\ 2(n+1)(i-1) + 2j - 3, & 2 \leq j \leq 2t + 1 \text{ and } i \text{ is even,} \\ 2(n+1)(i-1) + 2j - 1, & 2t + 2 \leq j \leq 4t - 1 \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f^*(v_{i,4t}v_{i,1}) = \begin{cases} 2(n+1)(i-1) + 4t - 1, & i \text{ is odd,} \\ 2(n+1)(i-1) + 8t - 1, & i \text{ is even.} \end{cases}$$

Thus  $f$  is an odd sum labeling of  $[P_m; C_n]$ . Hence  $[P_m; C_n]$  is an odd sum graph.  $\square$



**Figure 12:** An odd sum labeling of  $[P_5; C_8]$ .

**Proposition 2.9** *Quadrilateral snake  $Q_n$  is an odd sum graph for  $n \geq 1$ .*

*Proof* The vertex set and edge set of the Quadrilateral snake  $Q_n$  are  $V(Q_n) = \{u_i, v_j, w_j : 1 \leq i \leq n+1, 1 \leq j \leq n\}$  and  $E(Q_n) = \{u_i v_i, v_i w_i, u_i u_{i+1}, u_{i+1} w_i : 1 \leq i \leq n\}$  respectively. The labeling  $f : V(Q_n) \rightarrow \{0, 1, 2, \dots, 4n\}$  is defined as follows.

$$f(u_i) = \begin{cases} 4i - 4, & 1 \leq i \leq n+1 \text{ and } i \text{ is odd,} \\ 4i - 5, & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_i) = \begin{cases} 4i - 3, & 1 \leq i \leq n \text{ and } i \text{ is odd,} \\ 4i - 2, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f(w_i) = \begin{cases} 4i, & 1 \leq i \leq n \text{ and } i \text{ is odd,} \\ 4i - 1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows

$$f^*(u_i u_{i+1}) = 8i - 5, \quad 1 \leq i \leq n,$$

$$f^*(u_i v_i) = 8i - 7, \quad 1 \leq i \leq n,$$

$$f^*(v_i w_i) = 8i - 3, \quad 1 \leq i \leq n,$$

$$f^*(w_i u_{i+1}) = 8i - 1, \quad 1 \leq i \leq n.$$



For  $1 \leq i \leq m-1$ ,

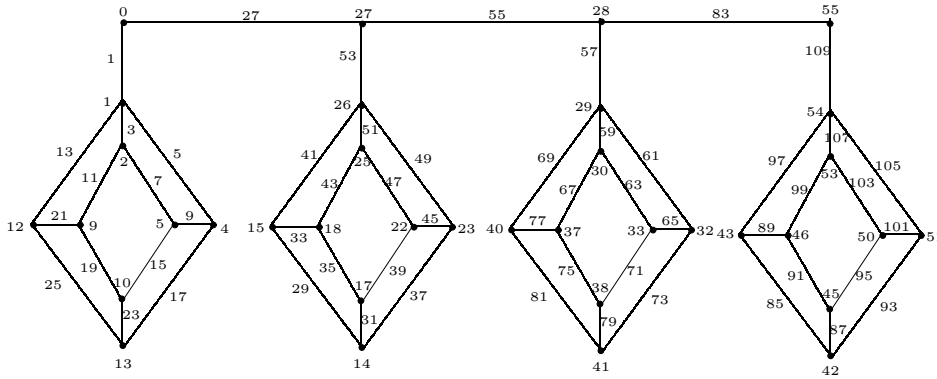
$$f^*(u_i u_{i+1}) = 28i - 1.$$

For  $1 \leq i \leq m$ ,

$$f^*(u_i v_{i,1}) = \begin{cases} 28i - 27, & i \text{ is odd,} \\ 28i - 3, & i \text{ is even.} \end{cases}$$

For $1 \leq i \leq m$ and $i$ is odd	For $1 \leq i \leq m$ and $i$ is even
$f^*(v_{i,1} v_{i,2}) = 28i - 23$	$f^*(v_{i,1} v_{i,2}) = 28i - 7$
$f^*(v_{i,1} v_{i,4}) = 28i - 25,$	$f^*(v_{i,1} v_{i,4}) = 28i - 5$
$f^*(v_{i,1} v_{i,6}) = 28i - 15$	$f^*(v_{i,1} v_{i,6}) = 28i - 15$
$f^*(v_{i,2} v_{i,3}) = 28i - 19$	$f^*(v_{i,2} v_{i,3}) = 28i - 11$
$f^*(v_{i,2} v_{i,7}) = 28i - 11$	$f^*(v_{i,2} v_{i,7}) = 28i - 19$
$f^*(v_{i,3} v_{i,4}) = 28i - 21$	$f^*(v_{i,3} v_{i,4}) = 28i - 9$
$f^*(v_{i,3} v_{i,8}) = 28i - 13$	$f^*(v_{i,3} v_{i,8}) = 28i - 17$
$f^*(v_{i,4} v_{i,5}) = 28i - 17$	$f^*(v_{i,4} v_{i,5}) = 28i - 13$
$f^*(v_{i,5} v_{i,6}) = 28i - 7$	$f^*(v_{i,5} v_{i,6}) = 28i - 23$
$f^*(v_{i,5} v_{i,8}) = 28i - 9$	$f^*(v_{i,5} v_{i,8}) = 28i - 21,$
$f^*(v_{i,6} v_{i,7}) = 28i - 3$	$f^*(v_{i,6} v_{i,7}) = 28i - 27$
$f^*(v_{i,7} v_{i,8}) = 28i - 5$	$f^*(v_{i,7} v_{i,8}) = 28i - 25$

Thus  $f$  is an odd sum labeling of  $(P_m; Q_3)$ . Hence  $(P_m; Q_3)$  is an odd sum graph.  $\square$



**Figure 14:** An odd sum labeling of  $(P_4; Q_3)$ .

**Proposition 2.11** For all positive integers  $p$  and  $n$ , the graph  $T_p^{(n)}$  is an odd sum graph.

*Proof* Let  $v_i^{(j)}$ ,  $1 \leq i \leq p$  be the vertices of the  $j^{th}$  copy of the path on  $p$  vertices,  $1 \leq j \leq n$ . The graph  $T_p^{(n)}$  is formed by adding an edge  $v_i^{(j)} v_i^{(j+1)}$  between  $j^{th}$  and  $(j+1)^{th}$  copy of the path at some  $i$ ,  $1 \leq i \leq p$ . The labeling  $f : V(G) \rightarrow \{0, 1, 2, \dots, np - 1\}$  is defined as follows:

For  $1 \leq j \leq n$  and  $1 \leq i \leq p$ ,

$$f(v_i^{(j)}) = \begin{cases} p(j-1) + i - 1, & j \text{ is odd,} \\ pj - i, & j \text{ is even.} \end{cases}$$

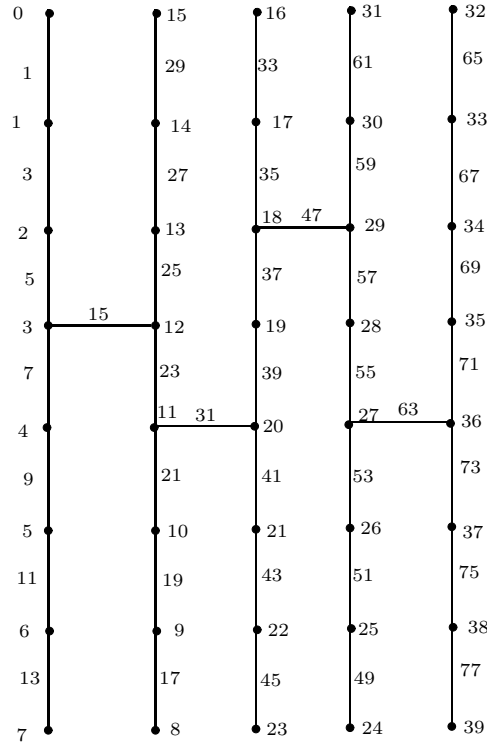
The induced edge labeling is obtained as follows:

For  $1 \leq j \leq n$  and  $1 \leq i \leq p-1$ ,

$$f^*(v_i^{(j)} v_{i+1}^{(j)}) = \begin{cases} 2p(j-1) + 2i - 1, & j \text{ is odd,} \\ 2pj - 2i - 1, & j \text{ is even} \end{cases} \text{ and}$$

$$f^*(v_i^{(j)} v_i^{(j+1)}) = 2pj - 1.$$

Thus  $f$  is an odd sum labeling of the graph  $T_p^{(n)}$ . Hence  $T_p^{(n)}$  is an odd sum graph.  $\square$



**Figure 15:** An odd sum labeling of  $T_8^{(5)}$ .

**Proposition 2.12** *The graph  $H_n \odot mK_1$  is an odd sum graph for all positive integers  $m$  and  $n$ .*

*Proof* Let  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  be the vertices on the path of length  $n-1$ . Let  $x_{i,k}$  and  $y_{i,k}$ ,  $1 \leq k \leq m$ , be the pendant vertices at  $u_i$  and  $v_i$  respectively, for  $1 \leq i \leq n$ . Define  $f : V(H_n \odot mK_1) \rightarrow \{0, 1, 2, \dots, 2n(m+1) - 1\}$  as follows:

For  $1 \leq i \leq n$ ,

$$f(u_i) = \begin{cases} i + m(i - 1), & i \text{ is odd,} \\ i(m + 1) - 2, & i \text{ is even} \end{cases} \quad \text{and}$$

$$f(v_i) = \begin{cases} f(u_i) + n(m + 1) + m - 2, & i \text{ is odd and } n \text{ is odd,} \\ f(u_i) + n(m + 1) - m + 2, & i \text{ is even and } n \text{ is odd,} \\ f(u_i) + n(m + 1), & n \text{ is even.} \end{cases}$$

For  $1 \leq i \leq n$  and  $1 \leq k \leq m$ ,

$$f(x_{i,k}) = \begin{cases} (m + 1)(i - 1) + 2k - 2, & i \text{ is odd,} \\ (m + 1)(i - 2) + 2k + 1, & i \text{ is even} \end{cases} \quad \text{and}$$

$$f(y_{i,k}) = \begin{cases} f(x_{i,k}) + n(m + 1) - m + 2, & i \text{ is odd and } n \text{ is odd,} \\ f(x_{i,k}) + n(m + 1) + m - 2, & i \text{ is even and } n \text{ is odd,} \\ f(x_{i,k}) + n(m + 1), & n \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows:

For  $1 \leq i \leq n - 1$ ,

$$f^*(u_i u_{i+1}) = 2i(m + 1) - 1 \quad \text{and}$$

$$f^*(v_i v_{i+1}) = f^*(u_i u_{i+1}) + 2n(m + 1).$$

For  $1 \leq i \leq n$  and  $1 \leq k \leq m$ ,

$$f^*(u_i x_{i,k}) = 2(m + 1)(i - 1) + 2k - 1 \quad \text{and}$$

$$f^*(v_i y_{i,k}) = f^*(u_i x_{i,k}) + 2n(m + 1).$$

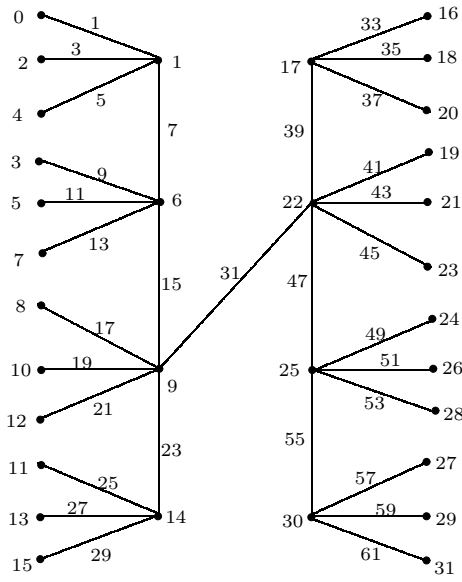
When  $n$  is odd,

$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = 2n(m + 1) - 1.$$

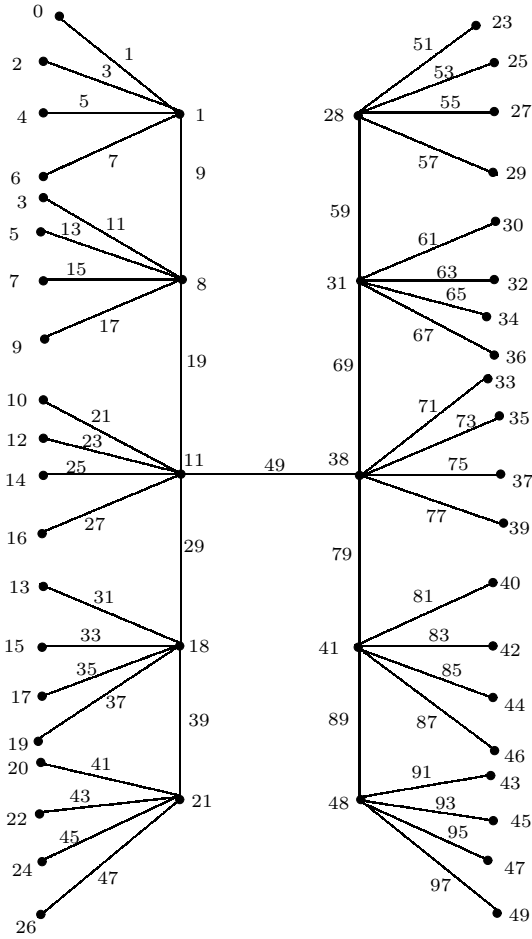
When  $n$  is even,

$$f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = 2n(m + 1) - 1.$$

Thus  $f$  is an odd sum labeling of  $H_n \odot mK_1$ . Hence  $H_n \odot mK_1$  is an odd sum graph for all positive integers  $m$  and  $n$ .  $\square$



**Figure 16:** An odd sum labeling of  $H_4 \odot 3K_1$ .



**Figure 17:** An odd sum labeling of  $H_5 \odot 4K_1$ .

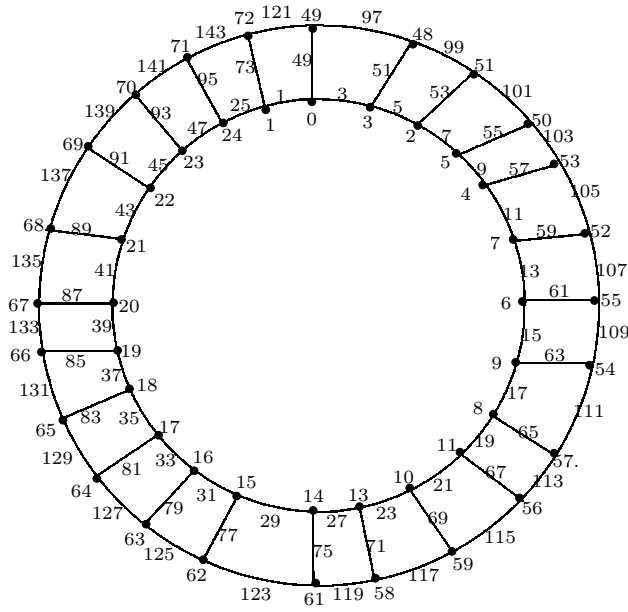
**Corollary 2.13** For any positive integer  $m$ , the bistar graph  $B(m, m)$  is an odd sum graph.

*Proof* By taking  $n = 1$  in Proposition 2.12, the result follows.  $\square$

**Proposition 2.14** For any even integer  $p \geq 4$ , the cyclic ladder  $P_2 \times C_p$  is an odd sum graph.

*Proof* Let  $u_1, u_2, \dots, u_p$  and  $v_1, v_2, \dots, v_p$  be the vertices of the inner and outer cycle which are joined by the edges  $\{u_i v_i : 1 \leq i \leq p\}$ .

**Case 1**  $p = 4m, m \geq 2$ .



**Figure 18:** An odd sum labeling of  $P_2 \times C_{24}$ .

The labeling  $f : V(P_2 \times C_p) \rightarrow \{0, 1, 2, \dots, 12m\}$  is defined as follows:

$$f(u_i) = \begin{cases} i - 1, & 1 \leq i \leq 2m - 1 \text{ and } i \text{ is odd,} \\ i + 1, & 2 \leq i \leq 4m - 2 \text{ and } i \text{ is even,} \\ i + 1, & 2m + 1 \leq i \leq 4m - 1 \text{ and } i \text{ is odd,} \end{cases}$$

$f(u_{4m}) = 1$  and

$$f(v_i) = \begin{cases} 8k + i, & 1 \leq i \leq 4m - 1 \text{ and } i \text{ is odd,} \\ 8k + i - 2, & 2 \leq i \leq 2m \text{ and } i \text{ is even,} \\ 8k + i, & 2m + 2 \leq i \leq 4m \text{ and } i \text{ is even.} \end{cases}$$



The induced edge labeling is obtained as follows.

$$f^*(u_i u_{i+1}) = \begin{cases} 2i+1, & 1 \leq i \leq 2m-1, \\ 2i+3, & 2m \leq i \leq 4m-2, \\ i+2, & i = 4m-1, \end{cases}$$

$$f^*(u_1 u_{4m}) = 1,$$

$$f^*(v_i v_{i+1}) = \begin{cases} 16m+2i-1, & 1 \leq i \leq 2m \\ 16m+2i+1, & 2m+1 \leq i \leq 4m-1, \end{cases}$$

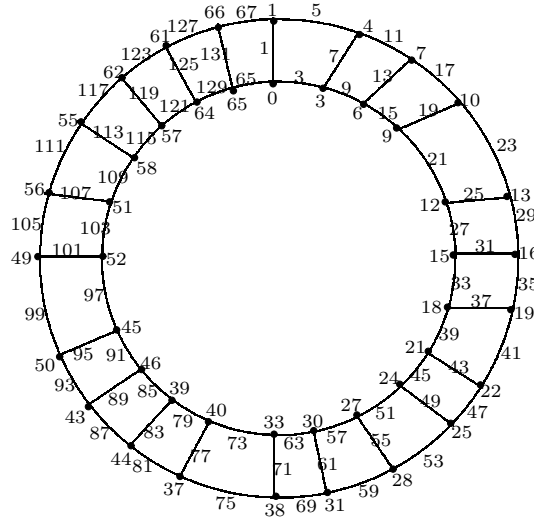
$$f^*(v_1 v_{4m}) = 20m+1,$$

$$f^*(u_i v_i) = \begin{cases} 8m+2i-1, & 1 \leq i \leq 2m, \\ 8m+2i+1, & 2m+1 \leq i \leq 4m-1 \text{ and } \end{cases}$$

$$f^*(u_{4m} v_{4m}) = 12m+1.$$

Thus  $f$  is an odd sum labeling of  $P_2 \times C_p$ . Hence  $P_2 \times C_p$  is an odd sum graph when  $p = 4m$ .

**Case 2**  $p = 4m+2, m \geq 1$ .



**Figure 19:** An odd sum labeling of  $P_2 \times C_{22}$ .

The labeling  $f : V(P_2 \times C_p) \rightarrow \{0, 1, 2, \dots, 12m\}$  is defined as follows:

$$f(u_i) = \begin{cases} 3i-3, & 1 \leq i \leq 2m+2, \\ 3i+1, & 2m+3 \leq i \leq 4m+1 \text{ and } i \text{ is odd,} \end{cases}$$

$$f(u_i) = \begin{cases} 3i - 3, & 2m + 4 \leq i \leq 4m \text{ and } i \text{ is even,} \\ 3i - 1, & i = 4m + 2 \text{ and} \end{cases}$$

$$f(v_i) = \begin{cases} 3i - 2, & 1 \leq i \leq 2m + 1, \\ 3i + 2, & 2m + 2 \leq i \leq 4m \text{ and } i \text{ is even,} \\ 3i - 2, & 2m + 3 \leq i \leq 4m + 1 \text{ and } i \text{ is odd,} \\ 3i, & i = 4m + 2. \end{cases}$$

The induced edge labels are given as

$$f^*(u_i u_{i+1}) = \begin{cases} 6i - 3, & 1 \leq i \leq 2m + 1, \\ 6i + 1, & 2m + 2 \leq i \leq 4m, \\ 6i + 3, & i = 4m + 1, \end{cases}$$

$$f^*(u_1 u_{4m+2}) = 12m + 5,$$

$$f^*(v_i v_{i+1}) = \begin{cases} 6i - 1, & 1 \leq i \leq 2m, \\ 6i + 3, & 2m + 1 \leq i \leq 4m, \\ 6i + 1, & i = 4m + 1, \end{cases}$$

$$f^*(v_1 v_{4m+2}) = 12m + 7 \text{ and}$$

$$f^*(u_i v_i) = \begin{cases} 6i - 5, & 1 \leq i \leq 2m + 1, \\ 6i - 1, & 2m + 2 \leq i \leq 4m + 2. \end{cases}$$

Thus  $f$  is an odd sum labeling of  $P_2 \times C_p$ . Whence  $P_2 \times C_p$  is an odd sum graph if  $p = 4m + 2$ .  $\square$

## References

- [1] F.Buckley and F.Harary, *Distance in Graphs*, Addison-Wesley, Reading, 1990.
- [2] R.Balakrishnan, A.Selvam and V.Yegnanarayanan, On felicitous labelings of graphs, *Graph Theory and its Applications*, (1996), 47–61.
- [3] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, **17** (2011), # DS6.
- [4] K.Manickam and M.Marudai, Odd mean labelings of graphs, *Bulletin of Pure and Applied Sciences*, **25E**(1) (2006), 149–153.
- [5] R.Ponraj, J.Vijaya Xavier Parthipan and R.Kala, Some results on pair sum labeling of graphs, *International Journal of Mathematical Combinatorics*, **4** (2010), 53–61.
- [6] S.Somasundaram and R.Ponraj, Mean labelings of graphs, *National Academy Science Letter*, **26** (2003), 210–213.
- [7] Selvam Avadayappan and R. Vasuki, Some results on mean graphs, *Ultra Scientist of Physical Sciences*, **21**(1)M (2009), 273–284.

## **Controllability of Fractional Stochastic Differential Equations With State-Dependent Delay**

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**Abstract:** In this paper, the approximate controllability for a class of nonlinear fractional stochastic differential equations with state-dependent delays in Hilbert space is studied. The result is extended to study the approximate controllability of fractional stochastic systems with state-dependent delays and resolvent operators. A set of sufficient conditions are established to obtain the required result by employing semigroup theory, fixed point technique and fractional calculus. In particular, the approximate controllability of nonlinear fractional stochastic control systems is established under the assumption that the corresponding linear control system is approximately controllable. Also, an example is presented to illustrate the applicability of the obtained theory.

**Key Words:** Approximate controllability, stochastic fractional differential equations, fixed point technique, state-dependent delay.

**AMS(2010):** 34A08, 93B05, 47H10

### **§1. Introduction**

Controllability is one of the important fundamental concepts in mathematical control theory and plays a vital role in both deterministic and stochastic control systems. In recent years, various controllability problems for different kinds of dynamical systems have been studied in many publications [1, 8, 9, 18, 19].. From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. However, the concept of exact controllability is usually too strong and has limited applicability. Approximate controllability is a weaker concept than complete controllability and it is completely adequate in applications [7, 23].. Recently, Wang [32] derived a set of sufficient conditions for the approximate controllability of differential equations with multiple delays by implementing some natural conditions such as growth conditions for the nonlinear term and compactness of the semigroup. Sakthivel and Anandhi [29] investigated the problem of approximate controllability for a class of nonlinear impulsive differential equations with state-dependent delay by using semigroup theory and fixed

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point technique.

On the other hand, the theory of fractional differential equations is emerging as an important area of investigation since it is richer in problems in comparison with corresponding theory of classical differential equations [17, 25, 26]. In fact, such models can be considered as an efficient alternative to the classical nonlinear differential models to simulate many complex processes. Recently, it has been proved that the differential models involving derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, physics, chemistry, aerodynamics, electrodynamics of complex medium, and so on [15, 17]. In recent years, existence results for fractional differential equations have been investigated in several papers [5,3]. More recently, Dabas and Chauhan studied the existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay.

In particular, the study of stochastic fractional differential equations has attracted great interest due to its applications in characterizing many problems in biology, electrical engineering and other areas of science. Many physical phenomena in evolution processes are modeled as stochastic fractional differential equations and existence results for such equations have been studied by several authors [4, 11, 30].

Most of the existing literature on controllability results for linear and nonlinear stochastic systems are without fractional derivatives [20, 21, 22, 24]. Only few papers deal with the controllability of fractional stochastic systems. Guendouzi and Hamada [12] studied the relative controllability of fractional stochastic dynamical systems with multiple delays in control. The authors derive a new set of sufficient conditions for the global relative controllability by fixed point technique and controllability Grammian matrix. Sakthivel et al. [28] discussed the approximate controllability of nonlinear fractional stochastic control system under the assumptions that the corresponding linear system is approximately controllable. Guendouzi and Idrissi [13] investigated the problem of approximate controllability for a class of dynamic control systems described by nonlinear fractional stochastic functional differential equations in Hilbert space driven by a fractional Brownian motion with Hurst parameter  $H > 1/2$ . Sakthivel et al. [27] studied the approximate controllability of neutral stochastic fractional integro-differential equation with infinite delay in a Hilbert space.

However, to the best of our knowledge, the approximate controllability problem for nonlinear fractional stochastic systems with state-dependent delay has not been investigated yet. Motivated by this consideration, in this paper we will study the approximate controllability problem for nonlinear fractional stochastic system, described by nonlinear fractional stochastic differential equations with state-dependent delay and control in Hilbert space, under the assumption that the associated linear system is approximately controllable. In fact, the results in this paper are motivated by the recent work of [29] and the fractional differential equations discussed in [6,28].

## §2. Preliminaries and Basic Properties

Let  $\mathcal{H}, \mathcal{K}$  be two separable Hilbert spaces and  $\mathcal{L}(\mathcal{K}, \mathcal{H})$  be the space of bounded linear operators

from  $\mathcal{K}$  into  $\mathcal{H}$ . For convenience, we will use the notation  $\|\cdot\|_{\mathcal{H}}$ ,  $\|\cdot\|_{\mathcal{K}}$  and  $\|\cdot\|$  to denote the norms in  $\mathcal{H}$ ,  $\mathcal{K}$  and  $\mathcal{L}(\mathcal{K}, \mathcal{H})$  respectively, and use  $(\cdot, \cdot)$  to denote the inner product of  $\mathcal{H}$  and  $\mathcal{K}$  without any confusion. Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$  be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and  $\mathcal{F}_0$  contains all  $\mathbf{P}$ -null sets.  $w = (w_t)_{t \geq 0}$  be a  $Q$ -Wiener process defined on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$  with the covariance operator  $Q$  such that  $\text{tr}Q < \infty$ . We assume that there exists a complete orthonormal system  $\{e_k\}_{k \geq 1}$  in  $\mathcal{K}$ , a bounded sequence of nonnegative real numbers  $\lambda_k$  such that  $Qe_k = \lambda_k e_k$ ,  $k = 1, 2, \dots$  and a sequence  $\{\beta_k\}_{k \geq 1}$  of independent Brownian motions such that

$$(w(t), e)_{\mathcal{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k} (e_k, e)_{\mathcal{K}} \beta_k(t), \quad e \in \mathcal{K}, t \in J := [0, b].$$

Let  $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{1/2}\mathcal{K}, \mathcal{H})$  be the space of all Hilbert-Schmidt operators from  $Q^{1/2}\mathcal{K}$  into  $\mathcal{H}$  with the inner product  $\langle \psi, \pi \rangle_{\mathcal{L}_2^0} = \text{tr}[\psi Q \pi^*]$ . Let  $\mathcal{L}^2(\mathcal{F}_b, \mathcal{H})$  be the Banach space of all  $\mathcal{F}_b$ -measurable square integrable random variables with values in the Hilbert space  $\mathcal{H}$ , and  $\mathbf{E}(\cdot)$  denote the expectation with respect to the measure  $\mathbb{P}$ .

The purpose of this paper is to investigate the approximate controllability for a class of nonlinear fractional stochastic differential equation with state-dependent delay and control of the form

$$\begin{aligned} {}^c D_t^\alpha [x(t) + g(t, x_{\varepsilon(t, x_t)})] &= A[x(t) + g(t, x_{\varepsilon(t, x_t)})] + Bu(t) + f(t, x_{\varepsilon(t, x_t)}) \\ &\quad + \sigma(t, x_{\varepsilon(t, x_t)}) \frac{dw(t)}{dt}, \quad t \in J \\ x_0 &= \phi \in \mathcal{B}, \end{aligned} \tag{2.1}$$

where  ${}^c D_t^\alpha$  is the Caputo fractional derivative of order  $\alpha$ ,  $0 < \alpha < 1$ ;  $x(\cdot)$  takes the value in the separable Hilbert space  $\mathcal{H}$ ;  $A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$  is the infinitesimal generator of an  $\alpha$ -resolvent family  $S_\alpha(t)_{t \geq 0}$ ; the control function  $u(\cdot)$  is given in  $\mathcal{L}_{\mathcal{F}}^2([0, b], \mathcal{U})$  of admissible control functions,  $\mathcal{U}$  is a Hilbert space.  $B$  is a bounded linear operator from  $\mathcal{U}$  into  $\mathcal{H}$ . The history  $x_t : (-\infty, 0] \rightarrow \mathcal{H}$ ,  $x_t(\theta) = x(t + \theta)$ ,  $\theta \leq 0$ , belongs to an abstract phase space  $\mathcal{B}$  defined axiomatically;  $g : J \times \mathcal{B} \rightarrow \mathcal{H}$ ,  $f : J \times \mathcal{B} \rightarrow \mathcal{H}$ ,  $\sigma : J \times \mathcal{B} \rightarrow \mathcal{L}_2^0$  and  $\varepsilon : J \times \mathcal{B} \rightarrow (-\infty, b]$  are appropriate functions to be specified later.

Let us recall the following known definitions. For more details see [17].

**Definition 2.1** *The fractional integral of order  $\alpha$  with the lower limit 0 for a function  $f$  is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0$$

*provided the right-hand side is pointwise defined on  $[0, \infty)$ , where  $\Gamma$  is the gamma function.*

**Definition 2.2** *Riemann-Liouville derivative of order  $\alpha$  with lower limit zero for a function  $f : [0, \infty) \rightarrow \mathbb{R}$  can be written as*

$${}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, n-1 < \alpha < n. \tag{2.2}$$

**Definition 2.3** *The Caputo derivative of order  $\alpha$  for a function  $f : [0, \infty) \rightarrow \mathbb{R}$  can be written*

as

$${}^c D^\alpha f(t) = {}^L D^\alpha \left( f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n-1 < \alpha < n. \quad (2.3)$$

If  $f(t) \in C^n[0, \infty)$ , then

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I^{n-\alpha} f^{(n)}(s), \quad t > 0, n-1 < \alpha < n.$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order  $\alpha > 0$  is given as

$$L\{{}^c D^\alpha f(t); s\} = s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \quad n-1 \leq \alpha < n.$$

**Definition 2.4** A two parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \quad \alpha, \beta \in \mathbb{C}, \mathcal{R}(\alpha) > 0,$$

where  $C$  is a contour which starts and ends at  $-\infty$  and encircles the disc  $|\mu| \leq |z|^{1/2}$  counter clockwise.

For short,  $E_\alpha(z) = E_{\alpha,1}(z)$ . It is an entire function which provides a simple generalization of the exponent function:  $E_1(z) = e^z$  and the cosine function:  $E_2(z^2) = \cosh(z)$ ,  $E_2(-z^2) = \cos(z)$ , and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha, \beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad \operatorname{Re} \lambda > \omega^{\frac{1}{\alpha}}, \omega > 0,$$

and for more details see [17].

**Definition 2.5** ([31]) A closed and linear operator  $A$  is said to be sectorial if there are constants  $\omega \in \mathbb{R}$ ,  $\theta \in [\frac{\pi}{2}, \pi]$ ,  $M > 0$ , such that the following two conditions are satisfied:

- $\rho(A) \subset \Sigma_{\theta, \omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\};$
- $\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \lambda \in \Sigma_{\theta, \omega}.$

**Definition 2.6** Let  $A$  be a closed and linear operator with the domain  $D(A)$  defined in a Banach space  $X$ . Let  $\rho(A)$  be the resolvent set of  $A$ . We say that  $A$  is the generator of an  $\alpha$ -resolvent family if there exist  $\omega \geq 0$  and a strongly continuous function  $S_\alpha : \mathbb{R}_+ \rightarrow L(X)$ , where  $L(X)$  is a Banach space of all bounded linear operators from  $X$  into  $X$  and the corresponding norm is denoted by  $\|\cdot\|$ , such that  $\{\lambda^\alpha : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$  and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{\lambda t} S_\alpha(t) x dt, \quad \operatorname{Re} \lambda > \omega, x \in X, \quad (2.4)$$

where  $S_\alpha(t)$  is called the  $\alpha$ -resolvent family generated by  $A$ .

**Definition 2.7** Let  $A$  be a closed and linear operator with the domain  $D(A)$  defined in a Banach space  $X$  and  $\alpha > 0$ . We say that  $A$  is the generator of a solution operator if there exist  $\omega \geq 0$  and a strongly continuous function  $S_\alpha : \mathbb{R}_+ \rightarrow L(X)$  such that  $\{\lambda^\alpha : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$  and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t) x dt, \quad \operatorname{Re} \lambda > \omega, x \in X, \quad (2.5)$$

where  $S_\alpha(t)$  is called the solution operator generated by  $A$ .

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on  $\alpha$ -resolvent family and solution operators, we refer the reader to [17].

In this paper, we assume that the operator  $A$  is sectorial of type  $\omega$  with  $\pi(1 - \alpha/2) < \theta < \pi$ . Then  $A$  is the generator of a solution operator given by

$$T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i} \int_{\widehat{B}_r} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda,$$

and the operator

$$S_\alpha = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i} \int_{\widehat{B}_r} e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda$$

is the  $\alpha$ -resolvent family generated by  $A$ , where  $\widehat{B}_r$  denotes the Bromwich path (see [6,31]).

Recently, it has been proven in [31] that if  $\alpha \in (0,1)$  and  $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$  is a sectorial operator, then for any  $x$  in a Banach space  $X$  and  $t > 0$ , we have

$$\|S_\alpha(t)\| \leq C e^{\omega t} (1 + t^{\alpha-1}), \quad t > 0, \quad \omega > \omega_0,$$

where  $C > 0$  depending solely on  $\theta$  and  $\alpha$ .

In this work, we will employ an axiomatic definition of the abstract phase space  $\mathcal{B}$  introduced by Hale and Kato [14]. We first define  $\widetilde{\mathcal{H}}((-\infty, b]; \mathcal{H})$  the Banach space of all continuous and  $\mathcal{F}_b$ -measurable  $\mathcal{H}$ -valued function  $x$ .

**Axiom 2.8**  $\mathcal{B}$  is a linear space that denotes the family of  $\mathcal{F}_0$ -measurable function from  $(-\infty, 0]$  into  $\mathcal{H}$ , endowed with norm  $\|\cdot\|_{\mathcal{B}}$ , which satisfies the following axioms:

(1) If  $x \in \widetilde{\mathcal{H}}$  is continuous on  $[0, b]$ ,  $b > 0$ , and  $x_0 \in \mathcal{B}$ , then for every  $t \in [0, b]$  the following conditions hold

(1-1)  $x_t \in \mathcal{B}$ ;

(1-2)  $\|x(t)\|_{\widetilde{\mathcal{H}}} \leq \delta \|x_t\|_{\mathcal{B}}$ ;

(1-3)  $\|x_t\|_{\mathcal{B}} \leq \mu(t) \sup_{0 \leq s \leq t} \|x(s)\|_{\widetilde{\mathcal{H}}} + \nu(t) \|x_0\|_{\mathcal{B}}$ , where  $\delta > 0$  is a constant;  $\mu, \nu : [0, \infty) \rightarrow [1, \infty)$ ,  $\mu$  is continuous,  $\nu$  is locally bounded;  $\delta, \mu$  and  $\nu$  are independent of  $x(\cdot)$ .

(2) For the function  $x(\cdot)$  in i.,  $x_t$  is a  $\mathcal{B}$ -valued continuous functions on  $[0, b]$ ;

(3) The space  $\mathcal{B}$  is complete.

Let  $x_b(x_0; u)$  be the state of (2.1) at terminal time  $b$  corresponding to the control  $u$  and the initial value  $x_0 = \phi \in \mathcal{B}$ . Introduce the set  $\mathcal{R}(b, \phi) = \{x_b(\phi; u)(0) : u(\cdot) \in \mathcal{L}_{\mathcal{F}}^2([0, b], \mathcal{U})\}$ , which is called the reachable set of system (1) at terminal time  $b$  and its closure in  $\mathcal{H}$  is denoted by  $\overline{\mathcal{R}(b, \phi)}$ .

**Definition 2.9** *The system (2.1) is said to be approximately controllable on  $J$  if  $\overline{\mathcal{R}(b, \phi)} = \mathcal{H}$ , that is, given an arbitrary  $\epsilon > 0$  it is possible to steer from the point  $\phi$  to within a distance  $\epsilon$  from all points in the state space  $\mathcal{H}$  at time  $b$ .*

In order to study the approximate controllability for the fractional control system (2.1), we introduce the approximate controllability of its linear part

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + (Bu)(t), & t \in J, \\ x(0) &= \phi \in \mathcal{B}. \end{aligned} \quad (2.6)$$

The approximate controllability for linear fractional control system (2.6) is a natural generalization of approximate controllability of linear first order control system [23]. It is convenient at this point to introduce the controllability operator associated with (2.6) as

$$\begin{aligned} \Gamma_0^b &= \int_0^b S_\alpha(b-s)BB^*S_\alpha^*(b-s)ds, \\ R(\kappa, \Gamma_0^b) &= (\kappa I + \Gamma_0^b)^{-1}, \quad \kappa > 0, \end{aligned}$$

where  $B^*$  denotes the adjoint of  $B$  and  $S_\alpha^*(t)$  is the adjoint of  $S_\alpha(t)$ . It is straightforward that the operator  $\Gamma_0^b$  is a linear bounded operator.

**Lemma 2.10**([23]) *The linear fractional control system (2.6) is approximately controllable on  $[0, b]$  if and only if  $\kappa R(\kappa, \Gamma_0^b) \rightarrow 0$  as  $\kappa \rightarrow 0^+$  in the strong operator topology.*

In order to establish the result, we need the following assumptions:

**(H1)** If  $\alpha \in (0, 1)$  and  $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$ , then for  $x \in \mathcal{H}$  and  $t > 0$  we have  $\|T_\alpha(t)\| \leq Me^{\omega t}$  and  $\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1})$ ,  $\omega > \omega_0$ . Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \quad \text{and} \quad \|S_\alpha(t)\| \leq t^{\alpha-1} \widetilde{M}_S,$$

where  $\widetilde{M}_T = \sup_{0 \leq t \leq b} \|T_\alpha(t)\|$ , and  $\widetilde{M}_S = \sup_{0 \leq t \leq b} Ce^{\omega t}(1 + t^{1-\alpha})$  (for more details, see [31]).

**(H2)** The function  $t \rightarrow \phi_t$  is well defined and continuous from the set  $\mathcal{Z}(\varepsilon^-) = \{\varepsilon(s, \tau) : (s, \tau) \in J \times \mathcal{B}, \varepsilon(s, \tau) \leq 0\}$  into  $\mathcal{B}$  and there exists a continuous and bounded function  $\varphi^\phi : \mathcal{Z}(\varepsilon^-) \rightarrow (0, \infty)$  such that, for every  $t \in \mathcal{Z}(\varepsilon^-)$

$$\|\phi_t\|_{\mathcal{B}} \leq \varphi^\phi(t) \|\phi\|_{\mathcal{B}}.$$

**(H3)** The function  $g : J \times \mathcal{B} \rightarrow \mathcal{H}$  is continuous and there exists some constant  $M_g > 0$  such that

$$\begin{aligned} \mathbf{E}\|g(t, \xi)\|_{\mathcal{H}}^2 &\leq M_g \left( \|\xi\|_{\mathcal{B}}^2 + 1 \right), & \xi \in \mathcal{B}, \\ \mathbf{E}\|g(t_2, \xi_2) - g(t_1, \xi_1)\|_{\mathcal{H}}^2 &\leq M_g \left( |t_2 - t_1|^2 + \|\xi_2 - \xi_1\|_{\mathcal{B}}^2 \right), & \xi_i \in \mathcal{B}, \quad i = 1, 2. \end{aligned}$$

**(H4)** The function  $f : J \times \mathcal{B} \rightarrow \mathcal{H}$  satisfies the following properties:

(1)  $f(t, \cdot) : \mathcal{B} \rightarrow \mathcal{H}$  is continuous for each  $t \in J$  and for each  $\xi \in \mathcal{B}$ ,  $f(\cdot, \xi) : J \rightarrow \mathcal{H}$  is strongly measurable;



(2) there exist a positive integrable functions  $m \in L^1([0, b])$  and a continuous nondecreasing function  $\Xi_f : [0, \infty) \rightarrow (0, \infty)$  such that for every  $(t, \xi) \in J \times \mathcal{B}$ , we have

$$\mathbf{E}\|f(t, \xi)\|_{\mathcal{H}}^2 \leq m(t)\Xi_f(\|\xi\|_{\mathcal{B}}^2), \quad \liminf_{q \rightarrow \infty} \frac{\Xi_f(q)}{q} = \Lambda < \infty.$$

(H5) The function  $\sigma : J \times \mathcal{B} \rightarrow \mathcal{L}_2^0$  satisfies the following properties:

(1)  $\sigma(t, \cdot) : \mathcal{B} \rightarrow \mathcal{L}_2^0$  is continuous for each  $t \in J$  and for each  $\xi \in \mathcal{B}$ ,  $\sigma(\cdot, \xi) : J \rightarrow \mathcal{L}_2^0$  is strongly measurable;

(2) there exist a positive integrable functions  $n \in L^1([0, b])$  and a continuous nondecreasing function  $\Xi_\sigma : [0, \infty) \rightarrow (0, \infty)$  such that for every  $(t, \xi) \in J \times \mathcal{B}$ , we have

$$\mathbf{E}\|\sigma(t, \xi)\|_{\mathcal{L}_2^0}^2 \leq n(t)\Xi_\sigma(\|\xi\|_{\mathcal{B}}^2), \quad \liminf_{q \rightarrow \infty} \frac{\Xi_\sigma(q)}{q} = \Upsilon < \infty.$$

(H6) For all  $\kappa > 0$  such that

$$\begin{aligned} & \left[ 4M_g\mu^{*^2} + 4\mu^{*^2}\Lambda\widetilde{M}_S^2\frac{b^{2\alpha}}{\alpha^2}\sup_{s \in J} m(s) + 4\mu^{*^2}\Upsilon\widetilde{M}_S^2\frac{b^{2\alpha}}{b(2\alpha-1)}\sup_{s \in J} n(s) \right] \\ & \times \left[ 5 + 30(\widetilde{M}_S^2M_B^2)^2\frac{(b^{2\alpha})^2}{\kappa^2\alpha^2} \right] < 1. \end{aligned}$$

**Lemma 2.11**([16]) *Let  $x \in \tilde{\mathcal{H}}$  be continuous on  $[0, b]$  and  $x_0 = \phi$ . If (H2) holds, then*

$$\|x_s\|_{\mathcal{B}} \leq \mu^* \sup_{\substack{0 \leq \theta \leq \max\{0, s\} \\ s \in \mathcal{Z}(\varepsilon) \cup J}} \|x(\theta)\|_{\mathcal{H}} + (\nu^* + \varphi^*)\|\phi\|_{\mathcal{B}},$$

where  $\varphi^* = \sup_{t \in \mathcal{Z}(\varepsilon)} \varphi^\phi(t)$ ,  $\nu^* = \sup_{t \in J} \nu(t)$ ,  $\mu^* = \sup_{t \in J} \mu(t)$ .

The following lemma is required to define the control function.

**Lemma 2.12**([23]) *For any  $\hat{x}_b \in \mathcal{L}^2(\mathcal{F}_b, \mathcal{H})$  there exists  $\hat{\varphi} \in \mathcal{L}_{\mathcal{F}}^2(\Omega; \mathcal{L}^2(J, \mathcal{L}_2^0))$  such that*

$$\hat{x}_b = \mathbf{E}\hat{x}_b + \int_0^b \hat{\varphi}(s)dw(s).$$

Now for any  $\kappa > 0$  and  $\hat{x}_b \in \mathcal{L}^2(\mathcal{F}_b, \mathcal{H})$ , we define the control function

$$\begin{aligned} u^\kappa(t) &= B^*S_\alpha^*(b-t)(\kappa I + \Gamma_0^b)^{-1} \\ &\times \left\{ \mathbf{E}\hat{x}_b + \int_0^b \hat{\varphi}(s)dw(s) - T_\alpha(b)[\phi(0) + g(0, x_{\varepsilon(0, \phi)})] + g(b, x_{\varepsilon(b, x_b)}) \right\} \\ &- B^*S_\alpha^*(b-t) \int_0^b (\kappa I + \Gamma_s^b)^{-1} S_\alpha(b-s) f(s, x_{\varepsilon(s, x_s)}) ds \\ &- B^*S_\alpha^*(b-t) \int_0^b (\kappa I + \Gamma_s^b)^{-1} S_\alpha(b-s) \sigma(s, x_{\varepsilon(s, x_s)}) dw(s). \end{aligned}$$

### §3. Controllability Results

In this section, we study the approximate controllability results for the systems (2.1) where the operator  $A$  is a sectorial type  $\omega$  with  $\pi(1-\alpha/2) < \theta < \pi$ . In particular, we establish approximate controllability of nonlinear fractional stochastic system (2.1) under the assumptions that the corresponding linear systems is approximately controllable.

**Theorem 3.1** *Assume that the assumptions (H1)-(H6) hold and  $S_\alpha(t)$  is compact, then the fractional stochastic system (2.1) has at least one mild solution.*

*Proof* Let  $\mathcal{C}((-\infty, b], \mathcal{H})$  be the space of all continuous  $\mathcal{H}$ -valued stochastic processes  $\{x(t), t \in (-\infty, b]\}$ . Consider the space  $\tilde{\mathcal{B}} = \{x : x \in \mathcal{C}((-\infty, b], \mathcal{H}), x(0) = \phi(0)\}$  endowed with seminorm  $\|\cdot\|_{\tilde{\mathcal{B}}}$  defined by

$$\|x\|_{\tilde{\mathcal{B}}} = \|\phi\|_{\mathcal{B}} + \sup_{-\infty < s \leq b} (\mathbf{E}\|x(s)\|^2)^{\frac{1}{2}}, \quad x \in \tilde{\mathcal{B}},$$

and the space  $\tilde{\mathcal{H}}$  defined in the previous section endowed with the norm  $\|x\|_{\tilde{\mathcal{H}}} = \sup_{t \in J} (\mathbf{E}\|x(t)\|_{\mathcal{H}}^2)^{\frac{1}{2}}$ .

In what follows, we assume that  $\varepsilon : [0, b] \times \mathcal{B} \rightarrow (-\infty, b]$  is continuous and  $\varepsilon(t, x_t) = 0$  for  $t = 0$ . For  $\kappa > 0$ , define the operator  $\mathcal{P} : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}$  by

$$\begin{aligned} (\mathcal{P}x)(t) &= T_\alpha(t)[\phi(0) + g(0, x_{\varepsilon(0, \phi)})] - g(t, x_{\varepsilon(t, x_t)}) \\ &\quad + \int_0^t S_\alpha(t-s)[Bu^\kappa(s) + f(s, x_{\varepsilon(s, x_s)})]ds \\ &\quad + \int_0^t S_\alpha(t-s)\sigma(s, x_{\varepsilon(s, x_s)})dw(s), \quad t \in J. \end{aligned}$$

It will be shown that the system (1) is approximately controllable if for all  $\kappa > 0$  there exists a fixed point of the operator  $\mathcal{P}$ .

For  $\phi \in \mathcal{B}$ , define

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T_\alpha(t)\phi(0), & t \in J. \end{cases}$$

Then  $y_0 = \phi$ .

For each  $z : J \rightarrow \mathcal{H}$  with  $z(0) = 0$ , we denote by  $\bar{z}$  the function defined by

$$\bar{z}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ z(t), & t \in J. \end{cases}$$

If  $x(\cdot)$  satisfies the system (2.1), then we can decompose  $x(\cdot)$  as  $x(t) = y(t) + \bar{z}(t)$ , which implies  $x_{\varepsilon(t, x_t)} = y_{\varepsilon(t, x_t)} + \bar{z}_{\varepsilon(t, x_t)}$ , for  $t \in J$  and the function  $z(\cdot)$  satisfies

$$\begin{aligned} z(t) &= T_\alpha(t)g(0, \phi) - g(t, y_{\varepsilon(t, \bar{z}_t)}) \\ &\quad + \bar{z}_{\varepsilon(t, \bar{z}_t)} + \int_0^t S_\alpha(t-s)Bu^\kappa(s)ds + \int_0^t S_\alpha(t-s)f(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)})ds \\ &\quad + \int_0^t S_\alpha(t-s)\sigma(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)})dw(s), \quad t \in J, \end{aligned}$$

where

$$\begin{aligned}
u^\kappa(t) &= B^* S_\alpha^*(b-t)(\kappa I + \Gamma_0^b)^{-1} \\
&\times \left\{ \mathbf{E} \hat{x}_b + \int_0^b \hat{\varphi}(s) dw(s) - T_\alpha(b)[\phi(0) + g(0, \phi)] + g(b, y_{\varepsilon(b, \bar{z}_b)} + \bar{z}_{\varepsilon(b, \bar{z}_b)}) \right\} \\
&- B^* S_\alpha^*(b-t) \int_0^b (\kappa I + \Gamma_s^b)^{-1} S_\alpha(b-s) f(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)}) ds \\
&- B^* S_\alpha^*(b-t) \int_0^b (\kappa I + \Gamma_s^b)^{-1} S_\alpha(b-s) \sigma(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)}) dw(s).
\end{aligned}$$

Set  $\tilde{\mathcal{B}}_0 = \{z \in \tilde{\mathcal{B}}, z_0 = 0 \in \mathcal{B}\}$ , and for any  $z \in \tilde{\mathcal{B}}_0$ , we have

$$\|z\|_{\tilde{\mathcal{B}}_0} = \|z_0\|_{\mathcal{B}} + \sup_{-\infty < s \leq b} (\mathbf{E} \|x(s)\|^2)^{\frac{1}{2}} = \sup_{-\infty < s \leq b} (\mathbf{E} \|x(s)\|^2)^{\frac{1}{2}}.$$

On the space  $\tilde{\mathcal{B}}_0$ , consider a set  $B_q = \{z \in \tilde{\mathcal{B}}_0 : \|z\|_{\tilde{\mathcal{B}}_0}^2 \leq q\}$  for some  $q \geq 0$ ; then, for each  $q$ ,  $B_q$  is clearly a bounded closed convex set in  $\tilde{\mathcal{B}}_0$ . For  $z \in B_q$ , from Lemma 2.11, we see that

$$\begin{aligned}
\|\bar{z}_t + y_t\|_{\mathcal{B}}^2 &\leq 2 \left( \|\bar{z}_t\|_{\mathcal{B}}^2 + \|y_t\|_{\mathcal{B}}^2 \right) \\
&\leq 4 \left( \mu^{*2} \sup_{\substack{0 \leq s \leq \max\{0, t\} \\ t \in \mathcal{Z}(\varepsilon) \cup J}} \mathbf{E} \|\bar{z}(s)\|^2 + (\nu^* + \varphi^*)^2 \|\bar{z}_0\|_{\mathcal{B}}^2 \right. \\
&\quad \left. + \mu^{*2} \sup_{\substack{0 \leq s \leq \max\{0, t\} \\ t \in \mathcal{Z}(\varepsilon) \cup J}} \mathbf{E} \|y(s)\|^2 + (\nu^* + \varphi^*)^2 \|y_0\|_{\mathcal{B}}^2 \right) \\
&\leq 4\mu^{*2} \left( q + \widetilde{M}_T^2 \mathbf{E} \|\phi(0)\|_{\mathcal{H}}^2 \right) + 4(\nu^* + \varphi^*)^2 \|\phi\|_{\mathcal{B}}^2.
\end{aligned} \tag{3.1}$$

Let  $\Phi : \tilde{\mathcal{B}}_0 \rightarrow \tilde{\mathcal{B}}_0$  be the operator defined by  $\Phi z$  such that

$$\Phi z(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ T_\alpha(t)g(0, \phi) - g(t, y_{\varepsilon(t, \bar{z}_t)} + \bar{z}_{\varepsilon(t, \bar{z}_t)}) + \int_0^t S_\alpha(t-s)Bu^\kappa(s)ds \\ + \int_0^t S_\alpha(t-s)f(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)})ds \\ + \int_0^t S_\alpha(t-s)\sigma(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)})dw(s), & t \in J. \end{cases}$$

Obviously, the operator  $\mathcal{P}$  has a fixed point if and only if  $\Phi$  has a fixed point. For the sake of convenience, we divide the proof into several steps.

**Step 1** We show that, for each  $\kappa > 0$ , there exists a positive number  $q$  such that  $\Phi(B_q) \subset B_q$ . If it is not true, then there exists  $\kappa > 0$  such that for every  $q > 0$  and  $t \in J$ , there exists a function  $z^q(t) \in B_q$ , but  $\Phi(z^q) \notin B_q$ , that is,  $\mathbf{E} \|(\Phi z^q)(t)\|_{\mathcal{H}}^2 > q$ . For such  $\kappa > 0$  and  $\bar{z} = z$  on

$J$ , we find that

$$\begin{aligned}
q &\leq \mathbf{E}\|\Phi(z^q)(t)\|_{\mathcal{H}}^2 \\
&\leq 5\mathbf{E}\|T_\alpha(t)g(0, \phi)\|_{\mathcal{H}}^2 + 5\mathbf{E}\|g(t, y_\varepsilon(t, \bar{z}_t^q) + \bar{z}_{\varepsilon(t, \bar{z}_t^q)}^q)\|_{\mathcal{H}}^2 \\
&\quad + 5\mathbf{E}\left\|\int_0^t S_\alpha(t-s)Bu^\kappa(s)ds\right\|_{\mathcal{H}}^2 + 5\mathbf{E}\left\|\int_0^t S_\alpha(t-s)f(s, y_\varepsilon(s, \bar{z}_s^q) + \bar{z}_{\varepsilon(s, \bar{z}_s^q)}^q)ds\right\|_{\mathcal{H}}^2 \\
&\quad + 5\mathbf{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s, y_\varepsilon(s, \bar{z}_s^q) + \bar{z}_{\varepsilon(s, \bar{z}_s^q)}^q)dw(s)\right\|_{\mathcal{H}}^2.
\end{aligned}$$

By Lemma 2.11 and assumptions (H1)-(H3), we have

$$\begin{aligned}
I_1 &= \mathbf{E}\|T_\alpha(t)g(0, \phi)\|_{\mathcal{H}}^2 + \mathbf{E}\|g(t, y_\varepsilon(t, \bar{z}_t^q) + \bar{z}_{\varepsilon(t, \bar{z}_t^q)}^q)\|_{\mathcal{H}}^2 \\
&\leq \widetilde{M}_T^2 M_g(1 + \|\phi\|_{\mathcal{B}}^2) + M_g(1 + \|y_\varepsilon(t, \bar{z}_t^q) + \bar{z}_{\varepsilon(t, \bar{z}_t^q)}^q\|_{\mathcal{B}}^2) \\
&\leq \widetilde{M}_T^2 M_g(1 + \|\phi\|_{\mathcal{B}}^2) + M_g(1 + 4\mu^{*2}(q + \widetilde{M}_T^2 \mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^* + \varphi^*)^2 \|\phi\|_{\mathcal{B}}^2).
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
I_2 &= \mathbf{E}\left\|\int_0^t S_\alpha(t-s)f(s, y_\varepsilon(s, \bar{z}_s^q) + \bar{z}_{\varepsilon(s, \bar{z}_s^q)}^q)ds\right\|_{\mathcal{H}}^2 \\
&\quad + \mathbf{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s, y_\varepsilon(s, \bar{z}_s^q) + \bar{z}_{\varepsilon(s, \bar{z}_s^q)}^q)dw(s)\right\|_{\mathcal{H}}^2 \\
&\leq \int_0^t \|S_\alpha(t-s)\| ds \int_0^t \|S_\alpha(t-s)\| \mathbf{E}\|f(s, y_\varepsilon(s, \bar{z}_s^q) + \bar{z}_{\varepsilon(s, \bar{z}_s^q)}^q)\|_{\mathcal{H}}^2 ds \\
&\quad + \int_0^t \|S_\alpha(t-s)\|^2 \mathbf{E}\|\sigma(s, y_\varepsilon(s, \bar{z}_s^q) + \bar{z}_{\varepsilon(s, \bar{z}_s^q)}^q)\|_{\mathcal{L}_2^0}^2 ds \\
&\leq \widetilde{M}_S^2 \frac{b^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1} m(s) \Xi_f(\|y_\varepsilon(t, \bar{z}_t^q) + \bar{z}_{\varepsilon(t, \bar{z}_t^q)}^q\|_{\mathcal{B}}^2) ds \\
&\quad + \widetilde{M}_S^2 \int_0^t (t-s)^{2(\alpha-1)} n(s) \Xi_\sigma(\|y_\varepsilon(t, \bar{z}_t^q) + \bar{z}_{\varepsilon(t, \bar{z}_t^q)}^q\|_{\mathcal{B}}^2) ds \\
&\leq \widetilde{M}_S^2 \frac{b^{2\alpha}}{\alpha^2} \Xi_f(4\mu^{*2}(q + \widetilde{M}_T^2 \mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^* + \varphi^*)^2 \|\phi\|_{\mathcal{B}}^2) \sup_{s \in J} m(s) \\
&\quad + \widetilde{M}_S^2 \frac{b^{2\alpha-1}}{2\alpha-1} \Xi_\sigma(4\mu^{*2}(q + \widetilde{M}_T^2 \mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^* + \varphi^*)^2 \|\phi\|_{\mathcal{B}}^2) \sup_{s \in J} n(s) \\
&= \widetilde{M}_S^2 b^{2\alpha} \left[ \frac{M_f}{\alpha^2} + \frac{M_\sigma}{b(2\alpha-1)} \right].
\end{aligned} \tag{3.3}$$

Further, by using (H1)-(H5), Hölder inequality, Eq.(3.1) and Lemma 2.11, we get

$$\begin{aligned}
\mathbf{E}\|u^\kappa(s)\|_2 &\leq \frac{1}{\kappa^2} M_B^2 b^{2\alpha} \widetilde{M}_S^2 \left\{ 6\|\mathbf{E}\hat{x}_b + \int_0^b \hat{\varphi}(s)dw(s)\|^2 + 6\mathbf{E}\|T_\alpha(b)\phi(0)\|^2 \right. \\
&\quad + \mathbf{E}\|T_\alpha(b)g(0, \phi)\|^2 + 6\mathbf{E}\|g(b, y_\varepsilon(b, \bar{z}_b) + \bar{z}_{\varepsilon(b, \bar{z}_b)}^q)\|^2 \\
&\quad \left. + 6\mathbf{E}\left\|\int_0^b S_\alpha(b-s)f(s, y_\varepsilon(s, \bar{z}_s^q) + \bar{z}_{\varepsilon(s, \bar{z}_s^q)}^q)ds\right\|_{\mathcal{H}}^2 \right\}
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
& + \mathbf{E} \left\| \int_0^b S_\alpha(b-s) \sigma(s, y_{\varepsilon(s, \bar{z}_s^q)} + \bar{z}_{\varepsilon(s, \bar{z}_s^q)}^q) dw(s) \right\|_{\mathcal{H}}^2 \Bigg\} \\
& \leq \frac{6b^{2\alpha}}{\kappa^2} M_B^2 \widetilde{M}_S^2 \left[ 2\|\mathbf{E}\hat{x}_b\|^2 + 2 \int_0^b \mathbf{E}\|\hat{\varphi}(s)\|^2 ds + \widetilde{M}_T^2 \|\phi\|_B^2 + \widetilde{M}_T^2 M_g (1 + \|\phi\|_B^2) \right. \\
& \quad + M_g \left( 1 + 4\mu^{*2} (q + \widetilde{M}_T^2 \mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^* + \varphi^*)^2 \|\phi\|_B^2 \right) \\
& \quad \left. + \widetilde{M}_S^2 b^{2\alpha} \left( \frac{M_f}{\alpha^2} + \frac{M_\sigma}{b(2\alpha-1)} \right) \right], \tag{3.5}
\end{aligned}$$

where  $M_B = \|B\|$ .

Now, we have

$$\begin{aligned}
I_3 &= \mathbf{E} \left\| \int_0^t S_\alpha(t-s) Bu^\kappa(s) ds \right\|_{\mathcal{H}}^2 \\
&\leq \left( \widetilde{M}_S^2 M_B^2 \right)^2 \frac{6}{\kappa^2 \alpha^2} (b^{2\alpha})^2 \left[ 2\|\mathbf{E}\hat{x}_b\|^2 + 2 \int_0^b \mathbf{E}\|\hat{\varphi}(s)\|^2 ds + \widetilde{M}_T^2 \|\phi\|_B^2 + \widetilde{M}_T^2 M_g (1 + \|\phi\|_B^2) \right. \\
&\quad \left. + M_g \left( 1 + 4\mu^{*2} (q + \widetilde{M}_T^2 \mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^* + \varphi^*)^2 \|\phi\|_B^2 \right) + \widetilde{M}_S^2 b^{2\alpha} \left( \frac{M_f}{\alpha^2} + \frac{M_\sigma}{b(2\alpha-1)} \right) \right]. \tag{3.6}
\end{aligned}$$

Combining the estimates (3.2), (3.3) and (3.5) yields

$$\begin{aligned}
q &\leq \mathbf{E}\|\Phi(z^q)(t)\|_{\mathcal{H}}^2 \\
&\leq 5\widetilde{M}_T^2 M_g \left( 1 + \|\phi\|_B^2 \right) + 5M_g \left( 1 + 4\mu^{*2} (q + \widetilde{M}_T^2 \mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^* + \varphi^*)^2 \|\phi\|_B^2 \right) \\
&\quad + 5\widetilde{M}_S^2 b^{2\alpha} \left[ \frac{M_f}{\alpha^2} + \frac{M_\sigma}{b(2\alpha-1)} \right] + 5 \left( \widetilde{M}_S^2 M_B^2 \right)^2 \frac{6}{\kappa^2 \alpha^2} (b^{2\alpha})^2 \left[ 2\|\mathbf{E}\hat{x}_b\|^2 + 2 \int_0^b \mathbf{E}\|\hat{\varphi}(s)\|^2 ds \right. \\
&\quad \left. + \widetilde{M}_T^2 \|\phi\|_B^2 + \widetilde{M}_T^2 M_g (1 + \|\phi\|_B^2) + M_g \left( 1 + 4\mu^{*2} (q + \widetilde{M}_T^2 \mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^* + \varphi^*)^2 \|\phi\|_B^2 \right) \right. \\
&\quad \left. + \widetilde{M}_S^2 b^{2\alpha} \left( \frac{M_f}{\alpha^2} + \frac{M_\sigma}{b(2\alpha-1)} \right) \right]. \tag{3.7}
\end{aligned}$$

Dividing both sides of (3.6) by  $q$  and taking  $q \rightarrow \infty$ , we obtain

$$\begin{aligned}
& \left[ 4M_g \mu^{*2} + 4\mu^{*2} \Lambda \widetilde{M}_S^2 \frac{b^{2\alpha}}{\alpha^2} \sup_{s \in J} m(s) + 4\mu^{*2} \Upsilon \widetilde{M}_S^2 \frac{b^{2\alpha}}{b(2\alpha-1)} \sup_{s \in J} n(s) \right] \\
& \quad \times \left[ 5 + 30(\widetilde{M}_S^2 M_B^2)^2 \frac{(b^{2\alpha})^2}{\kappa^2 \alpha^2} \right] \geq 1,
\end{aligned}$$

which is a contradiction by assumption (H6). Thus, for some  $q > 0$ ,  $\Phi(B_q) \subset B_q$ .

**Step 2** We prove that for each  $\kappa > 0$ , the operator  $\Phi$  maps  $B_q$  into a relatively compact subset of  $B_q$ . First we prove that the set  $V(t) = \{(\Phi z)(t) : z \in B_q\}$  is relatively compact in  $\mathcal{H}$  for every  $t \in J$ . The case  $t = 0$  is obvious.

For  $0 < \epsilon < t \leq b$ , we define an operator  $\Phi^\epsilon$  on  $B_q$  by

$$\begin{aligned} (\Phi^\epsilon z)(t) = & T_\alpha(t)g(0, \phi) - g(t, y_\epsilon(t, \bar{z}_t) + \bar{z}_\epsilon(t, \bar{z}_t)) + S_\alpha(\epsilon) \int_0^{t-\epsilon} S_\alpha(t-s-\epsilon)Bu^\kappa(s)ds \\ & + S_\alpha(\epsilon) \int_0^{t-\epsilon} S_\alpha(t-s-\epsilon)f(s, y_\epsilon(s, \bar{z}_s) + \bar{z}_\epsilon(s, \bar{z}_s))ds \\ & + S_\alpha(\epsilon) \int_0^{t-\epsilon} S_\alpha(t-s-\epsilon)\sigma(s, y_\epsilon(s, \bar{z}_s) + \bar{z}_\epsilon(s, \bar{z}_s))dw(s). \end{aligned}$$

Since  $S_\alpha(t)$  is a compact operator, the set  $V_\epsilon(t) = \{(\Phi^\epsilon z)(t) : z(\cdot) \in B_q\}$  is relatively compact in  $\mathcal{H}$  for every  $\epsilon > 0$ . Also, for every  $z \in B_q$ , we have

$$\begin{aligned} & \mathbf{E}\|(\Phi z)(t) - (\Phi^\epsilon z)(t)\|_{\mathcal{H}}^2 \\ \leq & 3\mathbf{E}\left\|\int_{t-\epsilon}^t S_\alpha(t-s)Bu^\kappa(s)ds\right\|_{\mathcal{H}}^2 + 3\mathbf{E}\left\|\int_{t-\epsilon}^t S_\alpha(t-s)f(s, y_\epsilon(s, \bar{z}_s) + \bar{z}_\epsilon(s, \bar{z}_s))ds\right\|_{\mathcal{H}}^2 \\ & + 3\mathbf{E}\left\|\int_{t-\epsilon}^t S_\alpha(t-s)\sigma(s, y_\epsilon(s, \bar{z}_s) + \bar{z}_\epsilon(s, \bar{z}_s))dw(s)\right\|_{\mathcal{H}}^2 \\ \leq & 3\widetilde{M}_S^2 \frac{\epsilon^{2\alpha}}{\alpha} \int_{t-\epsilon}^t (t-s)^{\alpha-1} \left[ M_B^2 \frac{6}{\kappa^2 \alpha} \epsilon^{2\alpha} (M_B \widetilde{M}_S)^2 \hat{M} + m(s)\Xi_f(q') + n(s)\Xi_\sigma(q') \right] ds \\ \longrightarrow & 0 \quad \text{as } \epsilon \rightarrow 0^+, \text{ where} \end{aligned}$$

$$q' = 4\mu^*(q + \widetilde{M}_T^2 \mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^* + \varphi^*)^2 \|\phi\|_{\mathcal{B}}^2$$

and

$$\begin{aligned} \hat{M} = & \left[ 2\|\mathbf{E}\hat{x}_b\|^2 + 2 \int_0^b \mathbf{E}\|\hat{\varphi}(s)\|^2 ds + \widetilde{M}_T^2 \|\phi\|_{\mathcal{B}}^2 + \widetilde{M}_T^2 M_g(1 + \|\phi\|_{\mathcal{B}}^2) \right. \\ & \left. + M_g(1 + q') + \widetilde{M}_S^2 b^{2\alpha} \left( \frac{M_f}{\alpha^2} + \frac{M_\sigma}{b(2\alpha-1)} \right) \right]. \end{aligned}$$

This implies that there are relatively compact sets arbitrarily close to the set  $V(t)$  for each  $t \in (0, b]$ . Hence  $V(t) = \{(\Phi z)(t) : z \in B_q\}$  is relatively compact in  $\mathcal{H}$ .

**Step 3** We show that  $V = \{(\Phi z)(t) : z(\cdot) \in B_q\}$  is equicontinuous on  $[0, b]$ . Let  $0 < \epsilon < t < b$  and  $\delta > 0$  be such that  $\|S_\alpha(s_1) - S_\alpha(s_2)\| < \epsilon$ , for every  $s_1, s_2 \in J$  with  $|s_1 - s_2| < \delta$ . For any  $z \in B_q$  and  $0 \leq t_1 \leq t_2 \in J$ , we get

$$\begin{aligned} \mathbf{E}\|(\Phi z)(t_2) - (\Phi z)(t_1)\|_{\mathcal{H}}^2 \leq & 8\|T_\alpha(t_2) - T_\alpha(t_1)\|^2 \mathbf{E}\|g(0, \phi)\|_{\mathcal{H}}^2 \\ & + 8\mathbf{E}\|g(t_2, y_\epsilon(t_2, \bar{z}_{t_2}) + \bar{z}_\epsilon(t_2, \bar{z}_{t_2})) - g(t_1, y_\epsilon(t_1, \bar{z}_{t_1}) + \bar{z}_\epsilon(t_1, \bar{z}_{t_1}))\|_{\mathcal{H}}^2 \\ & + 8\mathbf{E}\left\|\int_0^{t_1} [S_\alpha(t_2-s) - S_\alpha(t_1-s)]Bu^\kappa(s)ds\right\|_{\mathcal{H}}^2 \\ & + 8\mathbf{E}\left\|\int_{t_1}^{t_2} S_\alpha(t_2-s)Bu^\kappa(s)ds\right\|_{\mathcal{H}}^2 \end{aligned}$$

$$\begin{aligned}
& +8\mathbf{E}\left\|\int_0^{t_1}[S_\alpha(t_2-s)-S_\alpha(t_1-s)]f(s,y_\varepsilon(s,\bar{z}_s)+\bar{z}_\varepsilon(s,\bar{z}_s))ds\right\|_{\mathcal{H}}^2 \\
& +8\mathbf{E}\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)f(s,y_\varepsilon(s,\bar{z}_s)+\bar{z}_\varepsilon(s,\bar{z}_s))ds\right\|_{\mathcal{H}}^2 \\
& +8\mathbf{E}\left\|\int_0^{t_1}[S_\alpha(t_2-s)-S_\alpha(t_1-s)]\sigma(s,y_\varepsilon(s,\bar{z}_s)+\bar{z}_\varepsilon(s,\bar{z}_s))dw(s)\right\|_{\mathcal{H}}^2 \\
& +8\mathbf{E}\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)\sigma(s,y_\varepsilon(s,\bar{z}_s)+\bar{z}_\varepsilon(s,\bar{z}_s))dw(s)\right\|_{\mathcal{H}}^2.
\end{aligned}$$

By assumptions (H1), (H3)-(H5) and Hölder's inequality, it follows that

$$\begin{aligned}
\mathbf{E}\|(\Phi z)(t_2) - (\Phi z)(t_1)\|_{\mathcal{H}}^2 & \leq 8M_g\|T_\alpha(t_2) - T_\alpha(t_1)\|^2(1 + \|\phi\|_{\mathcal{B}}^2) \\
& + 8M_g\left(|t_2 - t_1|^2 + \|(y_\varepsilon(t_2, \bar{z}_{t_2}) - y_\varepsilon(t_1, \bar{z}_{t_1})) + (\bar{z}_\varepsilon(t_2, \bar{z}_{t_2}) - \bar{z}_\varepsilon(t_1, \bar{z}_{t_1}))\|_{\mathcal{B}}^2\right) \\
& + 8\widetilde{M}_S^2 M_B^2 \frac{(t_2 - t_1)^\alpha}{\alpha} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathbf{E}\|u^\kappa(s)\|^2 ds \\
& + 8\epsilon^2 M_B^2 \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathbf{E}\|u^\kappa(s)\|^2 ds \\
& + 8\epsilon^2 \frac{t_1^\alpha}{\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} [m(s)\Xi_f(q') + n(s)\Xi_\sigma(q')] ds \\
& + 8\widetilde{M}_S^2 \frac{(t_2 - t_1)^\alpha}{\alpha} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} [m(s)\Xi_f(q') + n(s)\Xi_\sigma(q')] ds.
\end{aligned}$$

Therefore, for  $\epsilon$  sufficiently small, the right-hand side of the above inequality tends to zero as  $t_1 \rightarrow t_2$ , since the compactness of  $S_\alpha(t)$  implies the continuity in the uniform operator topology on  $J$ . Thus, the set  $V = \{(\Phi z)(t) : z(\cdot) \in B_q\}$  is equicontinuous.

By using a procedure similar to that used in [2], we can easily prove that the map  $\Phi(\cdot)$  is continuous on  $B_q$  which completes the proof that  $\Phi(\cdot)$  is completely continuous. Hence from the Schauder fixed point theorem  $\Phi$  has a fixed point and consequently the equation (1) has a mild solution on  $J$ .  $\square$

**Theorem 3.2** *Assume that the assumptions of Theorem 3.1 hold and linear system (2.6) is approximately controllable on  $J$ . In addition, the functions  $f$ ,  $g$  and  $\sigma$  are uniformly bounded on their respective domains. Further, if  $S_\alpha(t)$  is compact, then the fractional control system (2.1) is approximately controllable on  $J$ .*

*Proof* Let  $x^\kappa(\cdot)$  be a fixed point of  $\Phi$  in  $B_q$ . By using the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned}
x^\kappa(b) &= \hat{x}_b - \kappa(\kappa I + \Gamma_0^b)^{-1} \left[ \mathbf{E}\hat{x}_b + \int_0^b \hat{\varphi}(s)dw(s) - T_\alpha(b)[\phi(0) + g(0, \phi)] - g(b, x_{\varepsilon(b, x_b^\kappa)}^\kappa) \right] \\
&+ \kappa \int_0^b (\kappa I + \Gamma_s^b)^{-1} S_\alpha(b-s) f(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa) ds \\
&+ \kappa \int_0^b (\kappa I + \Gamma_s^b)^{-1} S_\alpha(b-s) \sigma(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa) dw(s).
\end{aligned} \tag{3.8}$$

By the assumption on Theorem 3.2, we have

$$\|f(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa)\|^2 + \|\sigma(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa)\|^2 \leq N_1 \quad \text{and} \quad \|g(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa)\|^2 \leq N_2.$$

If the linear system (2.6) is approximately controllable on every  $[0, s]$ ,  $0 \leq s \leq b$ , then by Lemma 2.10, the approximate controllability of (2.6) is equivalent to convergence of the operator  $\kappa(\kappa I + \Gamma_0^b)^{-1}$  to zero operator in the strong operator topology as  $\kappa \rightarrow 0^+$ , and moreover  $\|\kappa(\kappa I + \Gamma_s^b)^{-1}\| \leq 1$ .

Then there is a subsequence denoted by  $\{f(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa), \sigma(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa)\}$  weakly converging to say  $\{f(s), \sigma(s)\}$ . Thus, from the above equation, we have

$$\begin{aligned} \mathbf{E}\|x^\kappa(b) - \hat{x}_b\|^2 &\leq 7\|\kappa(\kappa I + \Gamma_0^b)^{-1}[\mathbf{E}\hat{x}_b - T_\alpha(b)(\phi(0) + g(0, \phi))]\|^2 \\ &\quad + 7\mathbf{E}\left(\int_0^b \|\kappa(\kappa I + \Gamma_0^b)^{-1}\hat{\varphi}(s)\|_{\mathcal{L}_2^0}^2 ds\right) + 7\mathbf{E}\|\kappa(\kappa I + \Gamma_0^b)^{-1}g(b, x_{\varepsilon(b, x_b^\kappa)}^\kappa)\|^2 \\ &\quad + 7\mathbf{E}\left(\int_0^b \|\kappa(\kappa I + \Gamma_s^b)^{-1}S_\alpha(b-s)[f(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa) - f(s)]\| ds\right)^2 \\ &\quad + 7\mathbf{E}\left(\int_0^b \|\kappa(\kappa I + \Gamma_s^b)^{-1}S_\alpha(b-s)f(s)\| ds\right)^2 \\ &\quad + 7\mathbf{E}\left(\int_0^b \|\kappa(\kappa I + \Gamma_s^b)^{-1}S_\alpha(b-s)[\sigma(s, x_{\varepsilon(s, x_s^\kappa)}^\kappa) - \sigma(s)]\|_{\mathcal{L}_2^0} ds\right)^2 \\ &\quad + 7\mathbf{E}\left(\int_0^b \|\kappa(\kappa I + \Gamma_s^b)^{-1}S_\alpha(b-s)\sigma(s)\|_{\mathcal{L}_2^0} ds\right)^2. \end{aligned}$$

Using the Lebesgue dominated convergence theorem and the compactness of  $S_\alpha(t)$ , we obtain  $\mathbf{E}\|x^\kappa(b) - \hat{x}_b\|^2 \rightarrow 0$  as  $\kappa \rightarrow 0^+$ . This gives the approximate controllability of (1). Hence the proof is complete.  $\square$

The mathematical formulation of many physical phenomena contain integro-differential equations, these integro-differential equations arise in various applications such as viscoelasticity, heat equations, fluid dynamics, chemical kinetics and so on. Motivated by this consideration, in this paper we construct the fractional control system in the following integro-differential framework

$$\begin{aligned} {}^c D_t^\alpha [x(t) + g(t, x_{\varepsilon(t, x_t)})] &= A[x(t) + g(t, x_{\varepsilon(t, x_t)})] + \int_0^t G(t-s)x(s)ds + Bu(t) \\ &\quad + f(t, x_{\varepsilon(t, x_t)}) + \sigma(t, x_{\varepsilon(t, x_t)}) \frac{dw(t)}{dt}, \\ \alpha &\in (0, 1), \quad t \in J := [0, b] \\ x_0 &= \phi \in \mathcal{B}, \quad x'(0) = 0, \end{aligned} \tag{3.9}$$

where  $A, (G(t))_{t \geq 0}$  are linear operators defined on Hilbert space  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ , and  ${}^c D_t^\alpha x(t)$  represent the Caputo derivative of order  $\alpha > 0$ .

Further, we assume that the integro-differential abstract Cauchy problem

$$\begin{aligned} {}^c D_t^\alpha x(t) &= Ax(t) + \int_0^t G(t-s)x(s)ds \\ x(0) &= x_0, \quad x'(0) = 0, \end{aligned} \tag{3.10}$$



has an associated  $\alpha$ -resolvent operator of bounded linear operators  $(\S_\alpha)_{t \geq 0}$  on  $\mathcal{H}$ .

One parameter family of bounded linear operators  $(\S_\alpha)_{t \geq 0}$  on  $\mathcal{H}$  is called  $\alpha$ -resolvent operator of (3.9) if the following conditions are verified.

(1) The function  $\S_\alpha(\cdot) : \mathbf{R}_+ \rightarrow L(\mathcal{H})$  is strongly continuous and  $\S_\alpha(0)x = x$  for all  $x \in \mathcal{H}$  and  $\alpha \in (0, 1)$ .

(2) For all  $x \in \mathcal{D}(A)$ ,  $x \in \mathcal{C}(\mathbf{R}_+, [\mathcal{D}(A)]) \cap \mathcal{C}'((0, \infty), \mathcal{H})$  and

$$\begin{aligned} {}^c D_t^\alpha \S_\alpha(t)x &= A\S_\alpha(t)x + \int_0^t G(t-s)\S_\alpha(s)x ds \\ {}^c D_t^\alpha \S_\alpha(t)x &= \S_\alpha(t)Ax + \int_0^t \S_\alpha(t-s)G(s)x ds, \end{aligned}$$

for every  $t \geq 0$ .

**Definition 3.3** Let  $\alpha \in (0, 1)$ , we define the family  $\hat{\S}_\alpha(t)$  by  $\hat{\S}_\alpha(t)x := \int_0^t \hat{f}_{\alpha-1}(t-s)\S_\alpha(s)x ds$ ,  $x \in \mathcal{H}$  and  $\hat{f}_\eta(t)t^{\eta-1}/\Gamma(\eta)$ ,  $\eta \geq 0$ ,  $t > 0$  and  $\Gamma$  is the gamma function.

**Definition 3.4** An  $\mathcal{F}_t$ -adapted stochastic process  $x : (\infty, b] \rightarrow \mathcal{H}$  is called a mild solution of the system (14) on  $J$  if  $x_0 = \Phi$ ,  $x_{\varepsilon(t, x_t)} \in \mathcal{B}$ ,  $x|_J \in \mathcal{C}(J, \mathcal{H})$  and

$$\begin{aligned} x(t) &= \S_\alpha(t)g(0, \phi) - g(t, x_{\varepsilon(t, x_t)}) + \int_0^t \hat{\S}_\alpha(t-s)Bu(s)ds \\ &\quad + \int_0^t \hat{\S}_\alpha(t-s)f(s, x_{\varepsilon(t, x_t)})ds + \int_0^t \hat{\S}_\alpha(t-s)\sigma(s, x_{\varepsilon(t, x_t)})dw(s), \quad t \in J. \end{aligned}$$

**Theorem 3.5** Let the assumptions (H1)-(H6) hold,  $\mathcal{Z}(\cdot) \in \mathcal{C}((0, b]; L(\mathcal{H}))$  and  $\hat{\S}_\alpha(t)$  is compact. Further, if the linear system corresponding to (3.8) is approximately controllable on  $J$ , then the system (3.8) is approximately controllable.

*Proof* For all  $\kappa > 0$ , define the operator  $\hat{\Phi} : \tilde{\mathcal{B}}_0 \rightarrow \tilde{\mathcal{B}}_0$  by  $\hat{\Phi}z$  such that

$$\hat{\Phi}z(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ \S_\alpha(t)g(0, \phi) - g(t, y_{\varepsilon(t, \bar{z}_t)} + \bar{z}_{\varepsilon(t, \bar{z}_t)}) + \int_0^t \hat{\S}_\alpha(t-s)Bu^\kappa(s)ds \\ \quad + \int_0^t \hat{\S}_\alpha(t-s)f(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)})ds \\ \quad + \int_0^t \hat{\S}_\alpha(t-s)\sigma(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)})dw(s), & t \in J, \end{cases}$$

where  $y_0 = \phi$ ,  $z : J \rightarrow \mathcal{H}$  and  $\bar{z}(t) = z(t)$  for  $t \in J$  with  $z(0) = 0$ , and

$$\begin{aligned} u^\kappa(t) &= B^* \hat{\S}_\alpha^*(b-t)(\kappa I + \Gamma_0^b)^{-1} \\ &\quad \times \left\{ \mathbf{E}\hat{x}_b + \int_0^b \hat{\varphi}(s)dw(s) - \S_\alpha(b)[\phi(0) + g(0, \phi)] + g(b, y_{\varepsilon(b, \bar{z}_b)} + \bar{z}_{\varepsilon(b, \bar{z}_b)}) \right\} \\ &\quad - B^* \hat{\S}_\alpha^*(b-t) \int_0^b (\kappa I + \Gamma_s^b)^{-1} \hat{\S}_\alpha(b-s)f(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)})ds \\ &\quad - B^* \hat{\S}_\alpha^*(b-t) \int_0^b (\kappa I + \Gamma_s^b)^{-1} \hat{\S}_\alpha(b-s)\sigma(s, y_{\varepsilon(s, \bar{z}_s)} + \bar{z}_{\varepsilon(s, \bar{z}_s)})dw(s). \end{aligned}$$

One can easily show that the operator  $\hat{\Phi}$  has a fixed point by employing the technique used in Theorem 3.1 with some changes. Further, in order to prove the approximate controllability result, we assume that the functions  $g, f$  and  $\sigma$  are continuous and uniformly bounded. The proof of this theorem is similar to that of Theorem 3.2, and hence it is omitted.  $\square$

#### §4. An Example

Consider the following fractional stochastic partial differential equation with state-dependent delay and control of the form

$$\begin{aligned}
{}^c D_t^\alpha \left[ z(t, y) + \int_{-\infty}^t H(s-t) z(s - \varepsilon_1(t) \varepsilon_2(\|z(t)\|), y) ds \right] &= \frac{\partial^2}{\partial y^2} \left[ z(t, y) + \int_{-\infty}^t H(s-t) z(s - \varepsilon_1(t) \varepsilon_2(\|z(t)\|), y) ds \right] + \mu(t, y) \\
&+ \int_{-\infty}^t K(s-t) z(s - \varepsilon_1(t) \varepsilon_2(\|z(t)\|), y) ds \\
&+ \left[ \int_{-\infty}^t V(s-t) z(s - \varepsilon_1(t) \varepsilon_2(\|z(t)\|), y) ds \right] \frac{d\beta(t)}{dt}, \\
z(t, 0) &= z(t, \pi) = 0, \quad t \in [0, 1], \quad z(\theta, y) = \phi(\theta, y), \quad \theta \leq 0, \quad y \in [0, \pi],
\end{aligned} \tag{4.1}$$

where  $\beta(t)$  is a standard cylindrical Wiener process in  $\mathcal{H}$  defined on a stochastic space  $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbf{P})$ ;  ${}^c D_t^\alpha$  is the Caputo fractional derivative of order  $0 < \alpha < 1$ . To represent this system in the abstract form, we consider the spaces  $\mathcal{H} = \mathcal{U} = L^2[0, \pi]$  and  $\mathcal{B} = \mathcal{C}_0 \times L^2(h, \mathcal{H})$  ( $h : (-\infty, -r] \rightarrow \mathbf{R}$  be a positive function). We define the operator  $A$  by  $Az = z''$  with the domain

$$\mathcal{D}(A) = \{z \in \mathcal{H}; z, z' \text{ are absolutely continuous, } z'' \in \mathcal{H} \text{ and } z(0) = z(\pi) = 0\}.$$

Then  $A$  generates a strongly continuous semigroup  $(S(t))_{t \geq 0}$  which is compact. Now we introduce the functions  $g(t, \xi)(y) = \int_{-\infty}^0 a(-s) \xi(s, y) ds$ ,  $f(t, \xi)(y) = \int_{-\infty}^0 \tilde{a}(-s) \xi(s, y) ds$  and  $\sigma = (t, \xi)(y) = \int_{-\infty}^0 \hat{a}(-s) \xi(s, y) ds$ , here  $\varepsilon(s, y) = \varepsilon_1(s) \varepsilon_2(\|\xi(0)\|)$ . Further, define the bounded linear operator  $B : \mathcal{U} \rightarrow \mathcal{H}$  by  $Bu(t)(y) = \mu(t, y)$ ,  $0 \leq y \leq \pi$ ,  $u \in \mathcal{U}$ , where  $\mu : [0, 1] \times [0, \pi] \rightarrow [0, \pi]$  is continuous. On the other hand, the linear system corresponding to (4.1) is approximately controllable (but not exactly controllable). Then, the system (4.1) can be written in the abstract form of (2.1) and all the conditions of Theorem 3.2 are satisfied. Further, if we impose suitable conditions on  $g, f, \sigma$  and  $B$  to verify assumptions of Theorem 3.2, then we can conclude that the fractional control system (16) is approximately controllable on  $[0, b]$ .

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## References

- [1] N.Abada, M.Benchohra and H.Hammouche, Existence and controllability results for non-densely defined impulsive semilinear functional differential inclusions, *J. Diff. Equ.*, 246 (2009), 3834-3863.
- [2] R.P.Agarwal, B.Andrade and G.Siracusa, On fractional integro-differential equations with state-dependent delay, *Comput. Math. Appl.*, 62 (2011), 1143-1149.
- [3] M.Benchohra, J.Henderson and S.K.Ntouyas, Existence results for impulsive semilinear neutral functional differential equations in Banach spaces, *Memoirs on Diff. Equ. Math. Phys.*, 25 (2002), 105-120.
- [4] J.Cui and L.Yan, Existence result for fractional neutral stochastic integro-differential equations with infinite delay, *J. Phys. A: Math. Theor.*, 44 (2011) 335201 (16pp)
- [5] J.Dabas, A.Chauhan and M.Kumar, Existence of the mild solutions for impulsive fractional equations with infinite delay, *Int. J. Differ. Equ.*, (2011) 20. Article ID 793023.
- [6] J.Dabas and A.Chauhan, Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay, *Math. Computer Modelling*, 57 (2013), 754-763.
- [7] X.Fu and K.Mei, Approximate controllability of semilinear partial functional differential systems, *Journal of Dynamical and Control Systems*, 15 (2009), 425-443.
- [8] L.Górniewicz, S.K.Ntouyas and D.O'Regan, Controllability of semilinear differential equations and inclusions via semigroup theory in Banach spaces, *Rep. Math. Phys.*, 56, (2005), 437-470.
- [9] L.Górniewicz, S.K.Ntouyas and D.O'Regan, Existence and controllability results for first and second order functional semilinear differential inclusions with nonlocal conditions, *Numer. Funct. Anal. Optim.*, 28, (2007), 53-82.
- [10] T.Guendouzi, Existence and controllability of fractional-order impulsive stochastic system with infinite delay, To appear in *Discussiones Mathematicae Differential Inclusions, Control and Optimization*, 33 (1) (2013), 65-87.
- [11] T.Guendouzi and I.Hamada, Existence and controllability result for fractional neutral stochastic integro-differential equations with infinite delay, *AMO-Advanced Modeling and Optimization*, 15, 2 (2013), 281-300.
- [12] T.Guendouzi and I. Hamada, Relative controllability of fractional stochastic dynamical systems with multiple delays in control, *Malaya Journal of Matematik*, 1(1)(2013), 86-97.
- [13] T.Guendouzi and S.Idrissi, Approximate controllability of fractional stochastic functional evolution equations driven by a fractional Brownian motion, *Romai J.*, v.8, no.2(2012), 103-117.
- [14] J.K.Hale and J.Kato, Phase spaces for retarded equations with infinite delay, *Funkc. Ekvacioj.*, 21 (1978), 11-41.
- [15] R.Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, River Edge, 2000.
- [16] Y.Hino, S.Murakami and T.Naito, Functional differential equations with infinite delay, *Lecture Notes in Mathematics*, Vol.1473, Springer, Berlin. (1991).
- [17] A.A.Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.

- [18] J.Klamka, Constrained controllability of semilinear systems with delays, *Nonlinear Dynam.*, 56(2009), 169-177.
- [19] J.Klamka, Constrained controllability of semilinear systems with delayed controls, *Bull. Pol. Ac. Tech.*, 56(2008), 333-337.
- [20] J.Klamka, Stochastic controllability of systems with variable delay in control, *Bull. Polon. A: Tech.*, 56 (2008), 279-284.
- [21] J.Klamka, Stochastic controllability of linear systems with delay in control, *Bulletin of the Polish Academy of Sciences: Technical Sciences*, 55 (2007), 23-29.
- [22] J.Klamka, Stochastic controllability of linear systems with state delays, *International Journal of Applied Mathematics and Computer Science*, 17 (2007), 5-13.
- [23] N.I.Mahmudov, Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces, *SIAM J. Control Optim.*, 42 (2003), 1604-1622.
- [24] N.I.Mahmudov and A. Denker, On controllability of linear stochastic systems, *International Journal of Control*, 73 (2000), 144-151.
- [25] K.S. Miller and B. Ross, —it An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, 1993.
- [26] I.Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [27] R.Sakthivel, R.Ganesh and S.Suganya, Approximate controllability of fractional neutral stochastic system with infinite delay, *R. Math. Physics*, 70 (2012), 291-311.
- [28] R.Sakthivel, S.Suganya and S. M.Anthoni, Approximate controllability of fractional stochastic evolution equations, *Comput. Math. Appl.*, 63 (2012), 660-668.
- [29] R.Sakthivel and E. R. Anandhi, Approximate controllability of impulsive differential equations with state-dependent delay, *Internat. J. Control*, 83 (2010), 387-393.
- [30] R.Sakthivel, P.Revathi and Yong Ren, Existence of solutions for nonlinear fractional stochastic differential equations, *Nonlinear Anal. TMA*, 81 (2013), 70-86.
- [31] X.B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal. TMA*, 74 (2011), 2003-2011.
- [32] L. W. Wang, Approximate controllability for integrodifferential equations with multiple delays, *J. Optim. Theory Appl.*, 143 (2009), 185-206.

## A Note on Minimal Dominating Signed Graphs

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**Abstract:** In this note, we study minimal dominating signed graphs and obtain structural characterization of minimal dominating signed graphs. Further, we characterize signed graphs  $S$  for which  $MD(S) \sim CMD(S)$ , where  $\sim$  denotes switching equivalence and  $MD(S)$  and  $CMD(S)$  are denotes the minimal dominating signed graph and common minimal dominating signed graph of  $S$  respectively.

**Key Words:** Signed graphs, balance, switching, complement, minimal dominating signed graphs, common minimal signed graphs, negation.

**AMS(2010):** 05C22

### §1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [5]; the non-standard will be given in this paper when required. We treat only finite simple graphs without self loops and isolates.

A signed graph is an ordered pair  $S = (S^u, \sigma)$ , where  $S^u$  is a graph  $G = (V, E)$ , called the underlying graph of  $S$  and  $\sigma : E \rightarrow \{+, -\}$  is a function from the edge set  $E$  of  $S^u$  into the set  $\{+, -\}$ , called the signature (or sign in short) of  $S$ . Alternatively, the signed graph can be written as  $S = (V, E, \sigma)$ , with  $V, E, \sigma$  in the above sense. Let  $E^+(S) = \{e \in E : \sigma(e) = +\}$  and  $E^-(S) = \{e \in E : \sigma(e) = -\}$ . The elements of  $E^+(S)$  and  $E^-(S)$  are called positive and negative edges of  $S$ , respectively. A signed graph is all-positive (respectively, all-negative) if all its edges are positive (negative).

A cycle in a signed graph  $S$  is said to be *positive* if it contains an even number of negative edges. A given signed graph  $S$  is said to be *balanced* if every cycle in  $S$  is positive (see [6]). In a signed graph  $S = (S^u, \sigma)$ , for any  $A \subseteq E$  the *sign*  $\sigma(A)$  is the product of the signs on the edges of  $A$ . For more new notions on signed graphs refer the papers ([11, 12, 15, 16], [18]-[24]).

A marked signed graph is an ordered pair  $S_\mu = (S, \mu)$ , where  $S = (S^u, \sigma)$  is a signed graph and  $\mu : V(S^u) \rightarrow \{+, -\}$  is a function from the vertex set  $V(S^u)$  of  $S^u$  into the set  $\{+, -\}$ ,

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called a *marking* of  $S$ . In particular,  $\sigma$  induces a unique marking  $\mu_\sigma$  defined by

$$\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e),$$

where  $E_v$  is the set of edges incident at  $v$  in  $S$ , is called the *canonical marking* of  $S$ . We shall denote by  $\mathcal{M}_S$  the set of all markings of  $S$ . A signed graph  $S$  together with one of its markings  $\mu$  is denoted by  $S_\mu$ .

The following characterization of balanced signed graphs is well known.

**Proposition 1.1**(E.Sampathkumar [14]) *A signed graph  $S = (G, \sigma)$  is balanced if, and only if, there exists a marking  $\mu$  of its vertices such that each edge  $uv$  in  $S$  satisfies  $\sigma(uv) = \mu(u)\mu(v)$ .*

Given a marking  $\mu$  of  $S$ , by switching  $S$  with respect to  $\mu$  we mean changing the sign of every edge of  $S$  to its opposite whenever its end vertices are of opposite signs in  $S_\mu$ . The signed graph obtained in this way is denoted by  $S_\mu(S)$  and is called the  $\mu$ -switched signed graph or just switched signed graph when the marking is clear from the context (Sampathkumar et al. [17]).

We say that signed graph  $S_1$  *switches to* signed graph  $S_2$  (or that they are *switching equivalent* to each other), written as  $S_1 \sim S_2$ , whenever there exists  $\mu \in \mathcal{M}_{S_1}$  such that  $S_\mu(S_1) \cong S_2$ , where “ $\cong$ ” denotes the isomorphism between any two signed graphs in the standard sense. Note that  $S_1 \sim S_2$  implies that  $(S_1)^u \cong (S_2)^u$ .

Two signed graphs  $S_1 = (G, \sigma)$  and  $S_2 = (G', \sigma')$  are said to be *weakly isomorphic* (see [26]) or *cycle isomorphic* (see [28]) if there exists an isomorphism  $f : G \rightarrow G'$  such that the sign of every cycle  $Z$  in  $S_1$  equals to the sign of  $f(Z)$  in  $S_2$ . The following result will also be useful in our further investigation.

**Proposition 1.2**(T.Zaslavsky [28]) *Two signed graphs  $S_1$  and  $S_2$  with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

In [17], the authors introduced the switching and cycle isomorphism for signed digraphs.

## §2. Minimal Dominating Signed Graph

Mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In 1862, de Jaenisch [3] attempted to determine the minimum number of queens required to cover an  $n \times n$  chess board. In 1892, W. W. Rouse Ball [13] reported three basic types of problems that chess players studied during that time.

The study of domination in graphs was further developed in the late 1950s and 1960s, beginning with Berge [1] in 1958. Berge wrote a book on graph theory, in which he introduced the “coefficient of external stability”, which is now known as the domination number of a graph. Oystein Ore [10] introduced the terms “dominating set” and “domination number” in his book on graph theory which was published in 1962. The problems described above were studied

in more detail around 1964 by brothers Yaglom and Yaglom [27]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [2] published a survey paper, in which the notation  $\gamma(G)$  was first used for the domination number of a graph  $G$ . Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

Let  $G = (V, E)$  be a graph. A set  $D \subseteq V$  is a dominating set of  $G$ , if every vertex in  $V - D$  is adjacent to some vertex in  $D$ . A dominating set  $D$  of  $G$  is minimal, if for any vertex  $v \in D$ ,  $D - \{v\}$  is not a dominating set of  $G$  (See, Ore [10]).

Let  $\mathcal{S}$  be a finite set and  $F = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n\}$  be a partition of  $S$ . Then the *intersection graph*  $\Omega(F)$  of  $F$  is the graph whose vertices are the subsets in  $F$  and in which two vertices  $\mathcal{S}_i$  and  $\mathcal{S}_j$  are adjacent if and only if  $\mathcal{S}_i \cap \mathcal{S}_j \neq \phi$ ,  $i \neq j$ .

Kulli and Janakiram [8] introduced a new class of intersection graphs in the field of domination theory. The *minimal dominating graph*  $MD(G)$  of a graph  $G$  is the intersection graph defined on the family of all minimal dominating sets of vertices in  $G$ .

We now extend the notion of  $MD(G)$  to the realm of signed graphs. The *minimal dominating signed graph*  $MD(S)$  of a signed graph  $S = (S^u, \sigma)$  is a signed graph whose underlying graph is  $MD(G)$  and sign of any edge  $PQ$  in  $MD(S)$  is  $\mu(P)\mu(Q)$ , where  $\mu$  is the canonical marking of  $S$ ,  $P$  and  $Q$  are any two minimal dominating sets of vertices in  $S^u$ . Further, a signed graph  $S = (G, \sigma)$  is called minimal dominating signed graph, if  $S \cong MD(S')$  for some signed graph  $S'$ . In this paper we will give a structural characterization of which signed graphs are common minimal dominating signed graph. The following result indicates the limitations of the notion  $CMD(S)$  introduced above, since the entire class of unbalanced signed graphs is forbidden to be minimal dominating signed graphs.

**Proposition 2.1** *For any signed graph  $S = (G, \sigma)$ , its minimal dominating signed graph  $MD(S)$  is balanced.*

*Proof* Since sign of any edge  $PQ$  in  $MD(S)$  is  $\mu(P)\mu(Q)$ , where  $\mu$  is the canonical marking of  $S$ , by Proposition 1.1,  $MD(S)$  is balanced.  $\square$

For any positive integer  $k$ , the  $k^{th}$  iterated minimal dominating signed graph  $MD(S)$  of  $S$  is defined as follows:

$$MD^0(S) = S, MD^k(S) = MD(MD^{k-1}(S))$$

**Corollary 2.2** *For any signed graph  $S = (G, \sigma)$  and any positive integer  $k$ ,  $MD^k(S)$  is balanced.*

**Proposition 2.3** *For any two signed graphs  $S_1$  and  $S_2$  with the same underlying graph, their minimal dominating signed graphs are switching equivalent.*

*Proof* Suppose  $S_1 = (S_1^u, \sigma)$  and  $S_2 = (S_2^u, \sigma')$  be two signed graphs with  $S_1^u \cong S_2^u$ . By Proposition 2.1,  $MD(S_1)$  and  $MD(S_2)$  are balanced and hence, the result follows from Proposition 1.2.  $\square$

In [25], the authors introduced the notion common minimal dominating signed graph of a signed graph as follows:

A *common minimal dominating signed graph*  $CMD(S)$  of a signed graph  $S = (G, \sigma)$  is such a signed graph whose underlying graph is  $CMD(G)$  and sign of any edge  $uv$  in  $CMD(S)$  is  $\mu(u)\mu(v)$ , where  $\mu$  is the canonical marking of  $S$ .

The following result restricts the class of minimal dominating graphs.

**Proposition 2.4** *For any signed graph  $S = (G, \sigma)$ , its common minimal dominating signed graph  $CMD(S)$  is balanced.*

We now characterize the signed graphs whose minimal dominating signed graphs and common minimal dominating signed graphs are switching equivalent. In case of graphs the following result is due to Kulli and Janakiram [9]:

**Proposition 2.5**(Kulli and Janakiram [9]) *If  $G$  is a  $(p-3)$ -regular graph and every minimal dominating set of  $G$  is independent, then  $MD(G) \cong CMD(G)$ .*

**Proposition 2.6** *For any signed graph  $S = (G, \sigma)$ ,  $MD(S) \sim CMD(S)$  if, and only if,  $G$  is a  $(p-3)$ -regular graph and every minimal dominating set of  $G$  is independent.*

*Proof* Suppose  $MD(S) \sim CMD(S)$ . This implies,  $MD(G) \cong CMD(G)$  and hence by Proposition 2.5, we see that the graph  $G$  must be  $(p-3)$ -regular graph and every minimal dominating set of  $G$  is independent.

Conversely, suppose that  $G$  is  $(p-3)$ -regular graph and every minimal dominating set of  $G$  is independent. Then  $MD(G) \cong CMD(G)$  by Proposition 2.5. Now, if  $S$  is a signed graph with underlying graph as  $(p-3)$ -regular graph and every minimal dominating set of  $G$  is independent, by Propositions 2.1 and 2.4,  $MD(S)$  and  $CMD(S)$  are balanced and hence, the result follows from Proposition 1.2.  $\square$

The notion of *negation*  $\eta(S)$  of a given signed graph  $S$  defined in [7] as follows:

$\eta(S)$  has the same underlying graph as that of  $S$  with the sign of each edge opposite to that given to it in  $S$ . However, this definition does not say anything about what to do with nonadjacent pairs of vertices in  $S$  while applying the unary operator  $\eta(\cdot)$  of taking the negation of  $S$ .

Proposition 2.6 provides easy solutions to other signed graph switching equivalence relations, which are given in the following result.

**Corollary 2.7** *For any signed graph  $S = (G, \sigma)$ ,  $MD(\eta(S)) \sim CMD(S)$  (or  $MD(S) \sim CMD(\eta(S))$  or  $MD(\eta(S)) \sim CMD(\eta(S))$ ) if, and only if,  $G$  is a  $(p-3)$ -regular graph and every minimal dominating set of  $G$  is independent.*

For a signed graph  $S = (G, \sigma)$ , the  $MD(S)$  is balanced (Proposition 2.1). We now examine, the conditions under which negation of  $MD(S)$  is balanced.



**Proposition 2.8** *Let  $S = (G, \sigma)$  be a signed graph. If  $MD(G)$  is bipartite then  $\eta(MD(S))$  is balanced.*

*Proof* Since, by Proposition 2.1,  $MD(S)$  is balanced, each cycle  $C$  in  $MD(S)$  contains even number of negative edges. Also, since  $MD(G)$  is bipartite, all cycles have even length; thus, the number of positive edges on any cycle  $C$  in  $MD(S)$  is also even. Hence  $\eta(MD(S))$  is balanced.  $\square$

### §3. Characterization of Minimal Dominating Signed Graphs

The following result characterizes signed graphs which are minimal dominating signed graphs.

**Proposition 3.1** *A signed graph  $S = (G, \sigma)$  is a minimal dominating signed graph if, and only if,  $S$  is a balanced signed graph and its underlying graph  $G$  is a  $MD(G)$ .*

*Proof* Suppose that  $S$  is balanced and its underlying graph  $G$  is a minimal dominating graph. Then there exists a graph  $H$  such that  $MD(H) \cong G$ . Since  $S$  is balanced, by Proposition 1.1, there exists a marking  $\mu$  of  $G$  such that each edge  $uv$  in  $S$  satisfies  $\sigma(uv) = \mu(u)\mu(v)$ . Now consider the signed graph  $S' = (H, \sigma')$ , where for any edge  $e$  in  $H$ ,  $\sigma'(e)$  is the marking of the corresponding vertex in  $G$ . Then clearly,  $MD(S') \cong S$ . Hence  $S$  is a common dominating signed graph.

Conversely, suppose that  $S = (G, \sigma)$  is a minimal dominating signed graph. Then there exists a signed graph  $S' = (H, \sigma')$  such that  $MD(S') \cong S$ . Hence by Proposition 2.1,  $S$  is balanced.  $\square$

### References

- [1] C.Berge, *Theory of Graphs and its Applications*, Methuen, London, 1962.
- [2] E.J.Cockayne and S.T.Hedetniemi, Towards a theory of domination in graphs, *Networks*, 7(1977), 247-261.
- [3] C.F.De Jaenisch, *Applications de l'Analyse mathématique au Jeu des Echecs*, 1862.
- [4] David Easley and Jon Kleinberg, *Networks, Crowds, and Markets: Reasoning About a Highly Connected World*, Cambridge University Press, 2010.
- [5] F.Harary, *Graph Theory*, Addison-Wesley Publishing Co., 1969.
- [6] F.Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, 2(1953), 143-146.
- [7] F.Harary, Structural duality, *Behav. Sci.*, 2(4) (1957), 255-265.
- [8] V.R.Kulli and B.Janakiram, The minimal dominating graph, *Graph Theory Notes of New York*, XXVIII (1995), 12-15.
- [9] V.R.Kulli and B.Janakiram, On common minimal dominating graphs, *Graph Theory Notes of New York*, XXXIV (1998), 9-10.
- [10] O.Ore, *Theory of Graphs*. Amer. Math. Soc. Colloq. Publ., 38, 1962.
- [11] R.Rangarajan and P.Siva Kota Reddy, The edge  $C_4$  signed graph of a signed graph, *South-east Asian Bulletin of Mathematics*, 34(6) (2010), 1077-1082.

- [12] R.Rangarajan, M.S.Subramanya and P.Siva Kota Reddy, Neighborhood signed graphs, *Southeast Asian Bulletin of Mathematics*, 36(3) (2012), 389-397.
- [13] W.W.Rouse Ball, *Mathematical Recreation and Problems of Past and Present Times*, 1892.
- [14] E.Sampathkumar, Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, 7(3) (1984), 91-93.
- [15] E.Sampathkumar, P.Siva Kota Reddy and M.S.Subramanya, Directionally  $n$ -signed graphs, *Ramanujan Math. Soc., Lecture Notes Series (Proc. Int. Conf. ICDM 2008)*, 13 (2010), 155-162.
- [16] E.Sampathkumar, P.Siva Kota Reddy and M.S.Subramanya, Directionally  $n$ -signed graphs-II, *International J. Math. Combin.*, 4 (2009), 89-98.
- [17] E.Sampathkumar, M.S.Subramanya and P.Siva Kota Reddy, Characterization of line sidi-graphs, *Southeast Asian Bulletin of Mathematics*, 35(2) (2011), 297-304.
- [18] P.Siva Kota Reddy and M.S.Subramanya, Note on path signed graphs, *Notes on Number Theory and Discrete Mathematics*, 15(4) (2009), 1-6.
- [19] P.Siva Kota Reddy and M.S.Subramanya, Signed graph equation  $L^k(S) \sim \overline{S}$ , *International J. Math. Combin.*, 4 (2009), 84-88.
- [20] P.Siva Kota Reddy, S.Vijay and V.Lokesha,  $n^{th}$  Power signed graphs, *Proceedings of the Jangjeon Math. Soc.*, 12(3) (2009), 307-313.
- [21] P.Siva Kota Reddy, S.Vijay and V.Lokesha,  $n^{th}$  Power signed graphs, *Proceedings of the Jangjeon Math. Soc.*, 12(3) (2009), 307-313.
- [22] P.Siva Kota Reddy, S.Vijay and V.Lokesha,  $n^{th}$  Power signed graphs-II, *International J. Math. Combin.*, 1 (2010), 74-79.
- [23] P.Siva Kota Reddy and S.Vijay, Total minimal dominating signed graph, *International J. Math. Combin.*, 3 (2010), 11-16.
- [24] P.Siva Kota Reddy and K.V.Madhusudhan, Negation switching equivalence in signed graphs, *International J. Math. Combin.*, 3 (2010), 85-90.
- [25] P.Siva Kota Reddy,  $t$ -Path Sigraths, *Tamsui Oxford J. of Math. Sciences*, 26(4) (2010), 433-441.
- [26] P.Siva Kota Reddy, E.Sampathkumar and M.S.Subramanya, Common-edge signed graph of a signed graph, *J. Indones. Math. Soc.*, 16(2) (2010), 105-112.
- [27] P.Siva Kota Reddy, B.Prashanth, and T.R.Vasanth Kumar, Antipodal signed directed graphs, *Advn. Stud. Contemp. Math.*, 21(4) (2011), 355-360.
- [28] P.Siva Kota Reddy, B.Prashanth and Kavita S.Permi, A note on antipodal signed graphs, *International J. Math. Combin.*, 1 (2011), 107-112.
- [29] P.Siva Kota Reddy, Kavita S.Permi and K.R.Rajanna, Combinatorial aspects of a measure of rank correlation due to Kendall and its relation to complete signed digraphs, *International J. Math. Combin.*, 1 (2012), 74-77.
- [30] P.Siva Kota Reddy and B.Prashanth,  $\mathcal{S}$ -Antipodal signed graphs, *Tamsui Oxford J. of Inf. Math. Sciences*, 28(2) (2012), 165-174.
- [31] P.Siva Kota Reddy and S.Vijay, The super line signed graph  $\mathcal{L}_r(S)$  of a signed Graph, *Southeast Asian Bulletin of Mathematics*, 36(6) (2012), 875-882.

- [32] P.Siva Kota Reddy and B.Prashanth, The common minimal dominating signed graph, *Transactions on Combinatorics*, 1(3) (2012), 39-46.
- [33] P.Siva Kota Reddy and U.K.Misra, The equitable associate signed graphs, *Bull. Int. Math. Virtual Inst.*, 3(1) (2013), 15-20.
- [34] P.Siva Kota Reddy, K.R.Rajanna and Kavita S Permi, The common minimal common neighborhood dominating signed graphs, *Transactions on Combinatorics*, 2(1) (2013), 1-8.
- [35] P.Siva Kota Reddy, Smarandache directionally  $n$ -signed graphs: A Survey, *International J. Math. Combin.*, 2 (2013), 34-43.
- [36] P.Siva Kota Reddy and U.K.Misra, Graphoidal signed graphs, *Advn. Stud. Contemp. Math.*, 23(3) (2013), 451-460.
- [37] P.Siva Kota Reddy and U.K.Misra, Directionally  $n$ -signed graphs-III: The notion of symmetric balance, *Transactions on Combinatorics*, 2(4) (2013), 53-62.
- [38] T.Sozánsky, Enumeration of weak isomorphism classes of signed graphs, *J. Graph Theory*, 4(2)(1980), 127-144.
- [39] A.M.Yaglom and I.M.Yaglom. Challenging mathematical problems with elementary solutions. Volume 1: *Combinatorial Analysis and Probability Theory*, 1964.
- [40] T.Zaslavsky, Signed graphs, *Discrete Appl. Math.*, 4(1)(1982), 47-74.

## Number of Spanning Trees for Shadow of Some Graphs

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**Abstract:** In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this paper we derive simple formulas of the complexity, number of spanning trees of shadow of some graphs, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

**Key Words:** Complexity of graphs, number of spanning trees, shadow graphs, Chebyshev polynomials.

**AMS(2010):** 05C05,05C50

### §1. Introduction

In this work we deal with simple and finite undirected graphs  $G = (V, E)$ , where  $V$  is the vertex set and  $E$  is the edge set. For a graph  $G$ , a spanning tree in  $G$  is a tree which has the same vertex set as  $G$ . The number of spanning trees in  $G$ , also called, the complexity of the graph, denoted by  $\tau(G)$ , is a well-studied quantity (for long time). A classical result of Kirchhoff [16] can be used to determine the number of spanning trees for  $G = (V, E)$ . Let  $V = \{v_1, v_2, \dots, v_n\}$ , then the Kirchhoff matrix  $H$  defined as  $n \times n$  characteristic matrix  $H = D - A$ , where  $D$  is the diagonal matrix of the degrees of  $G$  and  $A$  is the adjacency matrix of  $G$ ,  $H = [a_{ij}]$  defined as follows:

- (1)  $a_{ij} = -1$ ,  $v_i$  and  $v_j$  are adjacent and  $i \neq j$ ;
- (2)  $a_{ij}$  equals the degree of vertex  $v_i$  if  $i = j$ , and
- (3)  $a_{ij} = 0$  otherwise. All of co-factors of  $H$  are equal to  $\tau(G)$ .

There are other methods for calculating  $\tau(G)$ . Let  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$  denote the eigenvalues of  $H$  matrix of a  $p$  point graph. Then it is easily shown that  $\mu_p = 0$ . Furthermore, Kelmans and Chelnokov [15] shown that  $\tau(G) = 1/p \prod_{k=1}^{p-1} \mu_k$ . The formula for the number of

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spanning trees in a  $d$ -regular graph  $G$  can be expressed as  $\tau(G) = 1/p \prod_{k=1}^{p-1} (d - \lambda_k)$  where  $\lambda_0 = d, \lambda_1, \lambda_2, \dots, \lambda_{p-1}$  are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exists simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such result is due to Cayley [3] who showed that complete graph on  $n$  vertices,  $K_n$  has  $n^{n-2}$  spanning trees that he showed  $\tau(K_n) = n^{n-2}$ ,  $n \geq 2$ . Another result,  $\tau(K_{p,q}) = p^{q-1} q^{p-1}$ ,  $p, q \geq 1$ , where  $K_{p,q}$  is the complete bipartite graph with bipartite sets containing  $p$  and  $q$  vertices, respectively. It is well known, as in e.g., ([4], [18]). Another result is due to Sedlacek [19] who derived a formula for the wheel on  $n+1$  vertices,  $W_{n+1}$ , he showed that  $\tau(W_{n+1}) = (3 + \sqrt{5}/2)^n + (3 - \sqrt{5}/2)^n - 2$  for  $n \geq 3$ . Sedlacek [20] also later derived a formula for the number of spanning trees in a Mobius ladder. The Mobius ladder  $M_n, \tau(M_n) = \frac{n}{2}[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n + 2]$  for  $n \geq 2$ . Another class of graphs for which an explicit formula has been derived is based on a prism. Boesch, et al. [1] and [2]. Douad, ([5]-[14]) later derived formulas for the number of spanning trees for many graphs. Now, we can introduce the following lemma:

**Lemma 1.1** ([5])  $\tau(G) = \frac{1}{n^2} \det(nI - \bar{D} + \bar{A})$ , where  $\bar{A}, \bar{D}$  are the adjacency and degree matrices of  $\bar{G}$ , the complement of  $G$ , respectively and  $I$  is the  $n \times n$  unit matrix.

The advantage of these formula is to express  $\tau(G)$  directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

## §2. Chebyshev Polynomials

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, Yuanping, et al. [21].

Let  $A_n(x)$  be  $n \times n$  matrix such that

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & & \\ -1 & 2x & -1 & 0 & \\ 0 & \ddots & \ddots & & \ddots \\ & \ddots & \ddots & \ddots & -1 \\ & & 0 & -1 & 2x \end{pmatrix},$$

where all other elements are zeros. Further we recall that the Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x). \quad (2.1)$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n} \frac{d}{dx} T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)}. \quad (2.2)$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0. \quad (2.3)$$

It can then be shown from this recursion that by expanding  $\det A_n(x)$  one gets

$$U_n(x) = \det(A_n(x)), n \geq 1. \quad (2.4)$$

Furthermore, by using standard methods for solving the recursion (2.3), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}], n \geq 1, \quad (2.5)$$

where the identity is true for all complex  $x$  (except at  $x = \pm 1$  where the function can be taken as the limit). The definition of  $U_n(x)$  easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x) = 2^{n-1} \prod_{j=1}^{n-1} (x - \cos(\frac{j\pi}{n})).$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1} U_{n-1}(x)$$

These two results yield another formula for  $U_n(x)$ ,

$$U_{n-1}^2(x) = 4^{n-1} \prod_{j=1}^{n-1} (x^2 - \cos^2(\frac{j\pi}{n})).$$

Finally, simple manipulation of the above formula yields the following, which also will be extremely useful to us latter:

$$U_{n-1}^2(\sqrt{\frac{x+2}{4}}) = \prod_{j=1}^{n-1} (x - 2\cos(\frac{2j\pi}{n})).$$

Furthermore one can show that

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}[1 - T_{2n}] = \frac{1}{2(1-x^2)}[1 - T_n(2x^2 - 1)].$$

and

$$T_n(x) = \frac{1}{2}[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n].$$

Now let  $B_n(x), C_n(x), D_n(x)$  and  $E_n(x)$  be  $n \times n$  matrices.

**Lemma 2.1**([18])

(i)

$$B_n(x) = \begin{pmatrix} x & -1 & 0 & & \\ -1 & 1+x & -1 & 0 & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & -1 & x+1 & -1 \\ & & 0 & -1 & x \end{pmatrix} \Rightarrow \det(B_n(x)) = (x-1)U_{n-1}(\frac{1+x}{2}).$$

(ii)

$$C_n(x) = \begin{pmatrix} x & 0 & 1 & & \\ 0 & 1+x & 0 & \ddots & \\ 1 & 0 & \ddots & \ddots & 1 \\ & \ddots & \ddots & x+1 & 0 \\ & & 1 & 0 & x \end{pmatrix} \Rightarrow \det(C_n(x)) = (n+x-2)U_{n-1}\left(\frac{x}{2}\right), n \geq 3, x > 2.$$

(iii)

$$D_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & 0 \\ 0 & x & 0 & \ddots & 1 \\ 1 & 0 & x & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 0 & x \end{pmatrix} \Rightarrow \det(D_n(x)) = \frac{2(x+n-3)}{x-3} [T_n(\frac{x-1}{2})], n \geq 3, x \geq 2.$$

(iv)

$$E_n(x) = \begin{pmatrix} x & 1 & 1 & \cdots & \cdots & 1 \\ 1 & x & 1 & \ddots & & \vdots \\ 1 & \ddots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & & \ddots & \ddots & x & 1 \\ 1 & \cdots & \cdots & 1 & 1 & x \end{pmatrix} \Rightarrow \det(E_n(x)) = (x+n-1)(x-1)^{n-1}.$$

**Lemma 2.2**([17]) *Let  $A \in F^{n \times n}$ ,  $B \in F^{n \times m}$ ,  $C \in F^{m \times n}$  and  $D \in F^{m \times m}$  and assume that  $D$  is nonsingular matrix. Then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^{nm} \det(A - BD^{-1}C) \det D.$$

This formula gives some sort of symmetry in some matrices which facilitates our calculation of determinants.

### §3. Complexity of Some Graphs

A shadow graph  $D_2(G)$  of a graph  $G$  is obtained by taking two copies of  $G$  say  $G_1$  and  $G_2$  and join each vertex  $u_i$  in  $G_1$  to the neighbors of the corresponding vertex  $v_i$  in  $G_2$ .

**Theorem 3.1** *Let  $P_n$  be a path graph of order  $n$ . Then*

$$\tau(D_2(P_n)) = 2^{3n-4}; n \geq 2.$$

*Proof* Applying Lemma 1.1, we have

$$\begin{aligned} \tau(D_2(P_n)) &= \frac{1}{(2n)^2} \det(2nI - \bar{D} + \bar{A}) \\ &= \frac{1}{(2n)^2} \det \begin{pmatrix} 3 & 0 & 1 & \cdots & \cdots & 1 & 1 & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 5 & 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 5 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 0 & 3 & 1 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 & 3 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \vdots & 0 & 5 & 0 & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 5 & 0 \\ 1 & \cdots & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 3 \end{pmatrix} \\ &= \frac{1}{(2n)^2} \det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \frac{1}{(2n)^2} \det(A+B) \cdot \det(A-B), (AB=BA). \end{aligned}$$

A straightforward induction using properties of determinants and above mentioned definition of Chebyshev polynomial in Lemma 2.1, we have

$$\begin{aligned} \tau(D_2(P_n)) &= \frac{1}{(2n)^2} \det \begin{pmatrix} 4 & 0 & 2 & \cdots & 2 \\ 0 & 6 & 0 & \ddots & \vdots \\ 2 & 0 & \ddots & \ddots & 2 \\ \vdots & \ddots & \ddots & 6 & 0 \\ 2 & \cdots & 2 & 0 & 4 \end{pmatrix} \times \det \begin{pmatrix} 2 & 0 & \cdots & \cdots & 0 \\ 0 & 4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 4 & 0 \\ 0 & \cdots & \cdots & 0 & 2 \end{pmatrix} \\ &= \frac{1}{(2n)^2} \times 2^n n^2 \times 2^2 \times 4^{n-2} = 2^{3n-4}. \quad \square \end{aligned}$$

**Theorem 3.2** *Let  $C_n$  be a cycle graph of order  $n$ . Then*

$$\tau(D_2(C_n)) = n2^{3n-2}, n \geq 3.$$



*Proof* Applying Lemma 1.1, we have

$$\begin{aligned}
 \tau(D_2(C_n)) &= \frac{1}{(2n)^2} \det(2nI - \bar{D} + \bar{A}) \\
 &= \frac{1}{(2n)^2} \det \begin{pmatrix} 5 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 5 & 0 & 1 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 5 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 5 \end{pmatrix} \\
 &= \frac{1}{(2n)^2} \det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \frac{1}{(2n)^2} \det(A+B) \cdot \det(A-B), (AB=BA).
 \end{aligned}$$

A straightforward induction using properties of determinants and above mentioned definition of Chebyshev polynomial in Lemma 2.1, we have

$$\tau(D_2(C_n)) = \frac{1}{(2n)^2} \times 2^n n^3 \times 4^n = n 2^{3n-2}. \quad \square$$

**Theorem 3.3** *Let  $K_n$  be a complete graph of order  $n$ . Then*

$$\tau(D_2(K_n)) = 2^{2n-2} n^{n-2} (n-1)^n, n \geq 2.$$

*Proof* Applying Lemma 1.1, we have

$$\begin{aligned}
 \tau(D_2(K_n)) &= \frac{1}{(2n)^2} \det(2nI - \bar{D} + \bar{A}) = \frac{1}{(2n)^2} \det \begin{pmatrix} A & I \\ I & A \end{pmatrix} \\
 &= \frac{1}{(2n)^2} \det(A+I) \times \det(A-I),
 \end{aligned}$$

where

$$A = \begin{pmatrix} 2n-1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2n-1 \end{pmatrix}.$$

Thus,

$$\begin{aligned}
\tau(D_2(K_n)) &= \frac{1}{(2n)^2} \det \begin{pmatrix} 2n & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2n \end{pmatrix} \\
&\quad \times \det \begin{pmatrix} 2n-2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2n-2 \end{pmatrix} \\
&= \frac{1}{(2n)^2} \times (2n)^n (2n-2)^n = 2^{2n-2} n^{n-2} (n-1)^n. \quad \square
\end{aligned}$$

**Theorem 3.4** Let  $K_{n,m}$  be a complete bipartite graph. Then

$$\tau(D_2(K_{n,m})) = 2^{n+m-2} n^{2m-1} m^{2n-1}$$

*Proof* Applying Lemma 1.1, we have

$$\begin{aligned}
\tau(D_2(K_{n,m})) &= \frac{1}{(2(m+n))^2} \det(2(n+m)I - \bar{D} + \bar{A}) \\
&= \frac{1}{(2(n+m))^2} \det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A+B) \times \det(A-B), (AB=BA) \\
&= \frac{1}{(2(n+m))^2} \\
&\quad \times \det \begin{pmatrix} 2m+2 & 2 & \cdots & 2 & 0 & \cdots & \cdots & 0 \\ 2 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \vdots \\ 2 & \cdots & 2 & 2m+2 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 2n+2 & 2 & \cdots & 2 \\ \vdots & \ddots & \ddots & \vdots & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 2 \\ 0 & \cdots & \cdots & 0 & 2 & \cdots & 2 & 2n+2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& \times \det \begin{pmatrix} 2m & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2m & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 2n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 2n \end{pmatrix} \\
& = \frac{1}{(2(n+m))^2} \det \begin{pmatrix} 2m+2 & 2 & \cdots & 2 \\ 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2 & \cdots & 2 & 2m+2 \end{pmatrix} \\
& \quad \times \det \begin{pmatrix} 2n+2 & 2 & \cdots & 2 \\ 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2 & \cdots & 2 & 2n+2 \end{pmatrix} \\
& \quad \times \det \begin{pmatrix} 2m & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2m \end{pmatrix} \times \det \begin{pmatrix} 2n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2n \end{pmatrix} \\
& = \frac{1}{(2(m+n))^2} \times 2^{2n+2m} (m+n)^2 (m)^{n-1} (n)^{m-1} m^n n^m \\
& = 4^{m+n-1} (m)^{2n-1} (n)^{2m-1}.
\end{aligned}$$

□

**Theorem 3.5** *Let  $F_n$  be the fan graph of order  $n$ . Then*

$$\tau(D_2(F_n)) = \frac{16 \times 6^{n-2} n}{\sqrt{5}} ((3 + \sqrt{5})^n - (3 - \sqrt{5})^n), n \geq 2.$$

*Proof* Applying Lemma 1.1, we have

$$\begin{aligned}
\tau(D_2(F_n)) &= \frac{1}{(2(n+1))^2} \det(2n+1)I - \bar{D} + \bar{A}) \\
&= \frac{1}{(2(n+1))^2} \times
\end{aligned}$$

$$\begin{aligned}
& \times \det \begin{pmatrix} 2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 5 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & 0 & 7 & \ddots & \ddots & \ddots & \vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 7 & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & \cdots & 1 & 0 & 5 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 5 & 0 & 1 & \cdots & \cdots & 1 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 0 & 7 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & 7 & 0 \\ 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 5 \end{pmatrix} \\
&= \frac{1}{(2(n+1))^2} \det(A+B) \times \det(A-B), (AB=BA) \\
&= \frac{1}{(2(n+1))^2} \det \begin{pmatrix} 2n+2 & 0 & \cdots & \cdots & \cdots & 0 \\ 2 & 6 & 0 & 2 & \cdots & 2 \\ \vdots & 0 & 8 & \ddots & \ddots & \vdots \\ \vdots & 2 & \ddots & \ddots & \ddots & 2 \\ \vdots & \vdots & \ddots & \ddots & 8 & 0 \\ 0 & 2 & \cdots & 2 & 0 & 6 \end{pmatrix} \times \det \begin{pmatrix} 2n & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 4 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 6 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 6 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 4 \end{pmatrix} \\
&= \frac{1}{(2(n+1))^2} \times (2^{n+1}(n+1)) \det \begin{pmatrix} 3 & 0 & 1 & \cdot & 1 \\ 0 & 4 & 0 & \ddots & \vdots \\ 1 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & 4 & 0 \\ 1 & \cdots & 1 & 0 & 3 \end{pmatrix} \\
&\quad \times (2^{n+1}n) \det \begin{pmatrix} 2 & 0 & \cdots & \cdot & 0 \\ 0 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 & 0 \\ 0 & \cdots & \cdots & 0 & 2 \end{pmatrix}.
\end{aligned}$$

A straightforward induction using properties of determinants and above mentioned definition of Chebyshev polynomial in Lemma 2.1, we have

$$\tau(D_2(F_n)) = \frac{1}{n+1} \times 2^{2n} \times 6^{n-2} \times n \times (n+1) U_{n-1}\left(\frac{3}{2}\right) = \frac{16 \times 6^{n-2} n}{\sqrt{5}} ((3+\sqrt{5})^n - (3-\sqrt{5})^n). \quad \square$$

**Theorem 3.6** *Let  $W_n$  be the wheel graph. Then*

$$\tau(D_2(W_n)) = (6^n \times n)[(3+\sqrt{5})^n + (3-\sqrt{5})^n - 2^{n+1}], n \geq 3.$$

*Proof* Applying Lemma 1.1, we have

$$\begin{aligned} \tau(D_2(W_n)) &= \frac{1}{(2(n+1))^2} \det(2(n+1)I - \bar{D} + \bar{A}) = \frac{1}{2(n+1)^2} \\ &\times \det \begin{pmatrix} 2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 7 & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \cdots & 1 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 0 & 7 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 7 & 0 & 1 & \cdots & 1 & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & 0 & 1 & \cdots & 1 & 0 & 7 \end{pmatrix} \\ &= \frac{1}{(2n)^2} \det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \frac{1}{(2(n+1))^2} \det(A+B) \cdot \det(A-B), (AB=BA) \\ &= \frac{1}{(2(n+1))^2} \det \begin{pmatrix} 2n+2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 8 & 0 & 2 & \cdots & 2 & 0 \\ \vdots & 0 & 8 & \ddots & \ddots & \ddots & 2 \\ \vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 \\ \vdots & 2 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 2 & \cdots & 2 & 0 & 8 \end{pmatrix} \times \end{aligned}$$

$$\begin{aligned}
& \times \det \begin{pmatrix} 2n & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 6 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 6 \end{pmatrix} \\
& = \frac{1}{(2(n+1))^2} (2^{n+1}(n+1)) \times \det \begin{pmatrix} 4 & 0 & 1 & \cdot & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 4 \end{pmatrix} \\
& \times (2^{n+1}n) \times \det \begin{pmatrix} 3 & 0 & \cdots & \cdot & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 3 \end{pmatrix}.
\end{aligned}$$

A straightforward induction using properties of determinants and above mentioned definition of Chebyshev polynomial in Lemma 2.1, we have

$$\begin{aligned}
\tau(D_2(W_n)) &= \frac{1}{n+1} \times 2^{2n} \times 3^n \times n \times (n+1) [T_n(\frac{3}{2}) - 1] \\
&= (n \times 6^n) [(3 + \sqrt{5})^n + (3 - \sqrt{5})^n - 2^{n+1}]. \quad \square
\end{aligned}$$

#### §4. Conclusion

The number of spanning trees  $\tau(G)$  in graphs (networks) is an important invariant. Its evaluation is not only interesting from a mathematical perspective, but also important for reliability of a network and designing electrical circuits. Some computationally hard problems such as the traveling salesman problem can be solved approximately by using spanning trees. Due to the high dependence in network design and reliability on graph theory, we obtained theorems with proofs in this paper.

## References

- [1] Boesch F.T. and Bogdanowicz Z. R., The number of spanning trees in a Prism, *Inter. J. Comput. Math.*, 21, (1987), 229-243.
- [2] Boesch F.T. and Prodinger H., Spanning tree formulas and Chebyshev polynomials, *Graphs and Combinatorics*, 2(1986), 191-200.
- [3] Cayley G. A., A theorme on trees, *Quart. J. Math.*, 23 (1889), 276-378.
- [4] Clark L., On the enumeration of multipartite spanning trees of the complete graph, *Bull. of the ICA*, 38 (2003), 50-60.
- [5] Daoud S.N., Some applications of spanning trees in complete and complete bipartite graph, *American Journal of Applied Sci. Pub.*, 9(4) (2012), 584-592.
- [6] Daoud S.N., Complexity of some special named graphs and Chebyshev polynomials, *International Journal of Applied Mathematics and Statistics*, Vol.32, 2(2013), 77-84.
- [7] Daoud S. N. and Elsonbaty A., Complexity of trapezoidal graphs with different triangulations, *Journal of Combinatorial Number Theory*, Vol.4, 2(2013), 49-59.
- [8] Daoud S. N. and Elsonbaty A., Complexity of some graphs generated by ladder graph, *Journal Applied Mathematics and Statistics*, Vol.36, 6(2013), 87-94.
- [9] Daoud S. N., Chebyshev polynomials and spanning tree formulas, *International J.Math. Combin.*, 4(2012), 68-79.
- [10] Daoud S. N., Number of spanning trees for Splitting of some Graphs, *International J. Math. Sci. and Engg. Appls.*, Vol.7, II(2013), 169-179.
- [11] Daoud S.N., Number of spanning trees of corona of some special graphs, *International J. Math. Sci. and Engg. Appls.*, Vol.7, II(2013), 117-129.
- [12] Daoud S.N., Number of spanning trees of join of some special graphs, *European J. Scientific Research*, Vol.87, 2(2012), 170-181.
- [13] Daoud S. N., Some applications of spanning trees of circulant graphs  $C_6$  and their applications, *Journal of Math. and Statistics Sci. Pub.*, 8(1) (2012), 24-31.
- [14] Daoud S. N., Complexity of cocktail party and crown graph, *American Journal of Applied Sci. Pub.*, 9(2) (2012), 202-207.
- [15] Kelmans A. K. and Chelnokov V. M., A certain polynomials of a graph and graphs with an extremal number of trees, *J. Comb. Theory (B)* 16(1974), 197-214.
- [16] Kirchhoff G. G., Uber die Auflosung der Gleichungen, auf welche man bei der Untersuchung der Linearen Verteilung galvanischer Strome gefuhrt wird, *Ann. Phys. Chem.*, 72 (1847), 497 -508.
- [17] Marcus M., *A Survey of Matrix Theory and Matrix Inequalities*, Unvi. Allyn and Bacon. Inc. Boston, 1964.
- [18] Qiao N. S. and Chen B., The number of spanning trees and chains of graphs, *J. Applied Mathematics*, 9 (2007), 10-16.
- [19] Sedlacek J., On the skeleton of a graph or digraph. In *Combinatorial Structures and their Applications* (R. Guy, M. Hanani, N. Sauer and J. Schonheim, eds), Gordon and Breach, New York (1970), 387-391.
- [20] Sedlacek J., Lucas number in graph theory, In *Mathematics (Geometry and Graph theory)* (Czech), Univ. Karlova, Prague 111-115 (1970).

- [21] Yuanping Z., Xuerong Y., Mordecai J., Chebyshev polynomials and spanning trees formulas for circulant and related graphs, *Discrete Mathematics*, 298 (2005), 334-364.

### Corrigendum

The authors of paper *Special Kinds of Colorable Complements in Graphs*, Vol.3,2013, 35-43 should be B.Chaluvapaju, C.Nandeeshu Kumar and V.Chaitra.

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*To doubt everything or to believe everything are two equally convenient solutions; both dispense with the necessity of reflection.*

By Henri Poincare, a French mathematician and theoretical physicist.

## Author Information

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## Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

## Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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