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First Approximate Exponential Change of Finsler Metric

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Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which a first approximate exponential change of Finsler metric becomes a Projective change. The condition under which a first approximate exponential change of Finsler metric of Douglas space becomes a Douglas space have been also found. The exponential change of Finler metric has been studied [1].

Key Words: Exponential change, projective change, Douglas space.

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§1. Introduction

Let $F^n = (M^n, L)$ is a Finsler space, where L is Finsler function of x and $y = \dot{x}$ and M^n is n-dimensional smooth manifold. In the paper [1] exponential change of Finler metric, i.e. Finlsr metric L changed to $Le^{\beta/L}$ represented by $\overline{L}$ where $\beta = b_i(x)\, y^i$ is one form defined on the manifold M^n. The exponential change of Finsler metric is represented as

$$\overline{L} = L\left\{1 + \frac{\beta}{L} + \frac{1}{2!}\left(\frac{\beta}{L}\right)^2 + \frac{1}{3!}\left(\frac{\beta}{L}\right)^3 + \frac{1}{4!}\left(\frac{\beta}{L}\right)^4 + \dots\right\} \qquad \text{for } \ |\beta| < |L|.$$

Neglecting powers of β higher than 2, $\overline{L}$ approximates to $L + \beta + \frac{\beta^2}{2L}$, which will be called first approximate exponential change of Finler metric L. That is,

$$\overline{L} = L + \beta + \frac{\beta^2}{2L} \tag{1.1}$$

Then Finsler space $\overline{F}^n = (M^n, \overline{L})$ is said to be obtained from Finsler space $F^n = (M^n, L)$ by first approximate exponential change. The quantities corresponding to $\overline{F}^n$ is denoted by putting bar on those quantities.

Some basic tensor of $F^n = (M^n, L)$ are given as follows:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\, \partial y^j}, \qquad l_i = \frac{\partial L}{\partial y^i} = L_i \quad \text{and} \quad h_{ij} = g_{ij} - l_i\, l_j,$$

where g_{ij} is fundamental metric tensor, l_i is normalized element of support and h_{ij} is angular metric tensor.

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Partial derivative with respect to x^i and y^i will be denoted as ∂_i and $\dot{\partial}_i$ respectively and derivatives are written as

$$L_i = \frac{\partial L}{\partial y^i}, \qquad L_{ij} = \frac{\partial^2 L}{\partial y^j\, \partial y^i} \quad \text{and} \quad L_{ijk} = \frac{\partial^3 L}{\partial y^k\, \partial y^j\, \partial y^i}. \tag{1.2}$$

The equation of geodesic of a Finsler space [2] is

$$\frac{d^2 x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where G^i is positively homogeneous function of degree two in y^i and is given by

$$2G^i = \frac{g^{ij}}{2}(y^r \dot{\partial}_j \partial_r L^2 - \partial_j L^2).$$

Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of Finsler space $F^n = (M^n, L)$ is given by [2]

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \qquad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

Cartan connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ is constructed from L with the help of following axioms [3]:

(1) Cartan connection $C\Gamma$ is v-metrical;

(2) Cartan connection $C\Gamma$ is h-metrical;

(3) The $(v)v$ torsion tensor field S^1 of Cartan connection vanishes;

(4) The $(h)h$ torsion tensor field T of Cartan connection vanishes;

(5) The deflection Tensor field D of Cartan connection vanishes,

Denote the h and v-covariant derivative with respect to Cartan connection by $_{|k}$ and $|_k$. Let

$$G^i = G^i + D^i, \tag{1.3}$$

where D^i is difference tensor homogeneous function of second degree in y^i. Then $G^i_j = G^i_j + D^i_j$, $G^i_{jk} = G^i_{jk} + D^i_{jk}$, where $D^i_j = \frac{\partial D^i}{\partial y^j}$ and $D^i_{jk} = \frac{\partial D^i_j}{\partial y^k}$ are homogeneous function of degree 1 and 0 in y^i respectively.

§2. Difference Tensor D^j

From (1.1) and (1.2) we have,

$$\overline{L}_i = \left(1 - \frac{\beta^2}{2L^2}\right) L_i + \left(1 + \frac{\beta}{L}\right) b_i, \tag{2.1}$$

$$\overline{L}_{ij} = \left(1 - \frac{\beta^2}{2L^2}\right) L_{ij} + \frac{\beta^2}{L^3} L_i L_j - \frac{\beta}{L^2}(L_i b_j + L_j b_i) + \frac{1}{L} b_i b_j, \tag{2.2}$$

$$\overline{L}_{ijk} = \left(1 - \frac{\beta^2}{2L^2}\right) L_{ijk} + \frac{\beta^2}{L^3}(L_{ij} L_k + L_{ik} L_j + L_{jk} L_i) - \frac{\beta}{L^2}(L_{ij} b_k + L_{ik} b_j + L_{jk} b_i)$$

$$+\frac{2\beta}{L^3}(L_iL_jb_k+L_iL_kb_j+L_jL_kb_i)-\frac{1}{L^2}(L_ib_jb_k+L_jb_ib_k+L_kb_jb_i)-\frac{3\beta^2}{L^4}L_iL_jL_k, \tag{2.3}$$

$$\partial_j\overline{L}_i=\left(1-\frac{\beta^2}{2L^2}\right)\partial_jL_i+\frac{\beta}{L^3}(\beta L_i-Lb_i)\partial_jL+\frac{1}{L^2}(Lb_i-\beta L_i)\partial_j\beta+\left(1+\frac{\beta}{L}\right)\partial_jb_i, \tag{2.4}$$

$$\begin{aligned}\partial_k\overline{L}_{ij}=\Big\{&\left(1-\frac{\beta^2}{2L^2}\right)\partial_kL_{ij}+\Big(\frac{\beta^2}{L^3}L_{ij}-\frac{3\beta^2}{L^4}L_iL_j+\frac{2\beta}{L^3}(L_ib_j+L_jb_i)-\frac{1}{L^2}b_ib_j\Big)\partial_kL\\&-\left(\frac{1}{L^2}(L_ib_j+L_jb_i)+\frac{\beta}{L^2}L_{ij}-\frac{2\beta}{L^3}L_iL_j\right)\partial_k\beta+\frac{\beta}{L^3}(\beta L_i-Lb_i)\partial_kL_j\\&+\frac{\beta}{L^3}(\beta L_j-Lb_j)\partial_kL_i+\frac{1}{L^2}(Lb_i-\beta L_i)\partial_kb_j+\frac{1}{L^2}(Lb_j-\beta L_j)\partial_kb_i\Big\}.\end{aligned} \tag{2.5}$$

Now in $\overline{F}^n$ and F^n, we have

$$\overline{L}_{ij|k}=0\Rightarrow\qquad\partial_k\overline{L}_{ij}-\overline{L}_{ijr}\overline{G}^r_k-\overline{L}_{ir}\overline{F}^r_{jk}-\overline{L}_{jr}\overline{F}^r_{ik}=0, \tag{2.6}$$

$$L_{ij|k}=0\Rightarrow\qquad\partial_kL_{ij}-L_{ijr}G^r_k-L_{ir}F^r_{jk}-L_{jr}F^r_{ik}=0, \tag{2.7}$$

$$\overline{G}^r_k=G^r_k+D^r_k\quad\text{ and }\quad\overline{F}^r_{ik}=F^r_{ik}+D^{*r}_{ik}.$$

Putting the value from (2.2), (2.3), (2.5) and (2.7) in (2.6) and contract the resulting equation by y^k, we have

$$\begin{aligned}&\left\{\frac{1}{L^2}(L_ib_j+L_jb_i)+\frac{\beta}{L^2}L_{ij}-\frac{2\beta}{L^3}L_iL_j\right\}r_{00}-\frac{1}{L^2}(Lb_i-\beta L_i)(r_{j0}+s_{j0})\\&-\frac{1}{L^2}(Lb_j-\beta L_j)(r_{i0}+s_{i0})+2\overline{L}_{ijr}D^r+\overline{L}_{ir}D^r_j+\overline{L}_{jr}D^r_i=0,\end{aligned} \tag{2.8}$$

where '0' denotes contraction with y^k.

Now deal with following equations in $\overline{F}^n$ and F^n

$$\overline{L}_{i|j}=0\Rightarrow\qquad\partial_j\overline{L}_i-\overline{L}_{ir}\overline{G}^r_j-\overline{L}_r\overline{F}^r_{ij}=0, \tag{2.9}$$

$$L_{i|j}=0\Rightarrow\qquad\partial_jL_i-L_{ir}G^r_j-L_rF^r_{ij}=0. \tag{2.10}$$

Putting the value from (2.1), (2.2), (2.4) and (2.10) in (2.9), we have

$$\left(1+\frac{\beta}{L}\right)b_{i|j}=\overline{L}_{ir}D^r_j+\overline{L}_rD^{*r}_{ij}+\frac{1}{L^2}(\beta L_i-Lb_i)(r_{j0}+s_{j0}). \tag{2.11}$$

Since

$$2r_{ij}=b_{i|j}+b_{j|i}, \tag{2.12}$$

therefore putting the value from (2.11) in (2.12), we have

$$2\left(1+\frac{\beta}{L}\right)r_{ij}=\overline{L}_{ir}D^r_j+\overline{L}_{jr}D^r_i+2\overline{L}_rD^{*r}_{ij}+\frac{1}{L^2}(\beta L_i-Lb_i)(r_{j0}+s_{j0})+\frac{1}{L^2}(\beta L_j-Lb_j)(r_{i0}+s_{i0}). \tag{2.13}$$

Subtract (2.8) from (2.13) and contract the resulting equation by $y^i\,y^j$, we get

$$\left(1-\frac{\beta^2}{2L^2}\right)L_rD^r+\left(1+\frac{\beta}{L}\right)b_rD^r=\frac{1}{2}\left(1+\frac{\beta}{L}\right)r_{00}. \tag{2.14}$$

Since

$$2s_{ij} = b_{i|j} - b_{j|i}, \tag{2.15}$$

therefore putting the value from (2.11) in (2.15), we have

$$2\left(1+\frac{\beta}{L}\right)s_{ij} = \overline{L}_{ir}D^r_j - \overline{L}_{jr}D^r_i + \frac{1}{L^2}(\beta L_i - Lb_i)(r_{j0}+s_{j0})$$

$$-\frac{1}{L^2}(\beta L_j - Lb_j)(r_{i0}+s_{i0}). \tag{2.16}$$

Subtract (2.8) from (2.16) and contract the resulting equation by $y^j b^i$, we have

$$\beta\{3\beta^2 - 2(1+b^2)L^2\}L_r D^r - L\{3\beta^2 - 2(1+b^2)L^2\}b_r D^r$$

$$= L\{2L^2(\beta+L)s_0 + r_{00}(L^2b^2-\beta^2)\}, \tag{2.17}$$

Solution of algebraic equation (2.14) and (2.17) is given by

$$b_r D^r = \frac{2L^2(2L^2-\beta^2)(\beta+L)s_0 + \{(L^2b^2-\beta^2)(\beta^2+2\beta L+2L^2) + \beta(\beta+L)(2L^2-\beta^2)\}r_{00}}{2(\beta^2+2\beta L+2L^2)\{2(1+b^2)L^2-3\beta^2\}}, \tag{2.18}$$

$$L_r D^r = \frac{L(\beta+L)\{(2L^2-\beta^2)r_{00} - 4L^2(\beta+L)s_0\}}{2(\beta^2+2\beta L+2L^2)(2(1+b^2)L^2-3\beta^2)}. \tag{2.19}$$

Subtract (2.8) from (2.16) and contract the resulting equation by y^j, we have

$$\left(1+\frac{\beta}{L}\right)s_{i0} + \frac{1}{2L^2}(Lb_i - \beta L_i)r_{00} = \overline{L}_{ir}D^r. \tag{2.20}$$

Putting the value from (2.2) in (2.20) using $LL_{ir} = g_{ir} - L_iL_r$, $L_i = l_i$ and contracting the resulting equation by g^{ij}, we have

$$D^j = \frac{2L^2(\beta+L)}{(2L^2-\beta^2)}s^j_0 + \frac{2L^2\{(2L^2-\beta^2)r_{00} - 4L^2(\beta+L)s_0\}}{(2L^2-\beta^2)\{2(1+b^2)L^2-3\beta^2\}}\left[\frac{(2L^3-3\beta^2L-2\beta^3)}{L^2(\beta^2+2\beta L+2L^2)}y^j + b^j\right]. \tag{2.21}$$

Proposition 2.1 *Difference tensor of first approximate exponential change of Finsler metric L is given by equations (2.21).*

§3. Projective Change of Finsler Metric

Definition 3.1([4]) *A Finsler space $\overline{F}^n$ is called projective to Finsler space F^n if there is geodesics correspond between $\overline{F}^n$ and F^n. That is, $L \to \overline{L}$ is projective if $\overline{G}^i = \overline{G}^i + P(x,y)\,y^i$, where $P(x,y)$ is called projective factor, this is homogeneous scalar function of degree one in y^i.*

Putting $D^j = Py^j$ in equation (2.21), where P is projective factor and contracting the resulting equation by y_j, we have

$$P = \frac{(\beta + L)\{(2L^2 - \beta^2)r_{00} - 4L^2(\beta + L)s_0\}}{(\beta^2 + 2\beta L + 2L^2)\{2(1 + b^2)L^2 - 3^2\}}. \tag{3.1}$$

Putting $D^j = Py^j$ in equation (2.21) the value from (3.1) in (2.21), we get

$$\frac{2\{(2L^2 - \beta^2)r_{00} - 4L^2(\beta + L)s_0\}}{(2L^2 - \beta^2)\{2(1 + b^2)L^2 - 3\beta^2\}}(\beta y^j - L^2 b^j) = \frac{2L^2(\beta + L)}{(2L^2 - \beta^2)}s^j_0. \tag{3.2}$$

Contracting (3.2) by b_j, we have

$$r_{00} = \frac{2L^2(\beta + L)}{(\beta^2 - L^2 b^2)}s_0. \tag{3.3}$$

Putting the value from (3.3) in (3.1), we have

$$P = \frac{2L^2(L + \beta)^2}{(\beta^2 - L^2 b^2)\{(\beta^2 + 2\beta L + 2L^2)\}}s_0. \tag{3.4}$$

Eliminating P and r_{00} from (3.4), (3.3) and (2.21), we have

$$s^j_0 = \left\{b^j - \left(\frac{\beta}{L^2}\right)y^j\right\}\frac{L^2 s_0}{(L^2 b^2 - \beta^2)}. \tag{3.5}$$

Equation (3.3)and (3.5) are necessary condition for first approximate exponential change of Finsler metric to be projective.

Conversely, if condition (3.3)and (3.5) are satisfied, then put these value in (2.21), we have

$$D^j = \frac{2L^2(L + \beta)^2}{(\beta^2 - L^2 b^2)(\beta^2 + 2\beta L + 2L^2)}s_0\, y^j = Py^j.$$

That is $\overline{F}^n$ is projective to F^n.

Theorem 3.1 *The first approximate exponential change of Finsler space is projective iff equation (3.3) and (3.5) are satisfied and then projective factor P is given by* $P = \frac{2L^2(L+\beta)^2}{(\beta^2 - L^2b^2)\{(\beta^2+2\beta L+2L^2)\}}s_0$.

§4. Douglas Space

Definition 4.1([5]) *A Finsler space F^n is called Douglas space if $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i. In brief, homogeneous polynomial of degree r in y^i is denoted by hp(r).*

If we denote

$$B^{ij} = D^i y^j - D^j y^i \tag{4.1}$$

from equation (2.21), we have

$$B^{ij} = \frac{2L^2\{(2L^2 - \beta^2)r_{00} - 4L^2(\beta + L)s_0\}}{(2L^2 - \beta^2)\{2(1 + b^2)L^2 - 3\beta^2\}}(b^i y^j - b^j y^i) + \frac{2L^2(\beta + L)}{(2L^2 - \beta^2)}(s^i_0 y^j - s^j_0 y^i). \tag{4.2}$$

From (4.2), we see that B^{ij} is hp(3).

That is, if Douglas space is transformed to be Douglas space by first approximate exponential change of Finsler metric, then B^{ij} is hp(3) and if B^{ij} is hp(3) then Douglas space transformed by first approximate exponential change is Douglas space.

Theorem 4.1 *The first approximate exponential change of Douglas space is Douglas space iff* B^{ij} *given by (4.2) is hp(3).*

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