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# Finite Forms of
# Reciprocity Theorem of Ramanujan and its Generalizations

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**Abstract**: In his lost notebook, Ramanujan has stated a beautiful two variable reciprocity theorem. Its three and four variable generalizations were recently, given by Kang. In this paper, we give new and an elegant approach to establish all the three reciprocity theorems via their finite forms. Also we give some applications of the finite forms of reciprocity theorems.

**Key Words**: $q$-series, reciprocity theorems, bilateral extension, $q$-gamma, $q$-beta, eta-functions.

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## §1. Introduction

In his lost notebook [16], Ramanujan has stated the following beautiful two variable reciprocity theorem.

**Theorem** 1.1 *If $a, b$ are complex numbers other than 0 and $-q^{-n}$, then*

$$\rho(a,b)-\rho(b,a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b,\ bq/a,\ q)_\infty}{(-aq,\ \ -bq)_\infty}, \tag{1.1}$$

*where*

$$\rho(a,b)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n},$$

*and as usual*

$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty},\quad n \text{ is an integer}.$$

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In what follows, we assume $|q| < 1$ and employ the following notations

$$(a_1, a_2, a_3, \cdots, a_m)_n = (a_1)_n (a_2)_n (a_3)_n \cdots (a_m)_n,$$
$$(a_1, a_2, a_3, \cdots, a_n)_\infty = (a_1)_\infty (a_2)_\infty (a_3)_\infty \cdots (a_n)_\infty.$$

The first proof of (1.1) was given by Andrews [4] using his identity, which he has derived using many summation and transformation formulae for basic hypergeometric series and the well-known Jacobi's triple product identity, which in fact, is a special case of (1.1). Somashekara and Fathima [19] used Ramanujan's ${}_1\psi_1$ summation formula and Heine's transformation formula to establish an equivalent version of (1.1). Bhargava, Somashekara and Fathima [9] provided another proof of (1.1). Kim, Somashekara and Fathima [15] gave a proof of (1.1) using only $q$ - binomial theorem. Guruprasad and Pradeep [11] also have devised a proof of (1.1) using $q$ - binomial theorem. Adiga and Anitha [1]devised a proof of (1.1) along the lines of Ismail's proof of Ramanujan's ${}_1\psi_1$ summation formula. Berndt, Chan, Yeap and Yee [8] found the three different proofs of (1.1). The first one is similar to that of Somashekara and Fathima [19]. The second proof depends on Rogers-Fine identity and the third proof is combinatorial. Kang [14] constructed a proof of (1.1) along the lines of Venkatachaleingar's proof of Ramanujan's ${}_1\psi_1$ summation formula. Recently, Somashekara and Narasimha Murthy [21] have given a proof of (1.1) using Abel's lemma on summation by parts and Jacobi's triple product identity. For more details one may refer the book by Andrews and Berndt [5].

Kang, in her paper [14] has obtained the following three and four variable generalizations of (1.1).

**Theorem** 1.2 *If* $|c| < |a| < 1$ *and* $|c| < |b| < 1$, *then*

$$\rho_3(a, b; c) - \rho_3(b, a; c) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c,\ aq/b,\ bq/a,\ q)_\infty}{(-c/a, -c/b, -aq, -bq)_\infty}, \tag{1.2}$$

*where*

$$\rho_3(a, b; c) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b)_{n+1}}.$$

**Theorem** 1.3 *If* $|c|, |d| < |a|, |b| < 1$, *then*

$$\rho_4(a, b; c, d) - \rho_4(b, a; c, d) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c,\ d,\ cd/ab,\ aq/b,\ bq/a,\ q)_\infty}{(-c/a, -c/b, -d/a, -d/b, -aq, -bq)_\infty}, \tag{1.3}$$

*where*

$$\rho_4(a, b; c, d) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(c,\ d,\ cd/ab)_n \left(1 + \frac{cdq^{2n}}{b}\right) (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b, -d/b)_{n+1}}.$$

In fact, to derive (1.2), Kang [14] has employed Ramanujan's ${}_1\psi_1$ summation formula and Jackson's transformation of ${}_2\phi_1$ and ${}_2\phi_2$ series. Later, Adiga and Guruprasad [2] have given a proof of (1.2) using $q$ - binomial theorem and Gauss summation formula. Somashekara and Mamta [20] have obtained (1.2) using (1.1) by parameter augmentation method. One more proof of (1.2) was given by Zhang [23].

Kang [14] has established the four variable reciprocity theorem(1.3) by employing Andrews generalization of ${}_1\psi_1$ summation formula [4, Theorem 6], Sears transformation of ${}_3\phi_2$ series and a limiting case of Watson's transformation for a terminating very well-poised ${}_8\phi_7$ series. Adiga and Guruprasad [3] have derived (1.3) using an identity of Andrew's [4, Theorem 1], Ramanujan's ${}_1\psi_1$ summation formula and the Watson's transformation.

The main objective of this paper is to give finite forms of the reciprocity theorems (1.1), (1.2) and (1.3). To obtain our results, we begin with a known finite unilateral summation and then shift the summation index, say $k$ $(0 \leq k \leq 2n)$ by $n$ :

$$\sum_{k=0}^{2n} A(k) = \sum_{k=-n}^{n} A(k+n).$$

After some manipulations, we employ some well-known transformation formulae for the basic hypergeometric series. The same method has been extensively utilized by Bailey [6]-[7], Slater [18], Schlosser [17] and Jouhet and Schlosser [13].

We recall some standard definitions which we use in this paper. The $q$-gamma function $\Gamma_q(x)$, was introduced by Thomae [22] and later by Jackson [12] as

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}, \qquad 0 < q < 1. \tag{1.4}$$

A $q$-Beta function is defined by

$$B_q(x,y) = (1-q)\sum_{n=0}^{\infty} \frac{(q^{n+1})_\infty}{(q^{n+y})_\infty} q^{nx}.$$

A relation between $q$-Beta function and $q$-gamma function is given by

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{1.5}$$

The Dedekind eta function is defined by

$$\begin{aligned} \eta(\tau) &:= e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}), \qquad Im(\tau) > 0 \\ &:= q^{1/24}(q;q)_\infty, \qquad \text{where } e^{2\pi i\tau} = q. \end{aligned} \tag{1.6}$$

In Section 2, we state some standard identities for basic hypergeometric series which we use for our purpose. In Section 3, we establish the finite forms of two, three and four variable reciprocity theorems 1.1, 1.2 and 1.3. In Section 4, we give some applications of the finite forms of reciprocity theorems.

## §2. Some Standard Identities for Basic Hypergeometric Series

In this section, we list some standard summation and transformation formulae for the basic hypergeometric series which will be used in the remainder of this paper. Some identities involving $q$ - shifted factorials are

$$(a)_{-n} = \frac{1}{(aq^{-n})_n} = \frac{(-q/a)^n}{(q/a)_n} q^{\binom{n}{2}}, \tag{2.1}$$

$$(a)_{k+n} = (a)_n \ (aq^n)_k \ , \tag{2.2}$$

$$(aq^{-n})_n = (q/a)_n \left(\frac{-a}{q}\right)^n q^{-\binom{n}{2}}, \tag{2.3}$$

$$(aq^{-kn})_n = \frac{(q/a)_{kn}}{(q/a)_{(k-1)n}} (-a)^n \, q^{\binom{n}{2}-kn^2}. \tag{2.4}$$

$q$ - Chu- Vandermonde's Sum [10, equation (II.7), p.354]

$$\sum_{k=0}^{n} \frac{(q^{-n}, A)_k}{(q,\ C)_k} \left(Cq^n/A\right)^k = \frac{(C/A)_n}{(C)_n}. \tag{2.5}$$

$q$ - Pfaff- Saalschütz's Summation formula [10, equation (II.12), p.355]

$$\sum_{k=0}^{n} \frac{(q^{-n}, A,\ B)_k}{(q,\ C,\ ABq^{1-n}/C)_k} q^k = \frac{(C/A,\ C/B)_n}{(C,\ C/AB)_n}. \tag{2.6}$$

Jackson's $q$ - analogue of Dougall's ${}_7F_6$ Sum [10, equation (II.22), p.356]

$$\sum_{k=0}^{n} \frac{(A, qA^{1/2}, -qA^{1/2},\ B,\ C,\ D,\ E, q^{-n})_k}{(q, A^{1/2}, -A^{1/2}, Aq/B, Aq/C, Aq/D, Aq/E, Aq^{n+1})_k} q^k$$
$$= \frac{(Aq,\ Aq/BC,\ Aq/BD,\ Aq/CD)_n}{(Aq/B, Aq/C, Aq/D, Aq/BCD)_n}, \tag{2.7}$$

where $A^2q = BCDEq^{-n}$.

Sear's terminating transformation formula [10, equation (III.13), p.360]

$$\sum_{k=0}^{n} \frac{(q^{-n}, B,\ C)_k}{(q,\ D,\ E)_k} \left(DEq^n/BC\right)^k = \frac{(E/C)_n}{(E)_n} \sum_{k=0}^{n} \frac{(q^{-n}, C,\ D/B)_k}{(q,\ D,\ Cq^{1-n}/E)_k} q^k, \tag{2.8}$$

Watson's transformation for a terminating very well poised ${}_8\phi_7$ series [10, equation (III.19), p.361]

$$\sum_{k=0}^{n} \frac{(q^{-n}, A, B,\ C)_k}{(q,\ D,\ E, F)_k} q^k = \frac{(D/B, D/C)_n}{(D, D/BC)_n}$$
$$\times \sum_{k=0}^{n} \frac{(\sigma, q\sigma^{1/2}, -q\sigma^{1/2},\ B,\ C,\ E/A,\ F/A, q^{-n})_k}{(q, \sigma^{1/2}, -\sigma^{1/2}, E, F, EF/AB, EF/AC, EFq^n/A)_k} \left(EFq^n/BC\right)^k , \tag{2.9}$$

where $DEF = ABCq^{1-n}$ and $\sigma = EF/Aq$.

Bailey's terminating ${}_{10}\phi_9$ transformation formula [10, equation (III.28), p.363]

$$\sum_{k=0}^{n} \frac{(A, qA^{1/2}, -qA^{1/2},\ B,\ C,\ D,\ E,\ F\,, \lambda Aq^{n+1}/EF\,, q^{-n})_k}{(q, A^{1/2}, -A^{1/2}, Aq/B, Aq/C, Aq/D, Aq/E, Aq/F\,, EFq^{-n}/\lambda\,, Aq^{n+1})_k} q^k$$
$$= \frac{(Aq,\ Aq/EF,\ \lambda q/E,\ \lambda q/F)_n}{(Aq/E,\ Aq/F,\ \lambda q/EF,\ \lambda q)_n}$$
$$\times \sum_{k=0}^{n} \frac{(\lambda\,, q\lambda^{1/2}, -q\lambda^{1/2},\ B/A,\ C/A,\ D/A,\ E,\ F\,, \lambda Aq^{n+1}/EF,\ q^{-n})_k}{(q, \lambda^{1/2}, -\lambda^{1/2}, Aq/B, Aq/C, Aq/D, \lambda q/E, \lambda q/F\,, EFq^{-n}/A\,, \lambda q^{n+1})_k} q^k, \tag{2.10}$$

where $\lambda = qA^2/BCD$.

## §3. Main Identities

In this section, we establish the finite forms of reciprocity theorems.

**Theorem** 3.1 *If $a, b$ are complex numbers other than 0 and $-q^{-n}$, then*

$$\left(1+\frac{1}{b}\right)\sum_{k=0}^{n}\frac{(q^{-n},-bq^{n})_k}{(q^{1+n},-aq)_k}\left(aq^{1+n}/b\right)^k$$

$$-\left(1+\frac{1}{a}\right)(1-q^n)\sum_{k=0}^{n-1}\frac{(q^{-n+1},-aq^{n+1})_k}{(q^{1+n})_{k+1}(-bq)_k}\left(bq^n/a\right)^k$$

$$=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_n\ (bq/a)_{n-1}\ (q)_n}{(-aq)_n\ (-bq)_{n-1}(q^{1+n})_n}. \tag{3.1}$$

*Proof* Replace $n$ by $2n$ in (2.5) to obtain

$$\sum_{k=0}^{2n}\frac{(q^{-2n},A)_k}{(q,\ C)_k}\left(Cq^{2n}/A\right)^k=\frac{(C/A)_{2n}}{(C)_{2n}}. \tag{3.2}$$

Shift the summation index $k$ by $n$, so that the sum runs from $-n$ to $n$ and (3.2) takes the form

$$\frac{(q^{-2n},A)_n}{(q,\ C)_n}\left(Cq^{2n}/A\right)^n\sum_{k=-n}^{n}\frac{(q^{-n},Aq^n)_k}{(q^{1+n},\ Cq^n)_k}\left(Cq^{2n}/A\right)^k=\frac{(C/A)_{2n}}{(C)_{2n}}.$$

This can be written as

$$\sum_{k=-n}^{n}\frac{(q^{-n},Aq^n)_k}{(q^{1+n},\ Cq^n)_k}\left(Cq^{2n}/A\right)^k=\frac{(q,\ C)_n(C/A)_{2n}}{(A,\ q^{-2n})_n(C)_{2n}}\left(Cq^{2n}/A\right)^{-n}. \tag{3.3}$$

Now, replacing $A$ by $-b$ and $C$ by $-aq^{1-n}$ in (3.3), then using (2.2) and (2.3) in the resulting identity, we obtain

$$\sum_{k=-n}^{n}\frac{(q^{-n},-bq^n)_k}{(q^{1+n},\ -aq)_k}\left(aq^{1+n}/b\right)^k=\frac{\left(\frac{1}{b}-\frac{1}{a}\right)}{\left(1+\frac{1}{b}\right)}\frac{(q)_n\ (aq/b)_n(bq/a)_{n-1}}{(-aq)_n(-bq)_{n-1}(q^{1+n})_n}.$$

This can be written as

$$\left(1+\frac{1}{b}\right)\sum_{k=0}^{n}\frac{(q^{-n},-bq^n)_k}{(q^{1+n},-aq)_k}\left(aq^{1+n}/b\right)^k+\left(1+\frac{1}{b}\right)\sum_{k=1}^{n}\frac{(q^{-n},-1/a)_k}{(q^{1+n},-q^{1-n}/b)_k}q^k$$

$$=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_n\ (bq/a)_{n-1}\ (q)_n}{(-aq)_n\ (-bq)_{n-1}(q^{1+n})_n}. \tag{3.4}$$

Now, the first term on left side of (3.4) is same as the first term on the left side of (3.1). Therefore, to complete the proof, it suffices to show that the second term on the left side of (3.4) is same as the second term on left side of (3.1). To this end, we change $n\to n-1$ and then set $B=-aq^{n+1}, C=q, D=q^{2+n}$ and $E=-bq$ in (2.8) to obtain

$$\sum_{k=0}^{n-1}\frac{(q^{-n+1},-aq^{n+1})_k}{(q^{n+2},-bq)_k}\left(bq^n/a\right)^k=\frac{(-b)_{n-1}}{(-bq)_{n-1}}\sum_{k=0}^{n-1}\frac{(q^{-n+1},-q/a)_k}{(q^{n+2},-q^{2-n}/b)_k}q^k. \tag{3.5}$$

Multiply (3.5) throughout by $\dfrac{\left(1+\frac{1}{a}\right)(1-q^{-n})}{(1-q^{n+1})\left(1+\frac{q^{1-n}}{b}\right)}q$, to obtain

$$\frac{\left(1+\frac{1}{a}\right)(1-q^{-n})}{(1-q^{n+1})\left(1+\frac{q^{1-n}}{b}\right)}q\sum_{k=0}^{n-1}\frac{(q^{-n+1},-aq^{n+1})_k}{(q^{n+2},-bq)_k}\left(bq^n/a\right)^k$$

$$=\frac{(1+b)}{(1+bq^{n-1})}\sum_{k=0}^{n-1}\frac{(q^{-n},-1/a)_{k+1}}{(q^{n+1},-q^{1-n}/b)_{k+1}}q^{k+1}.$$

This on simplification yields

$$\left(1+\frac{1}{b}\right)\sum_{k=1}^{n}\frac{(q^{-n},-1/a)_k}{(q^{n+1},-q^{1-n}/b)_k}q^k$$
$$=-\left(1+\frac{1}{a}\right)(1-q^n)\sum_{k=0}^{n-1}\frac{(q^{-n+1},-aq^{n+1})_k}{(q^{1+n})_{k+1}(-bq)_k}\left(bq^n/a\right)^k,$$

completing the proof of (3.1). □

**Theorem** 3.2 *If* $|c|<|a|<1$ *and* $|c|<|b|<1$*, then*

$$\left(1+\frac{1}{b}\right)\left(1+\frac{cq^n}{b}\right)$$
$$\times\sum_{k=0}^{n}\frac{(q^{-n},c,-cq^n/a,-bq^n)_k}{(q^{1+n},-aq)_k(-c/b,cq^{2n})_{k+1}}(1-cq^{2k+n})\left(\frac{aq^{1+n}}{b}\right)^k-\left(1+\frac{1}{a}\right)(1-q^n)$$
$$\times\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ c,\ -aq^{n+1})_k(-cq^n/b)_{k+1}}{(q^{1+n},-c/a,\ cq^{2n})_{k+1}(-bq)_k}(1-cq^{2k+n+1})\left(\frac{bq^n}{a}\right)^k$$
$$=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(c)_{2n}(aq/b,-bq^n,q)_n\ (bq/a)_{n-1}}{(-c/a,-c/b,-aq,\ q^{1+n})_n\ (-bq)_{2n-1}}.\tag{3.6}$$

*Proof* Replace $n$ by $2n$ in (2.6) to obtain

$$\sum_{k=0}^{2n}\frac{(q^{-2n},A,\ B)_k}{(q,\ C,ABq^{1-2n}/C)_k}q^k=\frac{(C/A,\ C/B)_{2n}}{(C,\ C/AB)_{2n}}.\tag{3.7}$$

Shift the summation index $k$ by $n$, so that the sum runs from $-n$ to $n$ and (3.7) takes the form

$$\sum_{k=-n}^{n}\frac{(q^{-n},Aq^n,\ Bq^n)_k}{(q^{1+n},\ Cq^n,ABq^{1-n}/C)_k}q^k=\frac{(C/A,\ C/B)_{2n}(q,\ C,\ ABq^{1-2n}/C)_n}{(C,\ C/AB)_{2n}(\ A,\ B,\ q^{-2n})_n}q^{-n}.\tag{3.8}$$

Now, we replace $A$ by $-c/b$, $B$ by $-q^{-n}/a$ and $C$ by $-cq^{-n}/a$ in (3.8), and then use (2.2) and (2.3) in the resulting identity, to obtain

$$\sum_{k=-n}^{n}\frac{(q^{-n},-1/a,\ -cq^n/b)_k}{(q^{1+n},\ -c/a,-q^{1-n}/b)_k}q^k=\frac{\left(\frac{1}{b}-\frac{1}{a}\right)}{\left(1+\frac{1}{b}\right)}\frac{(c)_{2n}(aq/b,-bq^n,q)_n\ (bq/a)_{n-1}}{(-c/a,-c/b,-aq,\ q^{1+n})_n\ (-bq)_{2n-1}}.$$

This can be written as

$$\left(1+\frac{1}{b}\right)\sum_{k=0}^{n}\frac{(q^{-n},-aq/c,-bq^{n})_k}{(q^{1+n},-aq,-bq^{1-n}/c)_k}q^k+\left(1+\frac{1}{b}\right)\sum_{k=1}^{n}\frac{(q^{-n},-1/a-cq^{n}/b)_k}{(q^{1+n},-c/a,-q^{1-n}/b)_k}q^k$$
$$=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(c)_{2n}(aq/b,-bq^{n},q)_n\ (bq/a)_{n-1}}{(-c/a,-c/b,-aq,\ q^{1+n})_n\ (-bq)_{2n-1}}. \tag{3.9}$$

Now, set $A=-aq/c,\ B=q,\ C=-bq^{n},\ D=-bq^{1-n}/c, E=q^{n+1}$ and $F=-aq$ in (2.9) and multiply the resulting identity throughout by $(1+b^{-1})$, to obtain

$$\left(1+\frac{1}{b}\right)\sum_{k=0}^{n}\frac{(q^{-n},-aq/c,-bq^{n})_k}{(q^{1+n},-aq,-bq^{1-n}/c)_k}q^k$$
$$=\left(1+\frac{1}{b}\right)\left(1+\frac{cq^{n}}{b}\right)\sum_{k=0}^{n}\frac{(q^{-n},\ c,-cq^{n}/a,\ -bq^{n})_k,}{(q^{1+n},-aq)_k\ (-c/b,\ cq^{2n})_{k+1}}(1-cq^{2k+n})\left(\frac{aq^{1+n}}{b}\right)^k. \tag{3.10}$$

Next, change $n\to n-1$ in (2.9) and then set $A=-q/a,\ B=q,\ C=-cq^{n+1}/b,\ D=-q^{2-n}/b, E=q^{n+2}$ and $F=-cq/a$ to obtain

$$\sum_{k=0}^{n-1}\frac{(q^{-n+1},-q/a,\ -cq^{n+1}/b)_k}{(q^{2+n},\ -cq/a,-q^{2-n}/b)_k}q^k=\frac{(-q^{1-n}/b,\ q^{1-2n}/c)_{n-1}}{(-q^{2-n}/b,\ q^{-2n}/c)_{n-1}}$$
$$\times\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ c,\ -aq^{n+1},-cq^{1+n}/b)_k}{(q^{2+n},-cq/a,-bq,\ cq^{2n+1})_k}\frac{(1-cq^{2k+n+1})}{(1-cq^{n+1})}\left(\frac{bq^{n}}{a}\right)^k. \tag{3.11}$$

Multiply (3.11) throughout by $\dfrac{\left(1+\frac{1}{b}\right)(1-q^{-n})\left(1+\frac{1}{a}\right)(1+\frac{cq^{n}}{b})}{(1-q^{1+n})(1+\frac{c}{a})(1+\frac{q^{1-n}}{b})}q$ to obtain

$$\left(1+\frac{1}{b}\right)\sum_{k=0}^{n-1}\frac{(q^{-n},-1/a,\ -cq^{n}/b)_{k+1}}{(q^{1+n},\ -c/a,-q^{1-n}/b)_{k+1}}q^{k+1}$$
$$=\frac{\left(1+\frac{1}{a}\right)(1-q^{-n})\left(1+\frac{cq^{n}}{b}\right)\left(1-\frac{q^{-n-1}}{c}\right)}{(1-q^{1+n})\left(1+\frac{c}{a}\right)\left(1-\frac{q^{-2n}}{c}\right)}$$
$$\times\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ c,\ -aq^{n+1},-cq^{1+n}/b)_k}{(q^{2+n},-cq/a,-bq,\ cq^{2n+1})_k}\frac{(1-cq^{2k+n+1})}{(1-cq^{n+1})}\left(\frac{bq^{n}}{a}\right)^k. \tag{3.12}$$

On simplification (3.12) yields

$$\left(1+\frac{1}{b}\right)\sum_{k=1}^{n}\frac{(q^{-n},-1/a,-cq^{n}/b)_k}{(q^{1+n},\ -c/a,\ q^{1-n}/b)_k}q^k=-\left(1+\frac{1}{a}\right)(1-q^{n})$$
$$\times\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ c,\ -aq^{n+1})_k(-cq^{n}/b)_{k+1}}{(q^{1+n},-c/a,\ cq^{2n})_{k+1}(-bq)_k}(1-cq^{2k+n+1})\left(\frac{bq^{n}}{a}\right)^k. \tag{3.13}$$

Using (3.10) and (3.13) in (3.9), we obtain (3.6). □

**Theorem** 3.3 *If* $|c|,\ |d| < |a|,\ |b| < 1,$ *then*

$$\frac{(1+1/b)\left(1-aq^{n+1}/b\right)\left(1+cdq^{2n-1}/a\right)}{(1+q^n)}$$
$$\times\sum_{k=0}^{n}\frac{(q^{-n},\ c,\ d,\ cd/ab,\ aq^{1-n}/b, cdq^{2n}/b)_k}{(q^{1-2n},-aq)_k(-c/b,-d/b,-cdq^{n-1}/a,-cdq^{n}/b)_{k+1}}\left(1+\frac{cdq^{2k}}{b}\right)q^k$$
$$-\frac{\left(1+\frac{1}{a}\right)(1-aq^{n+1}/b)}{(1+q^n)(1+cdq^{n-1}/a)}$$
$$\times\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ c,\ d,\ cd/ab,\ bq^{-n}/a)_k(cdq^{2n-1}/a)_{k+1}}{(q^{1-2n},-bq,-cdq^{n+1}/a)_k(-c/a,-d/a,-cdq^{n}/b)_{k+1}}\left(1+\frac{cdq^{2k}}{a}\right)q^k$$
$$=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(c,\ d,\ cd/ab)_{2n}(aq/b)_{n+1}(q)_n}{(-c/a,-d/a,-aq)_n(-c/b,-d/b)_{n+1}(q^{1+n})_n}$$
$$\times\frac{(bq/a)_{n-1}(-cd/aq)_{3n}}{(-bq)_{n-1}(-cdq^n/b)_n(-cd/aq,-cdq^{n-1}/a)_{2n}}. \tag{3.14}$$

*Proof* Replace $n$ by $2n$ in (2.7) to obtain

$$\sum_{k=0}^{2n}\frac{(q^{-2n},\ A,\ B,\ C,\ D,\ A^2q^{2n+1}/BCD)_k}{(q,Aq/B,Aq/C,Aq/D,BCDq^{-2n}/A,Aq^{2n+1})_k}\frac{(1-Aq^{2k})}{(1-A)}q^k$$
$$=\frac{(Aq,\ Aq/BC,\ Aq/BD,\ Aq/CD)_{2n}}{(Aq/B,Aq/C,Aq/D,Aq/BCD)_{2n}}. \tag{3.15}$$

Shift the summation index $k$ by $n$, so that the sum runs from $-n$ to $n$ and (3.15) takes the form

$$\sum_{k=-n}^{n}\frac{(q^{-n},\ Aq^n,\ Bq^n,\ Cq^n,\ Dq^n,\ A^2q^{3n+1}/BCD)_k}{(q^{1+n},Aq^{1+n}/B,Aq^{1+n}/C,Aq^{1+n}/D,BCDq^{-n}/A,Aq^{3n+1})_k}\frac{(1-Aq^{2k+2n})}{(1-A)}q^k$$
$$=\frac{(Aq,\ Aq/BC,\ Aq/BD,\ Aq/CD)_{2n}}{(Aq/B,Aq/C,Aq/D,Aq/BCD)_{2n}}$$
$$\times\frac{(q,\ Aq/B,\ Aq/C,\ Aq/D,\ BCDq^{-2n}/A,\ Aq^{2n+1})_n}{(q^{-2n},\ A,\ B,\ C,\ D,\ A^2q^{2n+1}/BCD)_n}q^{-n}. \tag{3.16}$$

Replacing $A,\ B,\ C,\ D$ respectively by $Aq^{-2n},\ Bq^{-n},\ Cq^{-n},\ Dq^{-n}$ in (3.16) and simplifying using (2.2), (2.3) and (2.4), we obtain

$$\sum_{k=0}^{n}\frac{(q^{-n},\ Aq^{-n},\ B,\ C,\ D,\ A^2q^{2n+1}/BCD)_k}{(q^{1+n},Aq/B,Aq/C,Aq/D,BCDq^{-2n}/A,Aq^{n+1})_k}(1-Aq^{2k})q^k$$
$$-A\sum_{k=1}^{n}\frac{(q^{-n},\ q^{-n}/A,\ B/A,\ C/A,\ D/A,\ Aq^{2n+1}/BCD)_k}{(q^{1+n},q/B,q/C,q/D,BCDq^{-2n}/A^2,q^{n+1}/A)_k}\left(1-\frac{q^{2k}}{A}\right)q^k$$
$$=(1-A)\frac{(q,\ Aq,\ q/A)_n(Aq/BC,\ Aq/BD,\ Aq/CD)_{2n}}{(q/B,q/C,q/D,\ Aq/B,\ Aq/C,\ Aq/D,q^{1+n}\ A^2q^{2n+1}/BCD)_n}$$
$$\times\frac{(Aq/BCD)_{3n}}{(\ Aq/BCD,\ Aq^{1+n}/BCD)_{2n}}. \tag{3.17}$$

Setting $A = aq/b$, $B = -q/b$, $C = -aq/c$ and $D = -aq/d$ in (3.17) and then simplifying, we obtain

$$\sum_{k=0}^{n} \frac{(q^{-n},\ aq^{1-n}/b, -q/b, -aq/c, -aq/d, -cdq^{2n}/b)_k}{(q^{1+n}, -aq, -cq/b, -dq/b, -aq^{2-2n}/cd, aq^{n+2}/b)_k} \left(1 - \frac{aq^{2k+1}}{b}\right) q^k$$
$$- \frac{aq}{b} \sum_{k=1}^{n} \frac{(q^{-n},\ bq^{-n-1}/a, -1/a, -b/c, -b/d, -cdq^{2n-1}/a)_k}{(q^{1+n}, -b, -c/a, -d/a, -bq^{1-2n}/cd, bq^n/a)_k} \left(1 - \frac{bq^{2k-1}}{a}\right) q^k$$
$$= \left(1 - \frac{aq}{b}\right) \frac{(q,\ aq^2/b,\ b/a)_n (c,\ d,\ cd/ab)_{2n}}{(-b, -c/a, -d/a, -cq/b, -dq/b, -aq, q^{1+n}, -cdq^n/b)_n}$$
$$\times \frac{(-cd/aq)_{3n}}{(-cd/aq, -cdq^{n-1}/a)_{2n}}. \tag{3.18}$$

Multiply (3.18) throughout by $\dfrac{\left(1+\frac{1}{b}\right)}{\left(1+\frac{c}{b}\right)\left(1+\frac{d}{b}\right)}$ to obtain

$$\left(1 + \frac{1}{b}\right) \sum_{k=0}^{n} \frac{(q^{-n},\ aq^{1-n}/b, -q/b, -aq/c, -aq/d, -cdq^{2n}/b)_k}{(q^{1+n}, -aq, -aq^{2-2n}/cd, aq^{n+2}/b)_k (-c/b, -d/b)_{k+1}} \left(1 - \frac{aq^{2k+1}}{b}\right) q^k$$
$$- \frac{\left(1+\frac{1}{b}\right)\frac{aq}{b}}{\left(1+\frac{c}{b}\right)\left(1+\frac{d}{b}\right)}$$
$$\times \sum_{k=1}^{n} \frac{(q^{-n},\ bq^{-n-1}/a, -1/a, -b/c, -b/d, -cdq^{2n-1}/a)_k}{(q^{1+n}, -b, -c/a, -d/a, -bq^{1-2n}/cd, bq^n/a)_k} \left(1 - \frac{bq^{2k-1}}{a}\right) q^k$$
$$= \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c,\ d,\ cd/ab)_{2n} (aq/b)_{n+1} (q)_n}{(-c/a, -d/a, -aq)_n (-c/b, -d/b)_{n+1} (q^{1+n})_n}$$
$$\times \frac{(bq/a)_{n-1} (-cd/aq)_{3n}}{(-bq)_{n-1} (-cdq^n/b)_n (-cd/aq, -cdq^{n-1}/a)_{2n}}. \tag{3.19}$$

Now, set $A = aq/b$, $B = -q/b$, $C = -aq/c$, $D = -aq/d$, $E = aq^{1-n}/b$, $F = q$ and $\lambda = -cd/b$ in (2.10), to obtain

$$\sum_{k=0}^{n} \frac{(q^{-n},\ aq^{1-n}/b, -q/b, -aq/c, -aq/d, -cdq^{2n}/b)_k}{(q^{1+n}, -aq, -cq/b, -dq/b, -aq^{2-2n}/cd, aq^{n+2}/b)_k} \left(1 - \frac{aq^{2k+1}}{b}\right) q^k$$
$$= \frac{(1 - aq^{n+1}/b)(1 - q^n)(1 + cdq^{2n-1}/a)}{(1 - q^{2n})(1 + cdq^{n-1}/a)(1 + cdq^n/b)}$$
$$\times \sum_{k=0}^{n} \frac{(q^{-n},\ c,\ d,\ cd/ab,\ aq^{1-n}/b, -cdq^{2n}/b)_k}{(q^{1-2n}, -aq, -cq/b, -dq/b, -cdq^n/a, cdq^{n+1}/b)_k} \left(1 + \frac{cdq^{2k}}{b}\right) q^k. \tag{3.20}$$

Multiply (3.20) throughout by $\dfrac{\left(1+\frac{1}{b}\right)}{\left(1+\frac{c}{b}\right)\left(1+\frac{d}{b}\right)}$ to obtain

$$\left(1 + \frac{1}{b}\right) \sum_{k=0}^{n} \frac{(q^{-n},\ aq^{1-n}/b, -q/b, -aq/c, -aq/d, -cdq^{2n}/b)_k}{(q^{1+n}, -aq, -aq^{2-2n}/cd, aq^{n+2}/b)_k (-c/b, -d/b)_{k+1}} \left(1 - \frac{aq^{2k+1}}{b}\right) q^k$$
$$= \frac{\left(1+\frac{1}{b}\right)(1 - aq^{n+1}/b)(1 + cdq^{2n-1}/a)}{(1 + q^n)}$$

$$\times\sum_{k=0}^{n}\frac{(q^{-n},\ c,\ d,\ cd/ab,\ aq^{1-n}/b,-cdq^{2n}/b)_k}{(q^{1-2n},-aq)_k(-c/b,-d/b,-cdq^{n-1}/a,-cdq^n/b)_{k+1}}\left(1+\frac{cdq^{2k}}{b}\right)q^k. \tag{3.21}$$

Next, change $n\rightarrow n-1$ in (2.10) and then set $A=bq/a,\ B=-q/a,\ C=-bq/c,\ D=-bq/d, E=bq^{-n}/a, F=q$ and $\lambda=-cd/a$ to obtain

$$\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ bq^{-n}/a,-q/a,-bq/c,-bq/d,-cdq^{2n}/a)_k}{(q^{2+n},-bq,-cq/a,-dq/a,-bq^{2-2n}/cd,bq^{n+1}/a)_k}\left(1-\frac{bq^{2k+1}}{a}\right)q^k$$
$$=\frac{(1-bq^n/a)(1-q^{n+1})(1+cdq^{2n-1}/b)}{(1-q^{2n})(1+cdq^{n-1}/a)(1+cdq^n/b)}$$
$$\times\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ c,\ d,\ cd/ab,\ bq^{-n}/a,-cdq^{2n}/a)_k}{(q^{1-2n},-bq,-cq/a,-dq/a,-cdq^{n+1}/a,-cdq^{n+1}/b)_k}\left(1+\frac{cdq^{2k}}{a}\right)q^k. \tag{3.22}$$

Multiplying (3.22) throughout by

$$\frac{\left(1+\frac{1}{a}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{b}{d}\right)\left(1-q^{-n}\right)\left(1-\frac{bq^{-n-1}}{a}\right)\left(1+\frac{cdq^{2n-1}}{a}\right)\left(1+\frac{1}{b}\right)q}{(1+b)\left(1+\frac{c}{a}\right)\left(1+\frac{d}{a}\right)\left(1-q^{n+1}\right)\left(1+\frac{bq^{1-2n}}{cd}\right)\left(1-\frac{bq^n}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{d}{b}\right)},$$

we obtain

$$\frac{\left(1+\frac{1}{b}\right)}{\left(1+\frac{c}{b}\right)\left(1+\frac{d}{b}\right)}\sum_{k=0}^{n-1}\frac{(q^{-n},\ bq^{-n-1}/a,-1/a,-b/c,-b/d,-cdq^{2n-1}/a)_{k+1}}{(q^{1+n},-b,-c/a,-d/a,-bq^{1-2n}/cd,bq^n/a)_{k+1}}$$
$$\times\left(1-\frac{bq^{2k+1}}{a}\right)q^{k+1}=\frac{\left(1+\frac{1}{a}\right)(1-aq^{n+1}/b)(b/aq)}{(1+q^n)(1+cdq^{n-1}/a)}$$
$$\times\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ c,\ d,\ cd/ab,\ bq^{-n}/a)_k(-cdq^{2n-1}/a)_{k+1}}{(q^{1-2n},-bq,-cdq^{n+1}/a)_k(-c/a,-d/a,-cdq^n/b)_{k+1}}\left(1+cdq^{2k}/a\right)q^k. \tag{3.23}$$

Now, (3.23) can be written as

$$\frac{\left(1+\frac{1}{b}\right)\frac{aq}{b}}{\left(1+\frac{c}{b}\right)\left(1+\frac{d}{b}\right)}\sum_{k=1}^{n}\frac{(q^{-n},\ bq^{-n-1}/a,-1/a,-b/c,-b/d,-cdq^{2n-1}/a)_k}{(q^{1+n},-b,-c/a,-d/a,-bq^{1-2n}/cd,\ bq^n/a)_k}$$
$$\times\left(1-\frac{bq^{2k-1}}{a}\right)q^k=\frac{\left(1+\frac{1}{a}\right)(1-aq^{n+1}/b)}{(1+q^n)(1+cdq^{n-1}/a)}$$
$$\times\sum_{k=0}^{n-1}\frac{(q^{-n+1},\ c,\ d,\ cd/ab,\ bq^{-n}/a)_k(-cdq^{2n-1}/a)_{k+1}}{(q^{1-2n},-bq,-cdq^{n+1}/a)_k(-c/a,-d/a,-cdq^n/b)_{k+1}}\times\left(1+cdq^{2k}/a\right)q^k. \tag{3.24}$$

On using (3.21)and (3.24) in (3.19), we obtain (3.14). □

**Remark** 3.1 Letting $n\rightarrow\infty$ in (3.1), (3.6) and (3.14), we obtain (1.1), (1.2) and (1.3) respectively.

## §4. Some Applications of the Finite Forms of the Reciprocity Theorems

In this Section, we deduce finite forms of some $q$ - series identities along with the $q$ - gamma, $q$ - beta and eta function identities from (3.1) and (3.6).

**Corollary** 4.1 *(Finite form of Euler's identity)*

$$\sum_{k=0}^{n-1} \frac{(q^{-n+1}, -q^{n+2}/x)_k}{(q^{1+n}, q)_k} (-1)^k q^{nk-k} x^k = (-x)_{n-1}. \tag{4.1}$$

*Proof* Set $b = -1$ and $a = q/x$ in (3.1), and after some simplifications, we obtain (4.1). Let $n \to \infty$ in (4.1) to obtain the well-known Euler's Identity

$$\sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{(q)_k} = (-x)_\infty. \qquad \square$$

**Corollary** 4.2 *(Finite form of ${}_1\phi_1$- series [10, equation (II.5), p.354]*

$$\sum_{k=0}^{n-1} \frac{(q^{-n+1}, x - xq^{n+2}/y)_k (xq^n)_{k+1}}{(q^{1+n}, xq^{2n})_{k+1} (q, y)_k} (1 - xq^{2n+k+1})(y/x)^k q^{nk-k} x^k$$

$$= \frac{(xq^n)_{n-1} (y/x)_{n-1}}{(q^{1+n})_n (y)_{n-1}}. \tag{4.2}$$

*Proof* Set $b = -1, a = -xq/y$ and $c = x$ in (3.6), and after some simplifications, we obtain (4.2). If we let $n \to \infty$ in (4.2), gives

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} (x)_k (y/x)^k}{(q,\ y)_k} = \frac{(y/x)_\infty}{(y)_\infty}. \qquad \square$$

**Corollary** 4.3

$$(1 + bq^{n+1}) \sum_{k=0}^{n} \frac{(q^{-n}, -q^n/b)_k}{(-bq, q^{2n+1})_k} (1 - q^{2n+k+1})(-1)^k q^{nk+k} b^k = 1. \tag{4.3}$$

*Proof* Set $a = -1, c = q$ and $b = b^{-1}$ in (3.6), and after some simplifications, we obtain (4.3). If we let $n \to \infty$ in (4.3), gives

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2} b^k}{(-bq)_{k+1}} = 1. \qquad \square$$

If we set $b = 1$ in (4.3), we obtain

$$(1 + q^{n+1}) \sum_{k=0}^{n} \frac{(q^{-n}, -q^n)_k}{(-q, q^{2n+1})_k} (1 - q^{2n+k+1})(-1)^k q^{nk+k} = 1. \tag{4.4}$$

Letting $n \to \infty$ in (4.4), we obtain

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(-q)_{k+1}} = 1.$$

We define

$$\Gamma_{q,n}(x) := \frac{(q)_n}{(q^x)_n} (1 - q)^{1-x},$$

and

$$B_{q,n}(x,y) := \frac{(q,\ q^{x+y})_n}{(q^x,\ q^y)_n}(1-q).$$

Note that $\Gamma_{q,n}(x) \to \Gamma_q(x)$ and $B_{q,n}(x,y) \to B_q(x,y)$ as $n \to \infty$, which are define in (1.4) and (1.5).

**Corollary** 4.4

$$\Gamma_{q,n}(x) = \frac{(-q^{1+x}, q^{1+n})_n(1-q)^{1-x}}{2(-q)_n(-q)_{n-1}}\left[\sum_{k=0}^{n}\frac{(q^{-n}, q^{n+x})_k}{(q^{1+n}, -q^{1+x})_k}(-1)^k q^{nk+k}\right.$$

$$\left.+(1+q^x)(1-q^n)\sum_{k=0}^{n-1}\frac{(q^{-n+1}, -q^{n+x+1})_k}{(q^{1+n})_{k+1}(q^{x+1})_k}(-1)^k q^{nk}\right]. \tag{4.5}$$

*Proof* Set $a = q^x$ and $b = -q^x$ in (3.1), and after some simplifications, we obtain (4.5). □

**Corollary** 4.5

$$B_{q,n}(x,y) = \frac{(1-q)(1-q^x)(q^{1+x}, q^{1+n}, q^y)_n}{(q^{1+x-y}, q^{y-x}, q^{x+y+n})_n}$$
$$\times\left[(1-q^{n+x})\sum_{k=0}^{n}\frac{(q^{-n}, q^{x+y})_k(q^{n+y})_k^2}{(q^{1+n})_k(q^{2n+x+y})_{k+1}(q^x)_{k+1}^2}(1-q^{2k+n+x+y})q^{nk+k+kx-ky}\right.$$
$$-q^{y-x}(1-q^n)$$

$$\left.\times\sum_{k=0}^{n-1}\frac{(q^{-n+1}, q^{x+y}, q^{n+x+1})_k(q^{n+x})_{k+1}}{(q^{1+n}, q^y, q^{2n+x+y}, q^y)_{k+1}}(1-q^{2k+n+x+y+1})q^{nk+ky-kx}\right]. \tag{4.6}$$

*Proof* Set $a = -q^x, b = -q^y$ and $c = q^{x+y}$ in (3.6), and after some simplifications, we obtain (4.6). □

**Corollary** 4.6

$$\sum_{k=0}^{n}\frac{(q^{-n})_k}{(-q)_{k+1}}(-1)^k q^{nk+k} + \sum_{k=0}^{n-1}\frac{(q^{-n+1}, -q^{n+2})_k}{(q, q^{n+1})_{k+1}}(-1)^k q^{nk} = \frac{2(-q)_{n-1}}{(1+q^{n+1})(q^{n+1})_n}. \tag{4.7}$$

*Proof* Set $a = q$ and $b = -q$ in (3.1), and after some simplifications, we obtain (4.5). □

Letting $n \to \infty$ in (4.5), (4.6) and (4.7) and using (1.4), (1.5) and (1.6), we obtain respectively $q$ - gamma, $q$ - beta and eta function identities

$$\Gamma_q(x) = \frac{(-q^{1+x})_\infty(1-q)^{1-x}}{2(-q)_\infty^2}\left[\sum_{k=0}^{\infty}\frac{q^{k(k+1)/2}}{(-q^{x+1})_k} + (1+q^x)\sum_{k=0}^{\infty}\frac{q^{k(k+1)/2}}{(q^{x+1})_k}\right],$$

$$B_q(x,y) = \frac{(1-q)(1-q^x)(q^{1+x},q^y)_\infty}{(q^{1+x-y},q^{y-x})_\infty} \times \left[\sum_{k=0}^{n} \frac{(-1)^k q^{k(k+1)/2}(q^{x+y})_k}{(q^x)_k^2} q^{k+kx-ky} - q^{y-x}\sum_{k=0}^{n-1} \frac{(-1)^k q^{k(k+1)/2}(q^{x+y})_k}{(q^y)_k(q^y)_{k+1}} q^{ky-kx}\right],$$

$$\frac{\eta(2\tau)}{\eta(\tau)} = \frac{q^{-1/24}}{2}\left[\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(-q)_{k+1}} + \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(q)_{k+1}}\right].$$

**Conclusion** We see that the finite forms of reciprocity theorems are interesting and also preserve all the symmetries. A number of identities of the types (4.1) - (4.7) can be deduced from the finite forms of reciprocity theorems.

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