International J.Math. Combin. Vol.4(2013), 78-95

Controllability of Fractional Stochastic Differential Equations With State-Dependent Delay

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Abstract: In this paper, the approximate controllability for a class of nonlinear fractional stochastic differential equations with state-dependent delays in Hilbert space is studied. The result is extended to study the approximate controllability of fractional stochastic systems with state-dependent delays and resolvent operators. A set of sufficient conditions are established to obtain the required result by employing semigroup theory, fixed point technique and fractional calculus. In particular, the approximate controllability of nonlinear fractional stochastic control systems is established under the assumption that the corresponding linear control system is approximately controllable. Also, an example is presented to illustrate the applicability of the obtained theory.

Key Words: Approximate controllability, stochastic fractional differential equations, fixed point technique, state-dependent delay.

AMS(2010): 34A08, 93B05, 47H10

§1. Introduction

Controllability is one of the important fundamental concepts in mathematical control theory and plays a vital role in both deterministic and stochastic control systems. In recent years, various controllability problems for different kinds of dynamical systems have been studied in many publications [1, 8, 9, 18, 19].. From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. However, the concept of exact controllability is usually too strong and has limited applicability. Approximate controllability is a weaker concept than complete controllability and it is completely adequate in applications [7, 23].. Recently, Wang [32] derived a set of sufficient conditions for the approximate controllability of differential equations with multiple delays by implementing some natural conditions such as growth conditions for the nonlinear term and compactness of the semigroup. Sakthivel and Anandhi [29] investigated the problem of approximate controllability for a class of nonlinear impulsive differential equations with state-dependent delay by using semigroup theory and fixed

[1]Received June 16, 2013, Accepted November 27, 2013.

point technique.

On the other hand, the theory of fractional differential equations is emerging as an important area of investigation since it is richer in problems in comparison with corresponding theory of classical differential equations [17, 25, 26]. In fact, such models can be considered as an efficient alternative to the classical nonlinear differential models to simulate many complex processes. Recently, it have been proved that the differential models involving derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, physics, chemistry, aerodynamics, electrodynamics of complex medium, and so on [15, 17]. In recent years, existence results for fractional differential equations have been investigated in several papers [5,3]. More recently, Dabas and Chauhan studied the existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay.

In particular, the study of stochastic fractional differential equations has attracted great interest due to its applications in characterizing many problems in biology, electrical engineering and other areas of science. Many physical phenomena in evolution processes are modeled as stochastic fractional differential equations and existence results for such equations have been studied by several authors [4, 11, 30].

Most of the existing literature on controllability results for linear and nonlinear stochastic systems are without fractional derivatives [20, 21, 22, 24]. Only few papers deal with the controllability of fractional stochastic systems. Guendouzi and Hamada [12] studied the relative controllability of fractional stochastic dynamical systems with multiple delays in control. The authors derive a new set of sufficient conditions for the global relative controllability by fixed point technique and controllability Grammian matrix. Sakthivel et al. [28]discussed the approximate controllability of nonlinear fractional stochastic control system under the assumptions that the corresponding linear system is approximately controllable. Guendouzi and Idrissi [13] investigated the problem of approximate controllability for a class of dynamic control systems described by nonlinear fractional stochastic functional differential equations in Hilbert space driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. Sakthivel et al. [27] studied the approximate controllability of neutral stochastic fractional integro-differential equation with infinite delay in a Hilbert space.

However, to the best of our knowledge, the approximate controllability problem for nonlinear fractional stochastic systems with state-dependent delay has not been investigated yet. Motivated by this consideration, in this paper we will study the approximate controllability problem for nonlinear fractional stochastic system, described by nonlinear fractional stochastic differential equations with state-dependent delay and control in Hilbert space, under the assumption that the associated linear system is approximately controllable. In fact, the results in this paper are motivated by the recent work of [29] and the fractional differential equations discussed in [6,28].

§2. Preliminaries and Basic Properties

Let $\mathcal{H}, \mathcal{K}$ be two separable Hilbert spaces and $\mathcal{L}(\mathcal{K}, \mathcal{H})$ be the space of bounded linear operators

from $\mathcal{K}$ into $\mathcal{H}$. For convenience, we will use the notation $\|.\|_{\mathcal{H}}$, $\|.\|_{\mathcal{K}}$ and $\|.\|$ to denote the norms in $\mathcal{H}, \mathcal{K}$ and $\mathcal{L}(\mathcal{K}, \mathcal{H})$ respectively, and use $(.,.)$ to denote the inner product of $\mathcal{H}$ and $\mathcal{K}$ without any confusion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets. $w = (w_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathcal{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \ldots$ and a sequence $\{\beta_k\}_{k\geq 1}$ of independent Brownian motions such that
$$(w(t), e)_{\mathcal{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathcal{K}}\beta_k(t), \quad e \in \mathcal{K}, t \in J := [0, b].$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{1/2}\mathcal{K}, \mathcal{H})$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}\mathcal{K}$ into $\mathcal{H}$ with the inner product $\langle \psi, \pi \rangle_{\mathcal{L}_2^0} = tr[\psi Q \pi^\star]$. Let $\mathcal{L}^2(\mathcal{F}_b, \mathcal{H})$ be the Banach space of all $\mathcal{F}_b$-measurable square integrable random variables with values in the Hilbert space $\mathcal{H}$, and $\mathbf{E}(\cdot)$ denote the expectation with respect to the measure $\mathbb{P}$.

The purpose of this paper is to investigate the approximate controllability for a class of nonlinear fractional stochastic differential equation with state-dependent delay and control of the form

$$\begin{aligned} {}^cD_t^{\alpha}[x(t) + g(t, x_{\varepsilon(t,x_t)})] &= A[x(t) + g(t, x_{\varepsilon(t,x_t)})] + Bu(t) + f(t, x_{\varepsilon(t,x_t)}) \\ &\quad + \sigma(t, x_{\varepsilon(t,x_t)})\frac{dw(t)}{dt}, \quad t \in J \\ x_0 &= \phi \in \mathcal{B}, \end{aligned} \tag{2.1}$$

where ${}^cD_t^{\alpha}$ is the Caputo fractional derivative of order α, $0 < \alpha < 1$; $x(.)$ takes the value in the separable Hilbert space $\mathcal{H}$; $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of an α-resolvent family $S_\alpha(t)_{t\geq 0}$; the control function $u(\cdot)$ is given in $\mathcal{L}^2_{\mathcal{F}}([0, b], \mathcal{U})$ of admissible control functions, $\mathcal{U}$ is a Hilbert space. B is a bounded linear operator from $\mathcal{U}$ into $\mathcal{H}$. The history $x_t : (-\infty, 0] \to \mathcal{H}$, $x_t(\theta) = x(t + \theta)$, $\theta \leq 0$, belongs to an abstract phase space $\mathcal{B}$ defined axiomatically; $g : J \times \mathcal{B} \to \mathcal{H}$, $f : J \times \mathcal{B} \to \mathcal{H}$, $\sigma : J \times \mathcal{B} \to \mathcal{L}_2^0$ and $\varepsilon : J \times \mathcal{B} \to (-\infty, b]$ are appropriate functions to be specified later.

Let us recall the following known definitions. For more details see [17].

Definition 2.1 *The fractional integral of order α with the lower limit 0 for a function f is defined as*
$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t > 0, \alpha > 0$$
provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2 *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^{L}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}ds, \quad t > 0, n-1 < \alpha < n. \tag{2.2}$$

Definition 2.3 *The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written* *as*

$$^{c}D^{\alpha}f(t) = {}^{L}D^{\alpha}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^k(0)\right), \quad t > 0, n-1 < \alpha < n. \tag{2.3}$$

If $f(t) \in C^n[0,\infty)$, *then*

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^n(s)ds = I^{n-\alpha}f^n(s), \quad t > 0, n-1 < \alpha < n.$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{^{c}D^{\alpha}f(t); s\} = s^{\alpha}\hat{f}(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0); \quad n-1 \leq \alpha < n.$$

Definition 2.4 *A two parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi i}\int_C \frac{\mu^{\alpha-\beta}e^{\mu}}{\mu^{\alpha}-z}d\mu, \quad \alpha,\beta \in C, \mathcal{R}(\alpha) > 0,$$

where C is a contour which starts and ends at $-\infty$ end encircles the disc $|\mu| \leq |z|^{1/2}$ counter clockwise.

For short, $E_{\alpha}(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = \cos h(z)$, $E_2(-z^2) = \cos(z)$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^{\infty} e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^{\alpha})dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}-\omega}, \quad Re\lambda > \omega^{\frac{1}{\alpha}}, \omega > 0,$$

and for more details see [17].

Definition 2.5([31]) *A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:*

- $\rho(A) \subset \Sigma_{\theta,\omega} = \{\lambda \in C : \lambda \neq \omega, |arg(\lambda-\omega)| < \theta\}$;
- $\|R(\lambda, A)\| \leq \frac{M}{|\lambda-\omega|}$, $\lambda \in \Sigma_{\theta,\omega}$.

Definition 2.6 *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha} : \mathbb{R}_+ \to L(X)$, where $L(X)$ is a Banach space of all bounded linear operators from X into X and the corresponding norm is denoted by $\|.\|$, such that $\{\lambda^{\alpha} : Re\lambda > \omega\} \subset \rho(A)$ and*

$$(\lambda^{\alpha}I - A)^{-1}x = \int_0^{\infty} e^{\lambda t}S_{\alpha}(t)xdt, \quad Re\lambda > \omega, x \in X, \tag{2.4}$$

where $S_{\alpha}(t)$ is called the α-resolvent family generated by A.

Definition 2.7 *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)xdt, \quad Re\lambda > \omega, x \in X, \tag{2.5}$$

where $S_\alpha(t)$ is called the solution operator generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on α-resolvent family and solution operators, we refer the reader to [17].

In this paper, we assume that the operator A is sectorial of type ω with $\pi(1-\alpha/2) < \theta < \pi$. Then A is the generator of a solution operator given by

$$T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda,$$

and the operator

$$S_\alpha = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}(\lambda^\alpha - A)^{-1}d\lambda$$

is the α-resolvent family generated by A, where $\widehat{B}_r$ denotes the Bromwich path (see [6,31]).

Recently, it has been proven in [31] that if $\alpha \in (0,1)$ and $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$ is a sectorial operator, then for any x in a Banach space X and $t > 0$, we have

$$\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1}), \quad t > 0, \ \ \omega > \omega_0,$$

where $C > 0$ depending solely on θ and α.

In this work, we will employ an axiomatic definition of the abstract phase space $\mathcal{B}$ introduced by Hale and Kato [14]. We first define $\tilde{\mathcal{H}}((-\infty, b]; \mathcal{H})$ the Banach space of all continuous and $\mathcal{F}_b$-measurable $\mathcal{H}$-valued function x.

Axiom 2.8 *$\mathcal{B}$ is a linear space that denotes the family of $\mathcal{F}_0$-measurable function from $(-\infty, 0]$ into $\mathcal{H}$, endowed with norm $\|\cdot\|_\mathcal{B}$, which satisfies the following axioms:*

(1) *If $x \in \tilde{\mathcal{H}}$ is continuous on $[0, b]$, $b > 0$, and $x_0 \in \mathcal{B}$, then for every $t \in [0, b]$ the following conditions hold*

(1-1) $x_t \in \mathcal{B}$;

(1-2) $\|x(t)\|_{\tilde{\mathcal{H}}} \leq \delta\|x_t\|_\mathcal{B}$;

(1-3) $\|x_t\|_\mathcal{B} \leq \mu(t)\sup_{0\leq s\leq t}\|x(s)\|_{\tilde{\mathcal{H}}} + \nu(t)\|x_0\|_\mathcal{B}$*, where $\delta > 0$ is a constant; $\mu, \nu : [0, \infty) \to [1, \infty)$, μ is continuous, ν is locally bounded; δ, μ and ν are independent of $x(\cdot)$.*

(2) *For the function $x(\cdot)$ in i., x_t is a $\mathcal{B}$-valued continuous functions on $[0, b]$;*

(3) *The space $\mathcal{B}$ is complete.*

Let $x_b(x_0; u)$ be the state of (2.1) at terminal time b corresponding to the control u and the initial value $x_0 = \phi \in \mathcal{B}$. Introduce the set $\mathcal{R}(b, \phi) = \{x_b(\phi; u)(0) : u(\cdot) \in \mathcal{L}^2_\mathcal{F}([0, b], \mathcal{U})\}$, which is called the reachable set of system (1) at terminal time b and its closure in $\mathcal{H}$ is denoted by $\overline{\mathcal{R}(b, \phi)}$.

Definition 2.9 *The system* (2.1) *is said to be approximately controllable on* J *if* $\overline{\mathcal{R}(b,\phi)}=\mathcal{H}$, *that is, given an arbitrary* $\epsilon>0$ *it is possible to steer from the point* ϕ *to within a distance* ϵ *from all points in the state space* $\mathcal{H}$ *at time* b.

In order to study the approximate controllability for the fractional control system (2.1), we introduce the approximate controllability of its linear part

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t)+(Bu)(t), \qquad t\in J, \\ x(0) &= \phi\in\mathcal{B}. \end{aligned} \tag{2.6}$$

The approximate controllability for linear fractional control system (2.6) is a natural generalization of approximate controllability of linear first order control system [23]. It is convenient at this point to introduce the controllability operator associated with (2.6) as

$$\begin{aligned} \Gamma_0^b &= \int_0^b S_\alpha(b-s)BB^*S_\alpha^*(b-s)ds, \\ R(\kappa,\Gamma_0^b) &= (\kappa I+\Gamma_0^b)^{-1}, \quad \kappa>0, \end{aligned}$$

where B^* denotes the adjoint of B and $S_\alpha^*(t)$ is the adjoint of $S_\alpha(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

Lemma 2.10([23]) *The linear fractional control system* (2.6) *is approximately controllable on* $[0,b]$ *if and only if* $\kappa R(\kappa,\Gamma_0^b)\to 0$ *as* $\kappa\to 0^+$ *in the strong operator topology.*

In order to establish the result, we need the following assumptions:

(**H1**) If $\alpha\in(0,1)$ and $A\in\mathcal{A}^\alpha(\theta_0,\omega_0)$, then for $x\in\mathcal{H}$ and $t>0$ we have $\|T_\alpha(t)\|\le Me^{\omega t}$ and $\|S_\alpha(t)\|\le Ce^{\omega t}(1+t^{\alpha-1})$, $\omega>\omega_0$. Thus we have

$$\|T_\alpha(t)\|\le\widetilde{M}_T \quad\text{and}\quad \|S_\alpha(t)\|\le t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T=\sup_{0\le t\le b}\|T_\alpha(t)\|$, and $\widetilde{M}_S=\sup_{0\le t\le b}Ce^{\omega t}(1+t^{1-\alpha})$ (fore more details, see [31]).

(**H2**) The function $t\to\phi_t$ is well defined and continuous from the set $\mathcal{Z}(\varepsilon^-)=\{\varepsilon(s,\tau):(s,\tau)\in J\times\mathcal{B},\varepsilon(s,\tau)\le 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $\varphi^\phi:\mathcal{Z}(\varepsilon^-)\to(0,\infty)$ such that, for every $t\in\mathcal{Z}(\varepsilon^-)$

$$\|\phi_t\|_{\mathcal{B}}\le\varphi^\phi(t)\|\phi\|_{\mathcal{B}}.$$

(**H3**) The function $g:J\times\mathcal{B}\to\mathcal{H}$ is continuous and there exists some constant $M_g>0$ such that

$$\begin{aligned} \mathbb{E}\|g(t,\xi)\|_{\mathcal{H}}^2 &\le M_g\Big(\|\xi\|_{\mathcal{B}}^2+1\Big), \qquad \xi\in\mathcal{B}, \\ \mathbb{E}\|g(t_2,\xi_2)-g(t_1,\xi_1)\|_{\mathcal{H}}^2 &\le M_g\Big(|t_2-t_1|^2+\|\xi_2-\xi_1\|_{\mathcal{B}}^2\Big), \qquad \xi_i\in\mathcal{B}, \quad i=1,2. \end{aligned}$$

(**H4**) The function $f:J\times\mathcal{B}\to\mathcal{H}$ satisfies the following properties:

(1) $f(t,\cdot):\mathcal{B}\to\mathcal{H}$ is continuous for each $t\in J$ and for each $\xi\in\mathcal{B}$, $f(\cdot,\xi):J\to\mathcal{H}$ is strongly measurable;

(2) there exist a positive integrable functions $m \in L^1([0,b])$ and a continuous nondecreasing function $\Xi_f : [0,\infty) \to (0,\infty)$ such that for every $(t,\xi) \in J \times \mathcal{B}$, we have

$$\mathbb{E}\|f(t,\xi)\|^2_{\mathcal{H}} \le m(t)\Xi_f\Big(\|\xi\|^2_{\mathcal{B}}\Big), \qquad \liminf_{q\to\infty} \frac{\Xi_f(q)}{q} = \Lambda < \infty.$$

(H5) The function $\sigma : J \times \mathcal{B} \to \mathcal{L}^0_2$ satisfies the following properties:

(1) $\sigma(t,\cdot) : \mathcal{B} \to \mathcal{L}^0_2$ is continuous for each $t \in J$ and for each $\xi \in \mathcal{B}$, $\sigma(\cdot,\xi) : J \to \mathcal{L}^0_2$ is strongly measurable;

(2) there exist a positive integrable functions $n \in L^1([0,b])$ and a continuous nondecreasing function $\Xi_\sigma : [0,\infty) \to (0,\infty)$ such that for every $(t,\xi) \in J \times \mathcal{B}$, we have

$$\mathbb{E}\|\sigma(t,\xi)\|^2_{\mathcal{L}^0_2} \le n(t)\Xi_\sigma\Big(\|\xi\|^2_{\mathcal{B}}\Big), \qquad \liminf_{q\to\infty} \frac{\Xi_\sigma(q)}{q} = \Upsilon < \infty.$$

(H6) For all $\kappa > 0$ such that

$$\left[4M_g{\mu^*}^2 + 4{\mu^*}^2\Lambda\widetilde{M}^2_S\frac{b^{2\alpha}}{\alpha^2}\sup_{s\in J} m(s) + 4{\mu^*}^2\Upsilon\widetilde{M}^2_S\frac{b^{2\alpha}}{b(2\alpha-1)}\sup_{s\in J} n(s)\right] \times\left[5 + 30(\widetilde{M}^2_S M^2_B)^2\frac{(b^{2\alpha})^2}{\kappa^2\alpha^2}\right] < 1.$$

Lemma 2.11([16]) *Let $x \in \tilde{\mathcal{H}}$ be continuous on $[0,b]$ and $x_0 = \phi$. If (H2) holds, then*

$$\|x_s\|_{\mathcal{B}} \le \mu^* \sup_{\substack{0\le\theta\le\max\{0,s\}\\ s\in\mathcal{Z}(\varepsilon)\cup J}} \|x(\theta)\|_{\tilde{\mathcal{H}}} + (\nu^* + \varphi^*)\|\phi\|_{\mathcal{B}},$$

where $\varphi^* = \sup_{t\in\mathcal{Z}(\varepsilon)} \varphi^\phi(t)$, $\nu^* = \sup_{t\in J}\nu(t)$, $\mu^* = \sup_{t\in J}\mu(t)$.

The following lemma is required to define the control function.

Lemma 2.12([23]) *For any $\hat{x}_b \in \mathcal{L}^2(\mathcal{F}_b,\mathcal{H})$ there exists $\hat{\varphi} \in \mathcal{L}^2_{\mathcal{F}}(\Omega;\mathcal{L}^2(J,\mathcal{L}^0_2))$ such that*

$$\hat{x}_b = \mathbb{E}\hat{x}_b + \int_0^b \hat{\varphi}(s)dw(s).$$

Now for any $\kappa > 0$ and $\hat{x}_b \in \mathcal{L}^2(\mathcal{F}_b,\mathcal{H})$, we define the control function

$$\begin{aligned} u^\kappa(t) &= B^*S^*_\alpha(b-t)(\kappa I + \Gamma^b_0)^{-1} \\ &\quad\times\left\{\mathbb{E}\hat{x}_b + \int_0^b \hat{\varphi}(s)dw(s) - T_\alpha(b)[\phi(0) + g(0,x_{\varepsilon(0,\phi)})] + g(b,x_{\varepsilon(b,x_b)})\right\} \\ &\quad -B^*S^*_\alpha(b-t)\int_0^b (\kappa I + \Gamma^b_s)^{-1}S_\alpha(b-s)f(s,x_{\varepsilon(s,x_s)})ds \\ &\quad -B^*S^*_\alpha(b-t)\int_0^b (\kappa I + \Gamma^b_s)^{-1}S_\alpha(b-s)\sigma(s,x_{\varepsilon(s,x_s)})dw(s). \end{aligned}$$

§3. Controllability Results

In this section, we study the approximate controllability results for the systems (2.1) where the operator A is a sectorial type ω with $\pi(1-\alpha/2) < \theta < \pi$. In particular, we establish approximate controllability of nonlinear fractional stochastic system (2.1) under the assumptions that the corresponding linear systems is approximately controllable.

Theorem 3.1 *Assume that the assumptions* $(H1)$-$(H6)$ *hold and* $S_\alpha(t)$ *is compact, then the fractional stochastic system* (2.1) *has at least one mild solution.*

Proof Let $\mathcal{C}((-\infty, b], \mathcal{H})$ be the space of all continuous $\mathcal{H}$-valued stochastic processes $\{x(t), t \in (-\infty, b]\}$. Consider the space $\tilde{\mathcal{B}} = \{x : x \in \mathcal{C}((-\infty, b], \mathcal{H}), x(0) = \phi(0)\}$ endowed with seminorm $\|\cdot\|_{\tilde{\mathcal{B}}}$ defined by

$$\|x\|_{\tilde{\mathcal{B}}} = \|\phi\|_{\mathcal{B}} + \sup_{-\infty<s\leq b} (\mathbb{E}\|x(s)\|^2)^{\frac{1}{2}}, \qquad x \in \tilde{\mathcal{B}},$$

and the space $\tilde{\mathcal{H}}$ defined in the previous section endowed with the norm $\|x\|_{\tilde{\mathcal{H}}} = \sup_{t\in J}(\mathbb{E}\|x(t)\|^2_{\mathcal{H}})^{\frac{1}{2}}$. In what follows, we assume that $\varepsilon : [0, b] \times \mathcal{B} \to (-\infty, b]$ is continuous and $\varepsilon(t, x_t) = 0$ for $t = 0$. For $\kappa > 0$, define the operator $\mathcal{P} : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ by

$$\begin{aligned}(\mathcal{P}x)(t) \quad = \quad & T_\alpha(t)[\phi(0) + g(0, x_{\varepsilon(0,\phi)})] - g(t, x_{\varepsilon(t,x_t)}) \\ & + \int_0^t S_\alpha(t-s)[Bu^\kappa(s) + f(s, x_{\varepsilon(s,x_s)})]ds \\ & + \int_0^t S_\alpha(t-s)\sigma(s, x_{\varepsilon(s,x_s)})dw(s), \qquad t \in J.\end{aligned}$$

It will be shown that the system (1) is approximately controllable if for all $\kappa > 0$ there exists a fixed point of the operator $\mathcal{P}$.

For $\phi \in \mathcal{B}$, define

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T_\alpha(t)\phi(0), & t \in J. \end{cases}$$

Then $y_0 = \phi$.

For each $z : J \to \mathcal{H}$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ z(t), & t \in J. \end{cases}$$

If $x(\cdot)$ satisfies the system (2.1), then we can decompose $x(\cdot)$ as $x(t) = y(t) + \bar{z}(t)$, which implies $x_{\varepsilon(t,x_t)} = y_{\varepsilon(t,x_t)} + \bar{z}_{\varepsilon(t,x_t)}$, for $t \in J$ and the function $z(\cdot)$ satisfies

$$\begin{aligned}z(t) \quad = \quad & T_\alpha(t)g(0, \phi) - g(t, y_{\varepsilon(t,\bar{z}_t)} \\ & + \bar{z}_{\varepsilon(t,\bar{z}_t)}) + \int_0^t S_\alpha(t-s)Bu^\kappa(s)ds + \int_0^t S_\alpha(t-s)f(s, y_{\varepsilon(s,\bar{z}_s)} + \bar{z}_{\varepsilon(s,\bar{z}_s)})ds \\ & + \int_0^t S_\alpha(t-s)\sigma(s, y_{\varepsilon(s,\bar{z}_s)} + \bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s), \qquad t \in J,\end{aligned}$$

where

$$
\begin{aligned}
u^{\kappa}(t) &= B^*S_\alpha^*(b-t)(\kappa I+\Gamma_0^b)^{-1}\\
&\quad\times\left\{\mathbb{E}\hat{x}_b+\int_0^b\hat{\varphi}(s)dw(s)-T_\alpha(b)[\phi(0)+g(0,\phi)]+g(b,y_{\varepsilon(b,\bar{z}_b)}+\bar{z}_{\varepsilon(b,\bar{z}_b)})\right\}\\
&\quad -B^*S_\alpha^*(b-t)\int_0^b(\kappa I+\Gamma_s^b)^{-1}S_\alpha(b-s)f(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})ds\\
&\quad -B^*S_\alpha^*(b-t)\int_0^b(\kappa I+\Gamma_s^b)^{-1}S_\alpha(b-s)\sigma(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s).
\end{aligned}
$$

Set $\tilde{\mathcal{B}}_0=\{z\in\tilde{\mathcal{B}}, z_0=0\in\mathcal{B}\}$, and for any $z\in\tilde{\mathcal{B}}_0$, we have

$$
\|z\|_{\tilde{\mathcal{B}}_0}=\|z_0\|_{\mathcal{B}}+\sup_{-\infty<s\le b}(\mathbb{E}\|x(s)\|^2)^{\frac12}=\sup_{-\infty<s\le b}(\mathbb{E}\|x(s)\|^2)^{\frac12}.
$$

On the space $\tilde{\mathcal{B}}_0$, consider a set $B_q=\{z\in\tilde{\mathcal{B}}_0:\|z\|^2_{\tilde{\mathcal{B}}_0}\le q\}$ for some $q\ge0$; then, for each q, B_q is clearly a bounded closed convex set in $\tilde{\mathcal{B}}_0$. For $z\in B_q$, from Lemma 2.11, we see that

$$
\begin{aligned}
\|\bar{z}_t+y_t\|^2_{\mathcal{B}} &\le 2\Big(\|\bar{z}_t\|^2_{\mathcal{B}}+\|y_t\|^2_{\mathcal{B}}\Big)\\
&\le 4\Bigg(\mu^{*^2}\sup_{\substack{0\le s\le\max\{0,t\}\\ t\in\mathcal{Z}(\varepsilon)\cup J}}\mathbb{E}\|\bar{z}(s)\|^2+(\nu^*+\varphi^*)^2\|\bar{z}_0\|^2_{\mathcal{B}}\\
&\qquad+\mu^{*^2}\sup_{\substack{0\le s\le\max\{0,t\}\\ t\in\mathcal{Z}(\varepsilon)\cup J}}\mathbb{E}\|y(s)\|^2+(\nu^*+\varphi^*)^2\|y_0\|^2_{\mathcal{B}}\Bigg)\\
&\le 4\mu^{*^2}\Big(q+\widetilde{M}_T^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}}\Big)+4(\nu^*+\varphi^*)^2\|\phi\|^2_{\mathcal{B}}.
\end{aligned}
\tag{3.1}
$$

Let $\Phi:\tilde{\mathcal{B}}_0\to\tilde{\mathcal{B}}_0$ be the operator defined by Φz such that

$$
\Phi z(t)=\begin{cases}
0, & t\in(-\infty,0];\\
T_\alpha(t)g(0,\phi)-g(t,y_{\varepsilon(t,\bar{z}_t)}+\bar{z}_{\varepsilon(t,\bar{z}_t)})+\displaystyle\int_0^t S_\alpha(t-s)Bu^{\kappa}(s)ds & \\
+\displaystyle\int_0^t S_\alpha(t-s)f(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})ds & \\
+\displaystyle\int_0^t S_\alpha(t-s)\sigma(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s), & t\in J.
\end{cases}
$$

Obviously, the operator $\mathcal{P}$ has a fixed point if and only if Φ has a fixed point. For the sake of convenience, we divide the proof into several steps.

Step 1 We show that, for each $\kappa>0$, there exists a positive number q such that $\Phi(B_q)\subset B_q$. If it is not true, then there exists $\kappa>0$ such that for every $q>0$ and $t\in J$, there exists a function $z^q(t)\in B_q$, but $\Phi(z^q)\notin B_q$, that is, $\mathbb{E}\|(\Phi z^q)(t)\|^2_{\mathcal{H}}>q$. For such $\kappa>0$ and $\bar{z}=z$ on

J, we find that

$$\begin{aligned} q &\leq \mathbf{E}\|\Phi(z^q)(t)\|_{\mathcal{H}}^2 \\ &\leq 5\mathbf{E}\|T_\alpha(t)g(0,\phi)\|_{\mathcal{H}}^2 + 5\mathbf{E}\|g(t, y_{\varepsilon(t,\bar{z}_t^q)} + \bar{z}^q_{\varepsilon(t,\bar{z}_t^q)})\|_{\mathcal{H}}^2 \\ &\quad +5\mathbf{E}\left\|\int_0^t S_\alpha(t-s)Bu^\kappa(s)ds\right\|_{\mathcal{H}}^2 + 5\mathbf{E}\left\|\int_0^t S_\alpha(t-s)f(s, y_{\varepsilon(s,\bar{z}_s^q)} + \bar{z}^q_{\varepsilon(s,\bar{z}_s^q)})ds\right\|_{\mathcal{H}}^2 \\ &\quad +5\mathbf{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s, y_{\varepsilon(s,\bar{z}_s^q)} + \bar{z}^q_{\varepsilon(s,\bar{z}_s^q)})dw(s)\right\|_{\mathcal{H}}^2. \end{aligned}$$

By Lemma 2.11 and assumptions $(H1)$-$(H3)$, we have

$$\begin{aligned} I_1 &= \mathbf{E}\|T_\alpha(t)g(0,\phi)\|_{\mathcal{H}}^2 + \mathbf{E}\|g(t, y_{\varepsilon(t,\bar{z}_t^q)} + \bar{z}^q_{\varepsilon(t,\bar{z}_t^q)})\|_{\mathcal{H}}^2 \\ &\leq \widetilde{M}_T^2 M_g(1+\|\phi\|_{\mathcal{B}}^2) + M_g(1+\|y_{\varepsilon(t,\bar{z}_t^q)} + \bar{z}^q_{\varepsilon(t,\bar{z}_t^q)}\|_{\mathcal{B}}^2) \\ &\leq \widetilde{M}_T^2 M_g\Big(1+\|\phi\|_{\mathcal{B}}^2\Big) + M_g\Big(1+4{\mu^*}^2(q+\widetilde{M}_T^2\mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^*+\varphi^*)^2\|\phi\|_{\mathcal{B}}^2\Big). \end{aligned} \tag{3.2}$$

$$\begin{aligned} I_2 &= \mathbf{E}\left\|\int_0^t S_\alpha(t-s)f(s, y_{\varepsilon(s,\bar{z}_s^q)} + \bar{z}^q_{\varepsilon(s,\bar{z}_s^q)})ds\right\|_{\mathcal{H}}^2 \\ &\quad +\mathbf{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s, y_{\varepsilon(s,\bar{z}_s^q)} + \bar{z}^q_{\varepsilon(s,\bar{z}_s^q)})dw(s)\right\|_{\mathcal{H}}^2 \\ &\leq \int_0^t \|S_\alpha(t-s)\|ds \int_0^t \|S_\alpha(t-s)\|\mathbf{E}\|f(s, y_{\varepsilon(s,\bar{z}_s^q)} + \bar{z}^q_{\varepsilon(s,\bar{z}_s^q)})\|_{\mathcal{H}}^2 ds \\ &\quad + \int_0^t \|S_\alpha(t-s)\|^2\mathbf{E}\|\sigma(s, y_{\varepsilon(s,\bar{z}_s^q)} + \bar{z}^q_{\varepsilon(s,\bar{z}_s^q)})\|_{\mathcal{L}_2^0}^2 ds \\ &\leq \widetilde{M}_S^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}m(s)\Xi_f\Big(\|y_{\varepsilon(t,\bar{z}_t^q)} + \bar{z}^q_{\varepsilon(t,\bar{z}_t^q)}\|_{\mathcal{B}}^2\Big)ds \\ &\quad +\widetilde{M}_S^2\int_0^t (t-s)^{2(\alpha-1)}n(s)\Xi_\sigma\Big(\|y_{\varepsilon(t,\bar{z}_t^q)} + \bar{z}^q_{\varepsilon(t,\bar{z}_t^q)}\|_{\mathcal{B}}^2\Big)ds \\ &\leq \widetilde{M}_S^2\frac{b^{2\alpha}}{\alpha^2}\Xi_f\Big(4{\mu^*}^2(q+\widetilde{M}_T^2\mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^*+\varphi^*)^2\|\phi\|_{\mathcal{B}}^2\Big)\sup_{s\in J} m(s) \\ &\quad +\widetilde{M}_S^2\frac{b^{2\alpha-1}}{2\alpha-1}\Xi_\sigma\Big(4{\mu^*}^2(q+\widetilde{M}_T^2\mathbf{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4(\nu^*+\varphi^*)^2\|\phi\|_{\mathcal{B}}^2\Big)\sup_{s\in J} n(s) \\ &= \widetilde{M}_S^2 b^{2\alpha}\left[\frac{M_f}{\alpha^2} + \frac{M_\sigma}{b(2\alpha-1)}\right]. \end{aligned} \tag{3.3}$$

Further, by using $(H1)$-$(H5)$, Hölder inequality, Eq.(3.1) and Lemma 2.11, we get

$$\begin{aligned} \mathbf{E}\|u^\kappa(s)\|2 &\leq \frac{1}{\kappa^2}M_B^2 b^{2\alpha}\widetilde{M}_S^2\Bigg\{6\|\mathbf{E}\hat{x}_b + \int_0^b \hat{\varphi}(s)dw(s)\|^2 + 6\mathbf{E}\|T_\alpha(b)\phi(0)\|^2 \\ &\quad +\mathbf{E}\|T_\alpha(b)g(0,\phi)\|^2 + 6\mathbf{E}\|g(b, y_{\varepsilon(b,\bar{z}_b)} + \bar{z}_{\varepsilon(b,\bar{z}_b)})\|^2 \\ &\quad +6\mathbf{E}\left\|\int_0^b S_\alpha(b-s)f(s, y_{\varepsilon(s,\bar{z}_s^q)} + \bar{z}^q_{\varepsilon(s,\bar{z}_s^q)})ds\right\|_{\mathcal{H}}^2 \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&+\mathbb{E}\left\|\int_0^b S_\alpha(b-s)\sigma(s,y_{\varepsilon(s,\bar{z}_s^q)}+\bar{z}^q_{\varepsilon(s,\bar{z}_s^q)})dw(s)\right\|^2_{\mathcal{H}}\Bigg\}\\
\le\ &\frac{6b^{2\alpha}}{\kappa^2}M_B^2\widetilde{M}_S^2\Bigg[2\|\mathbb{E}\hat{x}_b\|^2+2\int_0^b\mathbb{E}\|\hat{\varphi}(s)\|^2ds+\widetilde{M}_T^2\|\phi\|^2_{\mathcal{B}}+\widetilde{M}_T^2M_g(1+\|\phi\|^2_{\mathcal{B}})\\
&+M_g\Big(1+4{\mu^*}^2(q+\widetilde{M}_T^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}})+4(\nu^*+\varphi^*)^2\|\phi\|^2_{\mathcal{B}}\Big)\\
&+\widetilde{M}_S^2b^{2\alpha}\left(\frac{M_f}{\alpha^2}+\frac{M_\sigma}{b(2\alpha-1)}\right)\Bigg],
\end{aligned}
\tag{3.5}
$$

where $M_B=\|B\|$.

Now, we have

$$
\begin{aligned}
I_3\ =\ &\mathbb{E}\left\|\int_0^t S_\alpha(t-s)Bu^\kappa(s)ds\right\|^2_{\mathcal{H}}\\
\le\ &\Big(\widetilde{M}_S^2M_B^2\Big)^2\frac{6}{\kappa^2\alpha^2}(b^{2\alpha})^2\Bigg[2\|\mathbb{E}\hat{x}_b\|^2+2\int_0^b\mathbb{E}\|\hat{\varphi}(s)\|^2ds+\widetilde{M}_T^2\|\phi\|^2_{\mathcal{B}}+\widetilde{M}_T^2M_g(1+\|\phi\|^2_{\mathcal{B}})\\
&+M_g\Big(1+4{\mu^*}^2(q+\widetilde{M}_T^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}})+4(\nu^*+\varphi^*)^2\|\phi\|^2_{\mathcal{B}}\Big)+\widetilde{M}_S^2b^{2\alpha}\left(\frac{M_f}{\alpha^2}+\frac{M_\sigma}{b(2\alpha-1)}\right)\Bigg].
\end{aligned}
\tag{3.6}
$$

Combining the estimates (3.2), (3.3) and (3.5) yields

$$
\begin{aligned}
q\ \le\ &\mathbb{E}\|\Phi(z^q)(t)\|^2_{\mathcal{H}}\\
\le\ &5\widetilde{M}_T^2M_g\Big(1+\|\phi\|^2_{\mathcal{B}}\Big)+5M_g\Big(1+4{\mu^*}^2(q+\widetilde{M}_T^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}})+4(\nu^*+\varphi^*)^2\|\phi\|^2_{\mathcal{B}}\Big)\\
&+5\widetilde{M}_S^2b^{2\alpha}\left[\frac{M_f}{\alpha^2}+\frac{M_\sigma}{b(2\alpha-1)}\right]+5\Big(\widetilde{M}_S^2M_B^2\Big)^2\frac{6}{\kappa^2\alpha^2}(b^{2\alpha})^2\Bigg[2\|\mathbb{E}\hat{x}_b\|^2+2\int_0^b\mathbb{E}\|\hat{\varphi}(s)\|^2ds\\
&+\widetilde{M}_T^2\|\phi\|^2_{\mathcal{B}}+\widetilde{M}_T^2M_g(1+\|\phi\|^2_{\mathcal{B}})+M_g\Big(1+4{\mu^*}^2(q+\widetilde{M}_T^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}})+4(\nu^*+\varphi^*)^2\|\phi\|^2_{\mathcal{B}}\Big)\\
&+\widetilde{M}_S^2b^{2\alpha}\left(\frac{M_f}{\alpha^2}+\frac{M_\sigma}{b(2\alpha-1)}\right)\Bigg].
\end{aligned}
\tag{3.7}
$$

Dividing both sides of (3.6) by q and taking $q\to\infty$, we obtain

$$
\begin{aligned}
&\left[4M_g{\mu^*}^2+4{\mu^*}^2\Lambda\widetilde{M}_S^2\frac{b^{2\alpha}}{\alpha^2}\sup_{s\in J}m(s)+4{\mu^*}^2\Upsilon\widetilde{M}_S^2\frac{b^{2\alpha}}{b(2\alpha-1)}\sup_{s\in J}n(s)\right]\\
&\qquad\times\left[5+30(\widetilde{M}_S^2M_B^2)^2\frac{(b^{2\alpha})^2}{\kappa^2\alpha^2}\right]\ge 1,
\end{aligned}
$$

which is a contradiction by assumption $(H6)$. Thus, for some $q>0$, $\Phi(B_q)\subset B_q$.

Step 2 We prove that for each $\kappa>0$, the operator Φ maps B_q into a relatively compact subset of B_q. First we prove that the set $V(t)=\{(\Phi z)(t): z\in B_q\}$ is relatively compact in $\mathcal{H}$ for every $t\in J$. The case $t=0$ is obvious.

For $0 < \epsilon < t \leq b$, we define an operator Φ^ϵ on B_q by

$$
\begin{aligned}
(\Phi^\epsilon z)(t) &= T_\alpha(t)g(0,\phi) - g(t, y_{\varepsilon(t,\bar{z}_t)} + \bar{z}_{\varepsilon(t,\bar{z}_t)}) + S_\alpha(\epsilon)\int_0^{t-\epsilon} S_\alpha(t-s-\epsilon)Bu^\kappa(s)ds \\
&\quad + S_\alpha(\epsilon)\int_0^{t-\epsilon} S_\alpha(t-s-\epsilon)f(s, y_{\varepsilon(s,\bar{z}_s)} + \bar{z}_{\varepsilon(s,\bar{z}_s)})ds \\
&\quad + S_\alpha(\epsilon)\int_0^{t-\epsilon} S_\alpha(t-s-\epsilon)\sigma(s, y_{\varepsilon(s,\bar{z}_s)} + \bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s).
\end{aligned}
$$

Since $S_\alpha(t)$ is a compact operator, the set $V_\epsilon(t) = \{(\Phi^\epsilon z)(t) : z(.) \in B_q\}$ is relatively compact in $\mathcal{H}$ for every $\epsilon > 0$. Also, for every $z \in B_q$, we have

$$
\begin{aligned}
& \mathbb{E}\|(\Phi z)(t) - (\Phi^\epsilon z)(t)\|^2_{\mathcal{H}} \\
\leq\ & 3\mathbb{E}\left\|\int_{t-\epsilon}^t S_\alpha(t-s)Bu^\kappa(s)ds\right\|^2_{\mathcal{H}} + 3\mathbb{E}\left\|\int_{t-\epsilon}^t S_\alpha(t-s)f(s, y_{\varepsilon(s,\bar{z}_s)} + \bar{z}_{\varepsilon(s,\bar{z}_s)})ds\right\|^2_{\mathcal{H}} \\
& + 3\mathbb{E}\left\|\int_{t-\epsilon}^t S_\alpha(t-s)\sigma(s, y_{\varepsilon(s,\bar{z}_s)} + \bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s)\right\|^2_{\mathcal{H}} \\
\leq\ & 3\widetilde{M}_S^2\frac{\epsilon^{2\alpha}}{\alpha}\int_{t-\epsilon}^t (t-s)^{\alpha-1}\Big[M_B^2\frac{6}{\kappa^2\alpha}\epsilon^{2\alpha}(M_B\widetilde{M}_S)^2\hat{M} + m(s)\Xi_f(q') + n(s)\Xi_\sigma(q')\Big]ds \\
& \longrightarrow 0 \qquad \text{as } \epsilon \to 0^+, \quad \text{where}
\end{aligned}
$$

$$
q' = 4{\mu^*}^2(q + \widetilde{M}_T^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}}) + 4(\nu^* + \varphi^*)^2\|\phi\|^2_{\mathcal{B}}
$$

and

$$
\begin{aligned}
\hat{M} \ =\ & \Big[2\|\mathbb{E}\hat{x}_b\|^2 + 2\int_0^b \mathbb{E}\|\hat{\varphi}(s)\|^2ds + \widetilde{M}_T^2\|\phi\|^2_{\mathcal{B}} + \widetilde{M}_T^2M_g(1+\|\phi\|^2_{\mathcal{B}}) \\
& + M_g(1+q') + \widetilde{M}_S^2b^{2\alpha}\left(\frac{M_f}{\alpha^2} + \frac{M_\sigma}{b(2\alpha-1)}\right)\Big].
\end{aligned}
$$

This implies that there are relatively compact sets arbitrarily close to the set $V(t)$ for each $t \in (0, b]$. Hence $V(t) = \{(\Phi z)(t) : z \in B_q\}$ is relatively compact in $\mathcal{H}$.

Step 3 We show that $V = \{(\Phi z)(t) : z(\cdot) \in B_q\}$ is equicontinuous on $[0, b]$. Let $0 < \epsilon < t < b$ and $\delta > 0$ be such that $\|S_\alpha(s_1) - S_\alpha(s_2)\| < \epsilon$, for every $s_1, s_2 \in J$ with $|s_1 - s_2| < \delta$. For any $z \in B_q$ and $0 \leq t_1 \leq t_2 \in J$, we get

$$
\begin{aligned}
\mathbb{E}\|(\Phi z)(t_2) - (\Phi z)(t_1)\|^2_{\mathcal{H}} &\leq 8\|T_\alpha(t_2) - T_\alpha(t_1)\|^2\mathbb{E}\|g(0,\phi)\|^2_{\mathcal{H}} \\
&\quad + 8\mathbb{E}\|g(t_2, y_{\varepsilon(t_2,\bar{z}_{t_2})} + \bar{z}_{\varepsilon(t_2,\bar{z}_{t_2})}) - g(t_1, y_{\varepsilon(t_1,\bar{z}_{t_1})} + \bar{z}_{\varepsilon(t_1,\bar{z}_{t_1})})\|^2_{\mathcal{H}} \\
&\quad + 8\mathbb{E}\left\|\int_0^{t_1}[S_\alpha(t_2-s) - S_\alpha(t_1-s)]Bu^\kappa(s)ds\right\|^2_{\mathcal{H}} \\
&\quad + 8\mathbb{E}\left\|\int_{t_1}^{t_2} S_\alpha(t_2-s)Bu^\kappa(s)ds\right\|^2_{\mathcal{H}}
\end{aligned}
$$

$$
\begin{aligned}
&+8\mathbb{E}\left\|\int_0^{t_1}[S_\alpha(t_2-s)-S_\alpha(t_1-s)]f(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})ds\right\|_{\mathcal{H}}^2\\
&+8\mathbb{E}\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)f(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})ds\right\|_{\mathcal{H}}^2\\
&+8\mathbb{E}\left\|\int_0^{t_1}[S_\alpha(t_2-s)-S_\alpha(t_1-s)]\sigma(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s)\right\|_{\mathcal{H}}^2\\
&+8\mathbb{E}\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)\sigma(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s)\right\|_{\mathcal{H}}^2.
\end{aligned}
$$

By assumptions $(H1)$, $(H3)$-$(H5)$ and Hölder's inequality, it follows that

$$
\begin{aligned}
\mathbb{E}\|(\Phi z)(t_2)-(\Phi z)(t_1)\|_{\mathcal{H}}^2 \le\ & 8M_g\|T_\alpha(t_2)-T_\alpha(t_1)\|^2\Big(1+\|\phi\|_{\mathcal{B}}^2\Big)\\
&+8M_g\Big(|t_2-t_1|^2+\|(y_{\varepsilon(t_2,\bar{z}_{t_2})}-y_{\varepsilon(t_1,\bar{z}_{t_1})})+(\bar{z}_{\varepsilon(t_2,\bar{z}_{t_2})}-\bar{z}_{\varepsilon(t_1,\bar{z}_{t_1})})\|_{\mathcal{B}}^2\Big)\\
&+8\widetilde{M}_S^2M_B^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\|u^\kappa(s)\|^2ds\\
&+8\epsilon^2M_B^2\int_0^{t_1}(t_1-s)^{\alpha-1}\mathbb{E}\|u^\kappa(s)\|^2ds\\
&+8\epsilon^2\frac{t_1^\alpha}{\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}[m(s)\Xi_f(q')+n(s)\Xi_\sigma(q')]ds\\
&+8\widetilde{M}_S^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}[m(s)\Xi_f(q')+n(s)\Xi_\sigma(q')]ds.
\end{aligned}
$$

Therefore, for ϵ sufficiently small, the right-hand side of the above inequality tends to zero as $t_1\to t_2$, since the compactness of $S_\alpha(t)$ implies the continuity in the uniform operator topology on J. Thus, the set $V=\{(\Phi z)(t): z(\cdot)\in B_q\}$ is equicontinuous.

By using a procedure similar to that used in [2], we can easily prove that the map $\Phi(\cdot)$ is continuous on B_q which completes the proof that $\Phi(\cdot)$ is completely continuous. Hence from the Schauder fixed point theorem Φ has a fixed point and consequently the equation (1) has a mild solution on J. □

Theorem 3.2 *Assume that the assumptions of Theorem 3.1 hold and linear system* (2.6) *is approximately controllable on J. In addition, the functions f, g and σ are uniformly bounded on their respective domains. Further, if $S_\alpha(t)$ is compact, then the fractional control system* (2.1) *is approximately controllable on J.*

Proof Let $x^\kappa(\cdot)$ be a fixed point of Φ in B_q. By using the stochastic Fubini theorem, it is easy to see that

$$
\begin{aligned}
x^\kappa(b) =\ & \hat{x}_b-\kappa(\kappa I+\Gamma_0^b)^{-1}\left[\mathbb{E}\hat{x}_b+\int_0^b\hat{\varphi}(s)dw(s)-T_\alpha(b)[\phi(0)+g(0,\phi)]-g(b,x^\kappa_{\varepsilon(b,x_b^\kappa)})\right]\\
&+\kappa\int_0^b(\kappa I+\Gamma_s^b)^{-1}S_\alpha(b-s)f(s,x^\kappa_{\varepsilon(s,x_s^\kappa)})ds\\
&+\kappa\int_0^b(\kappa I+\Gamma_s^b)^{-1}S_\alpha(b-s)\sigma(s,x^\kappa_{\varepsilon(s,x_s^\kappa)})dw(s).
\end{aligned}
\tag{3.8}
$$

By the assumption on Theorem 3.2, we have

$$\|f(s,x^{\kappa}_{\varepsilon(s,x^{\kappa}_s)})\|^2+\|\sigma(s,x^{\kappa}_{\varepsilon(s,x^{\kappa}_s)})\|^2\le N_1 \quad \text{and} \quad \|g(s,x^{\kappa}_{\varepsilon(s,x^{\kappa}_s)})\|^2\le N_2.$$

If the linear system (2.6) is approximately controllable on every $[0,s]$, $0\le s\le b$, then by Lemma 2.10, the approximate controllability of (2.6) is equivalent to convergence of the operator $\kappa(\kappa I+\Gamma_0^b)^{-1}$ to zero operator in the strong operator topology as $\kappa\to 0^+$, and moreover $\|\kappa(\kappa I+\Gamma_s^b)^{-1}\|\le 1$.

Then there is a subsequence denoted by $\{f(s,x^{\kappa}_{\varepsilon(s,x^{\kappa}_s)}),\sigma(s,x^{\kappa}_{\varepsilon(s,x^{\kappa}_s)})\}$ weakly converging to say $\{f(s),\sigma(s)\}$. Thus, from the above equation, we have

$$\begin{aligned}
\mathbf{E}\|x^{\kappa}(b)-\hat{x}_b\|^2 \le\ & 7\|\kappa(\kappa I+\Gamma_0^b)^{-1}[\mathbf{E}\hat{x}_b-T_\alpha(b)(\phi(0)+g(0,\phi))]\|^2\\
&+7\mathbf{E}\left(\int_0^b\|\kappa(\kappa I+\Gamma_0^b)^{-1}\hat{\varphi}(s)\|^2_{\mathcal{L}_2^0}ds\right)+7\mathbf{E}\|\kappa(\kappa I+\Gamma_0^b)^{-1}g(b,x^{\kappa}_{\varepsilon(b,x^{\kappa}_b)})\|^2\\
&+7\mathbf{E}\left(\int_0^b\|\kappa(\kappa I+\Gamma_s^b)^{-1}S_\alpha(b-s)[f(s,x^{\kappa}_{\varepsilon(s,x^{\kappa}_s)})-f(s)]\|ds\right)^2\\
&+7\mathbf{E}\left(\int_0^b\|\kappa(\kappa I+\Gamma_s^b)^{-1}S_\alpha(b-s)f(s)\|ds\right)^2\\
&+7\mathbf{E}\left(\int_0^b\|\kappa(\kappa I+\Gamma_s^b)^{-1}S_\alpha(b-s)[\sigma(s,x^{\kappa}_{\varepsilon(s,x^{\kappa}_s)})-\sigma(s)]\|_{\mathcal{L}_2^0}ds\right)^2\\
&+7\mathbf{E}\left(\int_0^b\|\kappa(\kappa I+\Gamma_s^b)^{-1}S_\alpha(b-s)\sigma(s)\|_{\mathcal{L}_2^0}ds\right)^2.
\end{aligned}$$

Using the Lebesgue dominated convergence theorem and the compactness of $S_\alpha(t)$, we obtain $\mathbf{E}\|x^{\kappa}(b)-\hat{x}_b\|^2\to 0$ as $\kappa\to 0^+$. This gives the approximate controllability of (1). Hence the proof is complete. □

The mathematical formulation of many physical phenomena contain integro-differential equations, these integro-differential equations arise in various applications such as viscoelasticity, heat equations, fluid dynamics, chemical kinetics and so on. Motivated by this consideration, in this paper we construct the fractional control system in the following integro-differential framework

$$\begin{aligned}
{}^cD_t^\alpha[x(t)+g(t,x_{\varepsilon(t,x_t)})] &= A[x(t)+g(t,x_{\varepsilon(t,x_t)})]+\int_0^t G(t-s)x(s)ds+Bu(t)\\
&\quad +f(t,x_{\varepsilon(t,x_t)})+\sigma(t,x_{\varepsilon(t,x_t)})\frac{dw(t)}{dt},\\
&\quad \alpha\in(0,1),\quad t\in J:=[0,b]\\
x_0 &= \phi\in\mathcal{B},\quad x'(0)=0,
\end{aligned}\tag{3.9}$$

where A, $(G(t))_{t\ge0}$ are linear operators defined on Hilbert space $(\mathcal{H},\|\cdot\|_{\mathcal{H}})$, and ${}^cD_t^\alpha x(t)$ represent the Caputo derivative of order $\alpha>0$.

Further, we assume that the integro-differential abstract Cauchy problem

$$\begin{aligned}
{}^cD_t^\alpha x(t) &= Ax(t)+\int_0^t G(t-s)x(s)ds\\
x(0) &= x_0,\quad x'(0)=0,
\end{aligned}\tag{3.10}$$

has an associated α-resolvent operator of bounded linear operators $(\S_\alpha)_{t\geq 0}$ on $\mathcal{H}$.

One parameter family of bounded linear operators $(\S_\alpha)_{t\geq 0}$ on $\mathcal{H}$ is called α-resolvent operator of (3.9) if the following conditions are verified.

(1) The function $\S_\alpha(\cdot):\mathbb{R}_+\to L(\mathcal{H})$ is strongly continuous and $\S_\alpha(0)x=x$ for all $x\in\mathcal{H}$ and $\alpha\in(0,1)$.

(2) For all $x\in\mathcal{D}(A)$, $x\in\mathcal{C}(\mathbb{R}_+,[\mathcal{D}(A)])\cap\mathcal{C}'((0,\infty),\mathcal{H})$ and

$$^cD_t^\alpha \S_\alpha(t)x = A\S_\alpha(t)x+\int_0^t G(t-s)\S_\alpha(s)xds$$

$$^cD_t^\alpha \S_\alpha(t)x = \S_\alpha(t)Ax+\int_0^t \S_\alpha(t-s)G(s)xds,$$

for every $t\geq 0$.

Definition 3.3 *Let $\alpha\in(0,1)$, we define the family $\hat{\S}_\alpha(t)$ by $\hat{\S}_\alpha(t)x:=\int_0^t \hat{f}_{\alpha-1}(t-s)\S_\alpha(s)xds$, $x\in\mathcal{H}$ and $\hat{f}_\eta(t)t^{\eta-1}/\Gamma(\eta)$, $\eta\geq 0$, $t>0$ and Γ is the gamma function.*

Definition 3.4 *An $\mathcal{F}_t$-adapted stochastic process $x:(\infty,b]\to\mathcal{H}$ is called a mild solution of the system (14) on J if $x_0=\Phi$, $x_{\varepsilon(t,x_t)}\in\mathcal{B}$, $x|J\in\mathcal{C}(J,\mathcal{H})$ and*

$$\begin{aligned} x(t) &= \S_\alpha(t)g(0,\phi)-g(t,x_{\varepsilon(t,x_t)})+\int_0^t \hat{\S}_\alpha(t-s)Bu(s)ds \\ &\quad+\int_0^t \hat{\S}_\alpha(t-s)f(s,x_{\varepsilon(t,x_t)})ds+\int_0^t \hat{\S}_\alpha(t-s)\sigma(s,x_{\varepsilon(t,x_t)})dw(s), \qquad t\in J. \end{aligned}$$

Theorem 3.5 *Let the assumptions (H1)-(H6) hold, $\mathcal{Z}(\cdot)\in\mathcal{C}((0,b];L(\mathcal{H}))$ and $\hat{\S}_\alpha(t)$ is compact. Further, if the linear system corresponding to (3.8) is approximately controllable on J, then the system (3.8) is approximately controllable.*

Proof For all $\kappa>0$, define the operator $\hat{\Phi}:\tilde{\mathcal{B}}_0\to\tilde{\mathcal{B}}_0$ by $\hat{\Phi}z$ such that

$$\hat{\Phi}z(t)=\begin{cases} 0, & t\in(-\infty,0]; \\ \S_\alpha(t)g(0,\phi)-g(t,y_{\varepsilon(t,\bar{z}_t)}+\bar{z}_{\varepsilon(t,\bar{z}_t)})+\int_0^t \hat{\S}_\alpha(t-s)Bu^\kappa(s)ds & \\ +\int_0^t \hat{\S}_\alpha(t-s)f(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})ds & \\ +\int_0^t \hat{\S}_\alpha(t-s)\sigma(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s), & t\in J, \end{cases}$$

where $y_0=\phi$, $z:J\to\mathcal{H}$ and $\bar{z}(t)=z(t)$ for $t\in J$ with $z(0)=0$, and

$$\begin{aligned} u^\kappa(t) &= B^*\hat{\S}_\alpha^*(b-t)(\kappa I+\Gamma_0^b)^{-1} \\ &\quad\times\left\{\mathbb{E}\hat{x}_b+\int_0^b\hat{\varphi}(s)dw(s)-\S_\alpha(b)[\phi(0)+g(0,\phi)]+g(b,y_{\varepsilon(b,\bar{z}_b)}+\bar{z}_{\varepsilon(b,\bar{z}_b)})\right\} \\ &\quad-B^*\hat{\S}_\alpha^*(b-t)\int_0^b(\kappa I+\Gamma_s^b)^{-1}\hat{\S}_\alpha(b-s)f(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})ds \\ &\quad-B^*\hat{\S}_\alpha^*(b-t)\int_0^b(\kappa I+\Gamma_s^b)^{-1}\hat{\S}_\alpha(b-s)\sigma(s,y_{\varepsilon(s,\bar{z}_s)}+\bar{z}_{\varepsilon(s,\bar{z}_s)})dw(s). \end{aligned}$$

One can easily show that the operator $\hat{\Phi}$ has a fixed point by employing the technique used in Theorem 3.1 with some changes. Further, in order to prove the approximate controllability result, we assume that the functions g, f and σ are continuous and uniformly bounded. The proof of this theorem is similar to that of Theorem 3.2, and hence it is omitted. □

§4. An Example

Consider the following fractional stochastic partial differential equation with state-dependent delay and control of the form

$$\begin{aligned} {}^cD_t^{\alpha}\Big[z(t,y)+\int_{-\infty}^{t}H(s-t)z(s-\varepsilon_1(t)\varepsilon_2(\|z(t)\|),y)ds\Big] &= \frac{\partial^2}{\partial y^2}\Big[z(t,y)+ \\ &\int_{-\infty}^{t}H(s-t)z(s-\varepsilon_1(t)\varepsilon_2(\|z(t)\|),y)ds\Big]+\mu(t,y) \\ &+\int_{-\infty}^{t}K(s-t)z(s-\varepsilon_1(t)\varepsilon_2(\|z(t)\|),y)ds \\ &+\Big[\int_{-\infty}^{t}V(s-t)z(s-\varepsilon_1(t)\varepsilon_2(\|z(t)\|),y)ds\Big]\frac{d\beta(t)}{dt}, \\ &z(t,0)=z(t,\pi)=0, \quad t\in[0,1], \quad z(\theta,y)=\phi(\theta,y), \quad \theta\le 0, \quad y\in[0,\pi], \end{aligned} \tag{4.1}$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space (Ω, $\{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P}$); ${}^cD_t^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$. To represent this system in the abstract form, we consider the spaces $\mathcal{H} = \mathcal{U} = L^2[0,\pi]$ and $\mathcal{B} = \mathcal{C}_0 \times L^2(h,\mathcal{H})$ ($h : (-\infty,-r] \to \mathbb{R}$ be a positive function). We define the operator A by $Az = z''$ with the domain

$$\mathcal{D}(A) = \{z \in \mathcal{H}; z, z' \text{ are absolutely continuous}, \ \ z'' \in \mathcal{H} \ \text{ and } \ z(0) = z(\pi) = 0\}.$$

Then A generates a strongly continuous semigroup $(S(t))_{t\ge 0}$ which is compact. Now we introduce the functions $g(t,\xi)(y) = \int_{-\infty}^{0} a(-s)\xi(s,y)ds$ $f(t,\xi)(y) = \int_{-\infty}^{0} \tilde{a}(-s)\xi(s,y)ds$ and $\sigma = (t,\xi)(y) = \int_{-\infty}^{0} \hat{a}(-s)\xi(s,y)ds$, here $\varepsilon(s,y) = \varepsilon_1(s)\varepsilon_2(\|\xi(0)\|)$. Further, define the bounded linear operator $B : \mathcal{U} \to \mathcal{H}$ by $Bu(t)(y) = \mu(t,y)$, $0 \le y \le \pi$, $u \in \mathcal{U}$, where $\mu : [0,1]\times[0,\pi] \to [0,\pi]$ is continuous. On the other hand, the linear system corresponding to (4.1) is approximately controllable (but not exactly controllable). Then, the system (4.1) can be written in the abstract form of (2.1) and all the conditions of Theorem 3.2 are satisfied. Further, if we impose suitable conditions on g, f, σ and B to verify assumptions of Theorem 3.2, then we can conclude that the fractional control system (16) is approximately controllable on $[0,b]$.

Acknowledgement

This work was supported by The National Agency of Development of University Research (ANDRU), Algeria (PNR-SMA 2011-2014).

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