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Computation of Four Orthogonal Polynomials Connected to Euler's Generating Function of Factorials

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Abstract: In the present paper, first we describe the orthogonality relations between denominator polynomials of $[n-1/n]$ Pade approximants and related power series expansion; next we derive a continued fraction expansion called regular C-fraction for Euler's generating function of factorials and finally four orthogonal polynomials are extracted from numerator as well as denominator polynomials of both even and odd order convergents of the regular C-fraction connected to Pade approximants.

Key Words: Euler's generating function for factorials, regular C-fraction, orthogonal polynomials, Pade approximants.

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§1. Introduction

There is a very interesting literature [2-3] which interprets that $[n-1]/n$ order Pade approximants provides an orthogonality relation between its denominator polynomials and the power series expansion. They are nothing but even order convergents [2-3,7,9] of a regular C-fraction expansion of the power series expansion. The denominator polynomials transformed to monic form are orthogonal polynomials with respect to a linear moment functional $L : \mathbb{P} \longrightarrow \mathbb{R}$ from the space of all polynomials over $\mathbb{R}$ into $\mathbb{R}$ which has n^{th} moment same as the coefficient of x^n in a known power series. According to Favard's theorem [4-6,8] the necessary and sufficient condition for orthogonality of $P_n(x)$ is to satisfy the following three term recurrence relation:

$$\begin{aligned}&P_{-1}(x) := 0, \qquad P_0(x) := 1,\\ &P_n(x) := (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, 3, 4, \cdots,\end{aligned} \tag{1}$$

where c_n 's are real and λ_n 's are non-zero numbers. The orthogonality relation [5-6,8] is given by

$$L\{P_m(x)P_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n. \end{cases} \tag{2}$$

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Motivated strongly by the above works, in the present paper, four orthogonal polynomials are extracted from numerator as well as denominator polynomials of both even and odd order convergence of a regular C-fraction connected to Pade approximants for the Euler's generating function of factorials. In Section two, main results of Pade approximation, related continued fractions and orthogonal polynomials are reviewed which will be useful in the next Sections. In Section three, we derive a continued fraction expansion called regular C-fraction for Euler's generating function of factorials. In the last Section, we take the help of the regular C-fraction expansion derived in the previous Section to compute four orthogonal polynomials.

§2. Main Results of Pade Approximation, Related Continued Fractions and Orthogonal Polynomials

In this section, we review some results which are used for next Sections. We begin with the definition of Pade approximants and a standard result.

1. A rational function

$$[m/n]_f(x) = \frac{p_0^{(m,n)} + p_1^{(m,n)}\ x\ + \cdots + p_m^{(m,n)}\ x^m}{1 + q_1^{(m,n)}\ x + \cdots + q_n^{(m,n)}\ x^n} = \frac{P_m^{(m,n)}(x)}{Q_n^{(m,n)}(x)}$$

is said to be (m,n) order Pade approximants [2-3] for a formal power series

$$f(x) := a_0 + a_1\ x + \cdots + a_N\ x^N + \cdots ,$$

if

$$\begin{aligned}(1 + q_1^{(m,n)}\ x + \cdots + q_n^{(m,n)}\ x^n) \quad \times \quad (a_0 + a_1\ x + \cdots + a_{m+n}x^{m+n})\\ - ({p_0}^{(m,n)} + {p_1}^{(m,n)}\ x + \cdots + {p_m}^{(m,n)}\ x^m) = \mathbf{O}(x^{m+n+1}),\end{aligned}$$

where $q_i^{(m,n)}, p_j^{(m,n)}$ may or may not be zero $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.
The standard result [2-3] states that $[m/n]_f(x)$ exists and unique if and only if the $n \times n$ Hankel determinant:

$$H_{m,n} = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix} \neq 0,$$

where $a_i = 0, \quad \text{if } i < 0.$

2. Let

$$[n-1/n]_f(x) \quad = \quad \frac{P_{n-1}^{(n-1,n)}(x)}{Q_n^{(n-1,n)}(x)} \text{ be } \ (n-1,n) \ \text{ Pade approximants for } f(x).$$

Then $Q_n^{(n-1,n)}(x)$ satisfies

$$a_{n+k} + a_{n+k-1}q_1^{(n-1,n)} + \cdots + a_{k+1}q_{n-1}^{(n-1,n)} + a_k q_n^{(n-1,n)} = 0 \qquad (3)$$

for $k = 0, 1, 2, \cdots, n-1$. Put

$$\tilde{q}_n^{(n-1,n)}(x) = x^n Q_n^{(n-1,n)}\left(\frac{1}{x}\right) = x^n + q_1^{(n-1,n)} x^{n-1} + \cdots + q_n^{(n-1,n)}.$$

For the power series

$$f(x) := a_0 + a_1\ x + \cdots + a_N\ x^N + \cdots,$$

define a linear moment generating function by

$$L_f(x^n) = a_n.$$

Then (3) simply states that

$$L_f(x^k \tilde{q}_n^{(n-1,n)}(x)) = 0, \quad n = 0, 1, 2, 3, \cdots, n-1. \tag{4}$$

Hence

$$L_f(\tilde{q}_m^{(n-1,n)}(x)\ \tilde{q}_n^{(n-1,n)}(x)) = 0, \quad \text{if} \ \ m \neq n.$$

Therefore $\{\tilde{q}_n^{(n-1,n)}(x)\}$ forms monic orthogonal polynomial with respect to L_f.

3. Let

$$[n/n]_f(x) \quad = \quad \frac{P_n^{(n,n)}(x)}{Q_n^{(n,n)}(x)}$$

be (n, n) Pade approximants for $f(x)$. Then, we have

$$\frac{\tilde{P}_{n-1}^{(n,n)}(x)}{Q_n^{(n,n)}(x)} = [n-1/n]_{f_1}(x),$$

where $f_1(x) = a_1 + a_2 x + \cdots + a_{n+1} x^n + \cdots = \dfrac{f(x) - a_0}{x}$. Using item 2, $\{\tilde{q}_n^{(n,n)}(x)\}$ forms monic orthogonal polynomial with respect to L_{f_1}.

4. Let

$$[n+1/n]_f(x) \quad = \quad \frac{P_{n+1}^{(n+1,n)}(x)}{Q_n^{(n+1,n)}(x)}$$

be $(n+1, n)$ Pade approximants for $f(x)$. Then, we have

$$\frac{\tilde{P}_{n-1}^{(n+1,n)}(x)}{Q_n^{(n+1,n)}(x)} = [n-1/n]_{f_2}(x),$$

where $f_2(x) = a_2 + a_3 x + \cdots + a_{n+1} x^n + \cdots = \dfrac{f(x) - a_0 - a_1 x}{x^2}$. Using item 2, $\{\tilde{q}_n^{(n+1,n)}(x)\}$ forms monic orthogonal polynomial with respect to L_{f_2}.

5. Let $[m/n]_f(x) = \dfrac{P_m^{(m,n)}(x)}{Q_n^{(m,n)}(x)}$ be (m, n) order Pade approximants for $f(x)$. Then

$$\begin{aligned} \frac{1}{f(x)} - \frac{Q_n^{(m,n)}(x)}{P_m^{(m,n)}(x)} \quad &= \quad -\left[f(x) - \frac{P_m^{(m,n)}(x)}{Q_n^{(m,n)}(x)}\right]\left[\frac{f(x)P_m^{(m,n)}(x)}{Q_n^{(m,n)}(x)}\right]^{-1} \\ &= \quad \mathbf{O}(x^{m+n+1}). \end{aligned}$$

Hence

$$\frac{Q_n^{(m,n)}(x)}{P_m^{(m,n)}(x)} = [n/m]_{\frac{1}{f}}(x).$$

6. Suppose

$$f(x) = \frac{c_0}{1}_{\ +}\ \frac{c_1 x}{1}_{\ + \cdots +}\ \frac{c_n x}{1}_{\ + \cdots}.$$

Then [2-3]

$$\frac{P_1}{Q_1} = \frac{c_0}{1},\quad \frac{P_2}{Q_2} = \frac{c_0}{1+c_1x}, \cdots,\quad \frac{P_{2n}}{Q_{2n}} = \frac{P_{2n-1}+c_{2n}P_{2n-2}}{Q_{2n-1}+c_{2n}Q_{2n-2}} = [n-1/n]_f(x),$$
$$\frac{P_{2n+1}}{Q_{2n+1}} = \frac{P_{2n}+c_{2n+1}P_{2n-1}}{Q_{2n}+c_{2n+1}Q_{2n-1}} = [n/n]_f(x).$$

§3. Derivation of Regular C-Fraction for Euler's Generating Function of Factorials

The generating function for factorial numbers with its asymptotic relation

$$E(x) = \int_0^\infty \frac{e^{-t}}{1+xt}dt \sim \sum_{n=0}^{\infty}(-1)^n n! x^n, \quad \text{as} \quad x \longrightarrow 0,$$

was first studied by Euler [2]. It has a remarkable regular C-fraction expansion:

$$\int_0^\infty \frac{e^{-t}}{1+xt}dt = \frac{1}{1}_{\ +}\ \frac{x}{1}_{\ +}\ \frac{x}{1}_{\ + \cdots +}\ \frac{nx}{1}_{\ +}\ \frac{nx}{1}_{\ + \cdots}\ . \tag{5}$$

Theorem 3.1 *Let* $I_{2n-1} = \int_0^\infty \frac{t^{n-1}e^{-t}}{(1+xt)^n}dt$ *and* $I_{2n} = \int_0^\infty \frac{t^n e^{-t}}{(1+xt)^n}dt$, $n = 0, 1, 2, 3, \cdots$. *Then*

$$\begin{aligned} I_{2n-2} - I_{2n-1} &= xI_{2n}, \\ n\ I_{2n-1} - I_{2n} &= n\ xI_{2n+1} \end{aligned}$$

and as a result

$$\frac{I_{2n-1}}{I_{2n-2}} = \frac{1}{1}_{\ +}\ \frac{n\ x}{1\ +\ n\ x\frac{I_{2n+1}}{I_{2n}}}, \qquad n = 1, 2, 3, \cdots,$$

which readily gives

$$\frac{I_1}{I_0} = \int_0^\infty \frac{e^{-t}}{1+xt}dt = \frac{1}{1}_{\ +}\ \frac{x}{1}_{\ +}\ \frac{x}{1}_{\ + \cdots +}\ \frac{nx}{1}_{\ +}\ \frac{nx}{1}_{\ + \cdots}\ .$$

Proof We can show that

$$\begin{aligned} I_{2n-2} - I_{2n-1} &= \int_0^\infty \frac{[(1+xt)t^{n-1} - t^{n-1}]e^{-t}}{(1+xt)^n}dt \\ &= xI_{2n} \\ \frac{I_{2n-1}}{I_{2n-2}} &= \frac{1}{1 + n\ x\frac{I_{2n}}{n\ I_{2n-1}}} \end{aligned}$$

and

$$\begin{aligned} nI_{2n-1} - I_{2n} &= \int_0^\infty \frac{d(t^n\ e^{-t})}{(1+xt)^n} \\ &= n\ xI_{2n+1}. \\ \frac{I_{2n}}{n\ I_{2n-1}} &= \frac{1}{1+n\ x\frac{I_{2n+1}}{I_{2n}}}. \end{aligned}$$

For $n = 1$, also the identities hold:

$$I_1 = \frac{1}{1+x\frac{I_2}{I_1}}, \qquad \frac{I_2}{I_1} = \frac{1}{1+x\dfrac{I_2}{I_1}}.$$

Therefore

$$\frac{I_{2n-1}}{I_{2n-2}} = \cfrac{1}{1+\cfrac{n\ x}{1+n\ x\dfrac{I_{2n+1}}{I_{2n}}}}, \quad n = 1, 2, 3, \cdots \ .$$

Hence

$$\frac{I_1}{I_0} = I_1 = \int_0^\infty \frac{e^{-t}}{1+xt}dt = \frac{1}{1}{}_{+}\frac{x}{1}{}_{+}\frac{x}{1}{}_{+\cdots+}\frac{nx}{1}{}_{+}\frac{nx}{1}{}_{+\cdots} \ . \qquad \square$$

§4. Computation of Four Orthogonal Polynomials

Let us consider (5),

$$E(x) = \frac{1}{1}{}_{+}\frac{x}{1}{}_{+}\frac{x}{1}{}_{+\cdots+}\frac{nx}{1}{}_{+}\frac{nx}{1}{}_{+\cdots} \ .$$

In the language of Pade approximants, the continued fraction provides diagonal Pade approximants [2-3] which are given by (please see item 6 of Section two)

$$\frac{A_1}{B_1} = \frac{1}{1} = \frac{P_0^{(0,0)}}{Q_0^{(0,0)}}, \qquad \frac{A_3}{B_3} = \frac{1+x}{1+2x} = \frac{P_1^{(1,1)}}{Q_1^{(1,1)}}, \ \cdots \ , \frac{A_{2n+1}}{B_{2n+1}} = \frac{P_n^{(n,n)}}{Q_n^{(n,n)}}$$

and lower diagonal Pade approximants are given by (please see item 6 of Section 2)

$$\frac{A_2}{B_2} = \frac{1}{1+x} = \frac{P_0^{(0,1)}}{Q_0^{(0,1)}}, \qquad \frac{A_4}{B_4} = \frac{1+3x}{1+4x+2x^2} = \frac{P_1^{(1,2)}}{Q_1^{(1,2)}}, \ \cdots \ , \frac{A_{2n+2}}{B_{2n+2}} = \frac{P_n^{(n-1,n)}}{Q_n^{(n-1,n)}}.$$

Let us make use of definitions of even and odd parts of continued fraction as given in [9]. $[n-1/n]$ Pade approximants can be computed using the even part of continued fraction (5):

$$\frac{1}{1+x}{}_{-}\frac{1^2\ x^2}{1+3x}{}_{-\cdots-}\frac{n^2\ x^2}{1+(2n+1)x}{}_{-\cdots},$$

we obtain even order convergents:

$$\frac{A_{2n+2}(x)}{B_{2n+2}(x)} = \frac{(1+(2n+1)x)A_{2n}(x) - n^2x^2A_{2n-2}(x)}{(1+(2n+1)x)B_{2n}(x) - n^2x^2B_{2n-2}(x)}$$

with

$$\frac{A_2}{B_2} = \frac{1}{1+x}, \qquad \frac{A_4}{B_4} = \frac{1+3x}{1+4x+2x^2}, \quad n = 2, 3, 4, \cdots \quad .$$

Similarly, $[n/n]$ Pade approximants can be computed using the odd part of continued fraction (5):

$$1 - \frac{x}{1+2x} - \frac{1.2\ x^2}{1+4x} - \dots - \frac{n(n+1)\ x^2}{1+(2n+2)x} - \dots,$$

we obtain odd order convergents:

$$\frac{A_{2n+3}(x)}{B_{2n+3}(x)} = \frac{(1+(2n+2)x)A_{2n+1}(x) - n(n+1)x^2 A_{2n-1}(x)}{(1+(2n+2)x)B_{2n+1}(x) - n(n+1)x^2 B_{2n-1}(x)}$$

with

$$\frac{A_1}{B_1} = \frac{1}{1}, \qquad \frac{A_3}{B_3} = \frac{1+x}{1+2x}, \quad n = 2, 3, 4, \cdots \quad .$$

The desired orthogonal polynomials are nothing but

$$\begin{aligned} p_n(x) &= x^n A_{2n+2}\left(\frac{1}{x}\right), \quad n = 0, 1, 2, \cdots, \\ q_n(x) &= x^n B_{2n}\left(\frac{1}{x}\right), \quad n = 0, 1, 2, \cdots, \\ r_n(x) &= x^n A_{2n+1}\left(\frac{1}{x}\right), \quad n = 0, 1, 2, \cdots, \\ s_n(x) &= x^n B_{2n+1}\left(\frac{1}{x}\right), \quad n = 0, 1, 2, \cdots, \end{aligned}$$

where we can select $B_0\left(\frac{1}{x}\right) := 1$. Now we can describe orthogonality of $q_n(x)$, $s_n(x), p_n(x)$ and $r_n(x)$ as follows:

(1) Consider the series

$$E(x) = 0! - 1!x + 2!x^2 - 3!x^3 + \cdots + (-1)^n n! x^n + \cdots \quad .$$

The linear moment generating function with respect to $E(x)$ denoted by L_E has n^{th} moment,

$$L_E\{x^n\} = (-1)^n n!.$$

The three term recurrence relation of $q_n(x)$ is

$$\begin{aligned} q_{n+1}(x) &= (x + 2n + 1)q_n(x) - n^2 q_{n-1}(x), \\ q_0(x) &= 1, \qquad q_1(x) = x + 1, \quad n = 1, 2, 3, \cdots \quad . \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of $q_n(x)$ is

$$L_E\{q_m(x)q_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = (k-1)^2, \quad k = 2, 3, \cdots, n+1$. Following [1], it is well known fact in the literature that

$$q_n(x) = n!L_n(-x) = \sum_{r=0}^{n} r! \left[\binom{n}{r}\right]^2 x^{n-r},$$

where $L_n(x)$ is Laguerre polynomials of order 0.

(2) Using item 3 of Section two, we obtain the series

$$E_1(x) = \frac{E(x)-1}{x} = 1! - 2!x + \cdots + (-1)^n (n+1)! x^n + \cdots \quad .$$

The linear moment generating function with respect to $E_1(x)$ denoted by L_{E_1} has n^{th} moment

$$L_{E_1}\{x^n\} = (-1)^n (n+1)!.$$

The three term recurrence relation of $s_n(x)$ is

$$\begin{aligned} s_{n+1}(x) &= (x+2n+2)s_n(x) - n(n+1)s_{n-1}(x), \\ s_0(x) &= 1, \qquad s_1(x) = x+2, \quad n = 1, 2, 3, \cdots \quad . \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of $s_n(x)$ is

$$L_{E_1}\{s_m(x)s_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = (k-1)k, \quad k = 2, 3, \cdots, n+1$. Following [1], it is well known fact in the literature that

$$s_n(x) = n!L_n^1(-x) = \sum_{r=0}^{n} r! \binom{n}{r}\binom{n+1}{r} x^{n-r},$$

where $L_n^1(x)$ is Laguerre polynomials of order 1.

(3) Using items 4 and 5 of Section 2, we obtain the series

$$\frac{1}{E(x)} = 1 + x - d_1x^2 + d_2x^3 - d_3x^4 + \cdots + (-1)^n d_n x^{n+1} + \cdots \quad .$$

$$E_2(x) = \frac{\frac{-1}{E(x)} + 1 + x}{x^2} = d_1 - d_2x + d_3x^2 - d_4x^3 + \cdots + (-1)^n d_{n+1}x^n + \cdots \quad .$$

The linear moment generating function with respect to $E_2(x)$ denoted by L_{E_2} has n^{th} moment

$$L_{E_2}\{x^n\} = (-1)^n d_{n+1}.$$

The three term recurrence relation of $p_n(x)$ is

$$\begin{aligned} p_{n+1}(x) &= (x+2n+3)p_n(x) - (n+1)^2 p_{n-1}(x), \\ p_0(x) &= 1, \qquad p_1(x) = x+3, \quad n = 1, 2, 3, \cdots \quad . \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of $p_n(x)$ is

$$L_{E_2}\{p_m(x)p_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = k^2, \quad k = 2, 3, \cdots, n+1$.

(4) Using item 3 and item 5 of Section 2, we obtain the series

$$E_3(x) = \frac{\frac{1}{x}(1 - E(x))}{E(x)} = 1 - d_1 x + d_2 x^2 - d_3 x^3 + \cdots + (-1)^n d_n x^n + \cdots \quad .$$

The linear moment generating function with respect to $E_3(x)$ denoted by L_{E_3} has n^{th} moment

$$L_{E_3}\{x^n\} = (-1)^n d_n.$$

The three term recurrence relation of $r_n(x)$ is

$$\begin{aligned} r_{n+1}(x) &= (x + 2n + 2) r_n(x) - n(n+1) r_{n-1}(x), \\ r_0(x) &= 1, \qquad r_1(x) = x + 1, \quad n = 1, 2, 3, \cdots \quad . \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of $r_n(x)$ is

$$L_{E_3}\{r_m(x) r_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1 \lambda_2 \cdots \lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = k(k-1), \quad k = 2, 3, \cdots, n+1$.

Suppose $r_n(x) = x^n + r_{n-1} x^{n-1} + \cdots + r_1 x + r_0$. Since $L_{E_3}\{r_0(x) r_n(x)\} = 0$, we can compute d_n using

$$d_n = -[r_{n-1} d_{n-1} + \cdots + r_1 d_1 + r_0], \quad d_0 = 1, \quad n = 1, 2, \cdots \quad .$$

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