International J.Math. Combin. Vol.4(2012), 97-102

p^*-Graceful Graphs

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Abstract: A labeling or numbering of a graph is an assignment of labels to the vertices of G that induces a number to each edge. In this paper we define p^*-graceful graphs and investigate some graphs based on this definition.

Key Words: Pentagonal numbers, p^*-graceful graphs, Comb graph, Twig graph, Banana trees.

AMS(2010): 05C78

§1. Introduction

Throughout this paper, by a graph we mean a simple finite graph without isolated vertices. For all the terminology and notations in Graph Theory, we follow [2] and for all terminology regarding labeling we follow [4].

Graceful labeling has been suggested by Bermond in [1]. A graph $G = (V, E)$ is numbered if each vertex v is assigned a non-negative integer $f(v)$ and each edge uv is attributed the absolute value of the difference of numbers of its end points, that is, $|f(u) - f(v)|$. The numbering is called graceful if further more, we have the following three conditions: (1) all the vertices are labeled with distinct integers; (2) the largest value of the vertex labels is equal to the number of edges, i.e $f(v) \in \{0, 1, \cdots, q\}$ for all $v \in V(G)$ and (3) the edges of G are distinctly labeled with integers from 1 to q.

In this paper we suggest a labeling called p^*-graceful labeling which is an analogue to graceful labeling and investigate the p^*-graceful nature of some graphs.

§2. P^* Graceful Labeling Graphs

Definition 2.1 *A labeling f of a graph G is one-one mapping from the vertex set of G into the set of integers. Let G be a graph with q edges. Let $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$ be an injective function. Define the function $f_p^* : E(G) \to \{\omega^p(1), \omega^p(2), \cdots, \omega^p(q)\}$ such that $f_p^*(u, v) = |f_p(u) - f_p(v)|$. So f_p is said to be pentagonal graceful labeling of G and G is called a $p^* -$ graceful graph. Here $\omega^p(q) = \dfrac{q(3q-1)}{2}$ is the q^{th} pentagonal number.*

[1]Received August 28, 2012. Accepted December 16, 2012.

Theorem 2.1 *All paths are p^* graceful graphs.*

Proof Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of P_n, the path on n vertices. Define $f_p : V(P_n) \to \{0, 1, \cdots, \omega^p(n-1)\}$ such that

$$f_p(v_1) = 0;$$
$$f_p(v_{2i}) = f_p(v_{2i-1}) + \omega^p(q-(2i-2)) \quad \text{where } i = 1, 2, \cdots, \left\lfloor \frac{q}{2} \right\rfloor;$$
$$f_p(v_{2i+1}) = f_p(v_{2i}) - \omega^p(q-(2i-1)) \quad \text{where } i = 1, 2, \cdots, \left\lfloor \frac{q}{2} \right\rfloor.$$

Obviously, f_p is nothing but a pentagonal graceful labeling on P_n with $f_p^*(P_n) = \{\omega^p(1), \omega^p(2), \cdots, \omega^p(n-1)\}$. □

Theorem 2.2 *The graph nK_2 is p^*-graceful.*

Proof Let each K_2 be labeled with u_i, v_i where $1 \leqslant i \leqslant n$. Define $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$ such that

$$f_p(u_i) = i - 1 \quad \text{if } 1 \leqslant i \leqslant n;$$
$$f_p(v_i) = w^p(n-(i-1)) + f_p(u_i) \quad \text{if } 1 \leqslant i \leqslant n.$$

Then $f_p(u_i) \neq f_p(u_j)$ for $i \neq j$. Otherwise, if $f_p(u_i) = f_p(u_j)$ then $i - 1 = j - 1$. Thus $i = j$, a contradiction.

Again, if $i \neq j, f_p(v_i) \neq f_p(v_j)$. Otherwise if $f_p(v_i) = f_p(v_j)$ then $\omega^p(q-(i-1)) + f_p(u_i) = \omega^p(q-(j-1)) + f_p(u_j)$, i.e, $\omega^p(q-(i-1)) - \omega^p(q-(j-1)) = f_p(u_j) - f_p(u_i) \neq 0$. Thus $\omega^p(q-(i-1)) \neq \omega^p(q-(j-1))$. Consequently, $f_p(v_i) \neq f_p(v_j)$. Hence f_p is one-one.

Also

$$\begin{aligned} |f_p(u_i) - f_p(v_i)| &= |f_p(u_i) - \omega^p(q-(i-1)) - f_p(u_i)| \\ &= \omega^p(q-(i-1)) \text{ for } i = 1, 2, \cdots, n. \end{aligned}$$

We will have $\omega^p(n), \omega^p(n-1), \cdots, \omega^p(1)$ as the edge labels. Hence the result. □

Definition 2.2 *Let T be a tree. Denote the tree obtained from T by considering two copies of T and adding an edge by $T_{(2)}$ and in general the graph obtained from $T_{(n-1)}$ and T by adding an edge between them is denoted by $T_{(n)}$. Now $T_{(1)}$ is just T.*

Corollary 2.1 *$T_{(n)}$ is a p^*-graceful graph.*

Let P_{2n+1} be a path on $2n+1$ vertices. Take $2m+1 = \alpha$ is isomorphic copies of P_{2n+1}. Let w be a vertex which is adjacent to one end vertex of each copy. The newly obtained graph is a star with $2m+1$ spokes in which each spoke is a path of length $2n+1$ and is denoted by $S_{2n+1,2m+1}$. The degree of w is $2n+1$ and all the other vertices are of degree either 2 or 1. So this is a trivalent tree. In [5] Mathew Varkey proved that $S_{2k+1,2m+1}$ is a prime graph for all k and m. Now we prove the following.

Theorem 2.3 *The star $S_{2n+1,2m+1}$ is p^*-graceful for all $n, m \geqslant 1$.*

Proof Let P_{2n+1} be a path of length $2n$. Consider $\alpha = 2m+1$ isomorphic copies of P_{2n+1}. Adjoin a new vertex w to one end vertex of each copy of P_{2n+1}. Let $v_{i1} : i = 1, 2, \cdots, 2m+1$ be the vertices in the first level and $v_{ij} : i = 1, 2, \cdots, 2m+1$ and $j = 2, 3, \cdots, 2n+1$ be the remaining vertices of $S_{2n+1,2m+1}$. Define a function f_p from the vertex set of $S_{2n+1,2m+1}$ to the set of all non-negative integers less than or equal to number of edges of $S_{2n+1,2m+1}$ such that

$f_p(w) = 0, f_p(v_{i1}) = \omega^p(q - i + 1);\ i = 1, 2, \cdots, 2m+1$ and

$f_p(v_{ij}) = f_p(v_{i,j-1}) + (-1)^{j-1}\omega^p(q - (2m+1)(j-1) - (i-1))$ for $i = 1, 2, \cdots, 2m+1; j = 2, \cdots, 2n$.

Clearly f is injective. Hence $S_{2n+1,2m+1}$ is p^*-graceful. □

Corollary 2.2 *The star $S_{2n+1,2n+1}$ is p^*-graceful for all n.*

Corollary 2.3 *The star $S_{2n,2n}$ is p^*-graceful for all $n \geqslant 1$.*

Definition 2.3 *A caterpillar is a tree with the property that the removal of its end points leaves a path.*

Theorem 2.4 *A caterpillar $S(n_1, n_2, \cdots, n_m)$ is p^*-graceful.*

Definition 2.4 *The eccentricity $e(v)$ of a vertex v in a tree T is defined as $\max\{d(v, u) : u \in V(T)\}$ and the radius of T is the minimum eccentricity of the vertices.*

Definition 2.5 *A centre of a tree is a vertex of minimum eccentricity.*

Definition 2.6 *The neighborhood of vertex u is the set $N(u)$ consisting of all the vertices v which are adjacent with u. The closed neighborhood is defined as $N[u]$ and is given by $N[u] = N(u) \cup \{u\}$.*

A result by Jordan states that every tree has centre consisting of one point or two adjacent points. In this section we consider trees with exactly one centre.

Let $\{\alpha_1 K_{1,n_1}; \alpha_2 K_{1,n_2}; \cdots \alpha_p K_{1,n_p}\}$ be a family of stars where $\alpha_i K_{1,n_i}$ denotes α_i disjoint isomorphic copies of K_{1,n_i} for $i = 1, 2, \cdots, p$ and $\alpha_i \geqslant 1$. Let H_{ij} be the j^{th} isomorphic copy of K_{1,n_i} and u_{ij} and v_{ijk} for $k = 1, 2, \cdots, n_i$ be the central and end vertices respectively of H_{ij}. Let w be a new vertex adjacent to u_{ij} for $j = 1, 2, \cdots, \alpha_i; i = 1, 2, \cdots, p$. We thus obtain a new tree of radius 2 with unique centre which we shall denote by $H_w^{(\alpha_1+\alpha_2+\cdots+\alpha_p)}$. [see 5]. Now we prove the following theorem.

Theorem 2.5 *$H_w^{(\alpha_1+\alpha_2+\cdots+\alpha_p)}$ is p^*-graceful.*

Proof Consider the family of stars $\alpha_i K_{1,n_i}$ for $i = 1, 2, \cdots, p$. Let H_{ij} be the j^{th} isomorphic copy of K_{1,n_i} and u_{ij} and v_{ijk} for $k = 1, 2, \cdots, n_i$ be the central and end vertices respectively of H_{ij}. Let w be a new vertex adjacent to u_{ij} for $i = 1, 2, \cdots, p; j = 1, 2, \cdots, \alpha_i$ of each star.

Consider the mapping $f_p : V \to \{0, 1, \cdots, \omega^p(q)\}$ (where V is the vertex set and q is the number of edges of $H_w^{(\alpha_1+\alpha_2+\cdots+\alpha_p)}$) defined as $f_p(w) = 1, f_p(u_{11}) = 0$

$$\begin{aligned} f_p(u_{ij}) &= 0 \quad \text{for} \quad i = j = 1 \\ &= \omega^p(j) + 1 \quad \text{for} \quad i = 1 \text{ and } j = 2, 3, \cdots, \alpha_1 \\ &= \omega^p(\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} + j) + 1 \quad \text{for} \quad i = 2, 3, \ldots, p; j = 1, 2, \cdots \alpha_i \end{aligned}$$

$$\begin{aligned} f_p(v_{ijk}) &= \omega^p(q - (k-1)) \quad \text{for} \quad i = j = 1; k = 1, 2, \cdots, n_1 \\ &= \omega^p(q - (j-1)n_1 - (k-1)) + f_p(u_{1j}) \text{ for } i = 1; j = 2, 3, \cdots, \alpha_1; k = 1, 2, \cdots, n_1 \\ &= \omega^p\left(q - \sum_{l=1}^{i-1} \alpha_l n_l - (j-1)n_i - (k-1)\right) + f_p(u_{ij}) \\ &\quad \text{for } i = 2, 3, \cdots, p; j = 1, 2, \cdots, \alpha_p; k = 1, 2, \cdots, n_p \end{aligned}$$

Thus f_p is a one-one mapping which induces the edge labels $\{\omega^p(1), \omega^p(2), \ldots, \omega^p(q)\}$. Hence f_p is p^*- graceful labeling. Hence the theorem. □

Consider the family of stars $\{\alpha_1 K_{1,n_1}; \alpha_2 K_{1,n_2}; \cdots \alpha_p K_{1,n_p}\}$ where $\alpha_i K_{1,n_i}$ denotes α_i disjoint isomorphic copies of K_{1,n_i} for $i = 1, 2, \cdots, p$ and $\alpha_i \geqslant 1$. Let H_{ij} be the j^{th} isomorphic copy of K_{1,n_i} and u_{ij} and v_{ijk} for $k = 1, 2 \cdots, n_i$ be the central and end vertices respectively of H_{ij}. Adjoin a new vertex w to one end vertex of each star. The tree thus obtained is a tree with unique centre and radius 3 and is denoted by $H_w^{*(\alpha_1+\alpha_2+\cdots+\alpha_p)}$. Trees of this kind are referred to as banana trees, by some authors.

Corollary 2.4 *$H_w^{*(\alpha_1+\alpha_2+\cdots+\alpha_p)}$ is p^*-graceful.*

Definition 2.7 *The comb graph is a graph obtained from a path P_n by attaching a pendant vertex to each vertex of P_n.*

Theorem 2.6 *The comb graph $G = P_n \Theta K_1$ is p^*-graceful.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of P_n and $v_1, v_2, \cdots, v_n$ be the pendant vertices attached to $u_i : i = 1, 2, \cdots, n$. Then the graph $G = P_n \Theta K_1$ has $2n$ vertices and $q = 2n - 1$ edges.

Define $f_p : V(G) \to \{0, 1, \cdots, \omega^p(2n-1)\}$ such that

$$\begin{aligned} f_p(u_i) &= \omega^p(q - (i-1) + f_p(u_{i-1}) && \text{when } i \text{ is even} \\ &= f_p(u_{i-1}) - \omega^p(q - (i-1)) && \text{when } i \neq 1 \text{ is odd} \\ &= 0 && \text{when } i = 1 \end{aligned}$$

$$\begin{aligned} f_p(v_i) &= f_p(u_i) - \omega^p(q - (n+i-2)) && \text{when } i \text{ is even} \\ &= f_p(u_i) + \omega^p(q - (n+i-2)) && \text{when } i \neq 1 \text{ is odd} \\ &= \omega^p(q) && \text{when } i = 1 \end{aligned}$$

Thus $f_p^* = \{\omega^p(1), \cdots, \omega^p(q)\}$. Hence G is p^*-graceful. □

Definition 2.8 *A twig is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertex of the path.*

Theorem 2.7 *The twig graphs are p^*-graceful.*

Proof Let $v_1, v_2, \cdots, v_n$ be the n vertices of P_n and $v_{ij}; i = 2, 3 \cdots, n-1;\ j = 1, 2$ be the pendant vertices attached to each v_i. Then the graph has $q = 3n - 5$ edges.

Define $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$
such that $f_p(v_1) = 0$

$$f_p(v_{2i}) = f_p(v_{2i-1}) + \omega^p(q - (2i-2)) \qquad i = 1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor;$$
$$f_p(v_{2i+1}) = f_p(v_{2i}) - \omega^p(q - (2i-1)) \qquad i = 1, 2, \cdots, \left\lfloor \frac{n-1}{2} \right\rfloor;$$
$$f_p(v_{ij}) = f_p(v_i) + (-1)^{i-1}\omega^p(q - (n-1) - 2(i-2) - (j-1)) \text{ for } i = 2, \cdots, n-1; j = 1, 2.$$

Hence $f_p^* = \{\omega^p(1), \cdots, \omega^p(q)\}$. Therefore G is p^*-graceful. □

Definition 2.9 *The graph $C_n \hat{\circ} K_{1,n}$ is obtained from C_n and $K_{1,n}$ by identifying any vertex of C_n with the central vertex of $K_{1,n}$.*

Theorem 2.8 *$C_3 \hat{\circ} K_{1,n}$ is p^*-graceful for $n \geqslant 5$.*

Proof Let $C_3 \hat{\circ} K_{1,n} = G$ and let v_1, v_2, v_3 be the vertices of C_3, $u_1, u_2, \cdots, u_n$ be the vertices of $K_{1,n}$. Let v_1 be the vertex to which $K_{1,n}$ is attached with. The mapping $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$ where $q = n + 3$, defined by

$$\begin{aligned} f_p(u_i) &= \omega^p(i) \text{ for } i = 1, 2, 3 \\ &= \omega^p(i+1) \text{ for } i = 4, 5 \\ &= \omega^p(i+3) \text{ for } i = 6, 7, \cdots, q \end{aligned}$$

and $f_p(v_1) = 0, f_p(v_2) = 22 = \omega^p(4), f_p(v_3) = 92 = \omega^p(8)$ is p^*-graceful.

Further, the theorem is true only for $n \geqslant 5$. Since $C_3 \hat{\circ} K_{1,n}$ to have a p^*-graceful labeling we should have the pentagonal numbers $\omega^p(4)$ and $\omega^p(8)$ for the vertices of C_3 which is possible only with $n \geqslant 5$. □

§3. Graphs That Are Not p^*-Graceful.

Theorem 3.1 *Wheels are not p^*-graceful.*

Proof As the central vertex of a wheel is attached to all other vertices, 0 cannot be assigned to it, for if, 0 is the central label then the attaching vertices , that is, all the remaining vertices should have pentagonal numbers as their respective labels which in turn leads to non-pentagonal numbers as edge labels, contradicting the definition. Again if we assign 0 to any other vertex, then we will have to label its adjacent three vertices with pentagonal numbers which again generates non-pentagonal numbers as edge labels. Thus in no way wheels are p^*-graceful. □

Definition 3.1 *The Helm H_n is the graph obtained from a wheel by attaching a pendant vertex at each vertex of the n-cycle.*

Corollary 3.1 *The Helm H_n is not p^*-graceful.*

Definition 3.2 *The Fan graph $P_n + K_1$ is a graph obtained by joining the path P_n with the complete graph K_1.*

Corollary 3.2 *The Fan graph $P_n + K_1$ is not p^*-graceful.*

Definition 3.3 *Let W_n be a wheel with $n+1$ vertices. Attach a pendant edge to each rim vertex of W_n. Join each pendant vertex with the central vertex of the wheel. This graph is called the Flower graph denoted by F_n.*

Corollary 3.3 *The Flower graph F_n is not p^*-graceful.*

Remark Further research on the topic is pursued.

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