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The Crossing Number of The Generalized Petersen Graph $P[3k-1,k]$

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Abstract: The crossing number of a graph is the least number of crossings of edges among all drawings of the graph in the plane. In this paper, we investigate the crossing number of the generalized Petersen graph $P[3k-1,k]$ and get the result that $k \leq cr(P[3k-1,k]) \leq k+1$ for $k \geq 3$.

Key Words: crossing number, generalized Petersen graph, Cartesian product, Smarandache $\mathscr{P}$-drawing.

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§1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. A *Smarandache $\mathscr{P}$-drawing* of a graph G for a graphical property $\mathscr{P}$ is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$, which is said to be *optimal* if $\mathscr{P}=G$ with minimized crossings. The crossing number $cr(G)$ of a simple graph G is defined as the minimum number of edge crossings in a drawing of G in the plane. A drawing with the minimum number of crossings(an optimal drawing) must be a good drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing. Moreover, no three edges cross in a point. Let D be a good drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$.

The generalized Petersen graph $P[m,n]$ is defined to be the graph of order $2m$ whose vertex set is $\{u_1,u_2,\cdots,u_m;x_1,x_2,\cdots,x_m\}$ and edge set is $\{u_iu_{i+1},u_ix_i,x_ix_{i+n}, i=1,2,\cdots,m;$ addition modulo m$\}$. The Cartesian product of two graph G_1 and G_2, denoted by $G_1 \times G_2$, has the vertex set $V(G_1 \times G_2)=V(G_1)\times V(G_2)$, edge set $E(G_1\times G_2)=\{(u_i,u_j)(u_h,u_k)|u_i=u_h$ and $v_jv_k \in E(G_2)$; or $v_j=v_k$ and $u_iu_h \in V(G_1)\}$. In a drawing D, if an edge is not crossed by any other edge, we say that it is clean in D; if it is crossed by at least one edge, we say that it is crossed in D. The following proposition is a trivial observation.

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Proposition 1.1 *If there exists a clean edge e in a drawing D and contracting it results in a new drawing D^*, then $cr(D) \geq cr(D^*)$.*

Proposition 1.2 *If there exists a crossed edge e in a drawing D and contracting it results in a new drawing D^*, then $cr(D) \geq cr(D^*) + 1$.*

Proposition 1.3 *If G_1 is a subgraph of G_2, then $cr(G_1) \leq cr(G_2)$.*

Proposition 1.4 *Let G_1 be a graph homeomorphic to graph G_2, then $cr(G_1) = cr(G_2)$.*

Crossing number is an important parameter, which manifest the nonplanar of a given graph. It has not only theory significance but also great practical significance, early in the eighties of the 19th century. Bhatt and Leithon [1,2] showed that the crossing number of a network(graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for the network. Szekly [3] solved the very difficult problem of Edörs in dispersed geometry by crossing number of a graph. At present, the crossing number of a graph has widely used in VLSI layout, dispersed geometry, number theory, biological project and so on.

Calculating the crossing number of a given graph is NP-complete [4]. Only the crossing number of very few families of graphs are known exactly, some of which are the crossing number of generalized Petersen graph. Guy and Harary (1967) have shown that, for $k \geq 3$, the graph $P[2k,k]$ is homeomorphic to the Möbius ladder M_{2k}, so that its crossing number is one, and it is well known that $P[2k,2]$ is planar. Exoo and Harary, etc. researched on the crossing number of some generalized Petersen graph. In [5] they showed that

$$cr(P[n,2]) \leqslant \begin{cases} 0, & \text{for even}, \quad n \geq 4, \\ 3, & \text{for odd}, \quad n \geq 7. \end{cases}$$

and they also proved that $cr(P[3,2]) = 0$ and $cr(P[5,2]) = 2$. R.B.Richter and G.Salazar investigated the crossing number of the generalized Petersen graph P[N,3] and in [6] proved that(the graph P[9,3] is not be included)

$$cr(P[3k+h,3]) \leqslant \begin{cases} k+h, & \text{for k} \geq 3, \quad h \in \{0,2\}, \\ k+3, & \text{for k} \geq 3, \quad h = 1. \end{cases}$$

S.Fiorini and J.B.Gauci [7] study the crossing number of the generalized Petersen graph P[3k,k] and prover that $cr(P[3k,k]) = k$ for $k \geq 4$.

In this paper, we investigate the crossing number of the generalized Petersen graph $P[3k-1,k]$ and get the main result that

$$k \leq cr(P[3k-1,k]) \leq k+1 \text{ for } k \geq 3.$$

§2. Cartesian Products

Let S_3 denote the star-graph $K_{1,3}$ and P_n the path-graph with $n+1$ vertices, and consider the graph of the Cartesian product $S_3 \times P_n$, denoting the vertices $(0,i),(1,i),(2,i)$ and $(3,i)$ by

x_i, a_i, b_i and c_i, respectively for $(i = 0, 1, \cdots, n)$, where the vertices x_i represent the hubs of the star. In the drawing of $S_3 \times P_n$, we delete the path $\Gamma = (x_0, x_1, \cdots, x_n)$ which passes through the hubs of the stars. We let the subgraph of $((S_3 \times P_n) - \Gamma)$ induced by the vertices x_i, a_i, b_i and c_i be denoted by S^i. Also, the subgraph induced by the vertices $x_i, a_i, b_i, c_i, x_{i+1}, a_{i+1}, b_{i+1},$ and c_{i+1} is denoted by H^i, so that H^i is made up of S^i and S^{i+1} together with the three edges connecting the two stars, as illustrated in Figure 1.

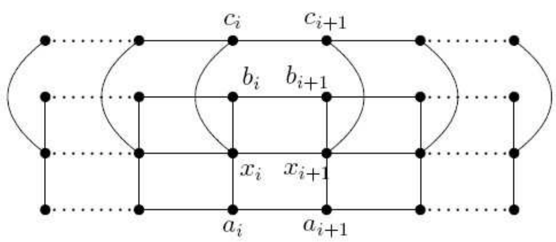

Figure 1 A good drawing of $S_3 \times P_n$

It is easy to obtain the following lemma 2.1 below, since the upper bound follows from the drawings of Figure 1, while the proof of the lower bound follows the same lines as that in Jendrol and Scerbova [8].

Lemma 2.1 *Let G denote the graph of Cartesian product $S_3 \times P_n (n \geq 1)$, with the path Γ joining the hubs of the stars deleted, that is, $G := ((S_3 \times P_n) - \Gamma)$. If D is a good drawing of G in which no star $S^i (i = 0, 1, \cdots, n)$ has a crossed edge, then $cr_D(G) = n - 1$.*

§3. The Generalized Petersen Graph $P[3k-1, k]$

The generalized Petersen graph $P[3k-1, k]$ of order $\{6k-2\}$ is made up of a principal cycle $C = \{u_1, u_2, \cdots, u_{3k-1}\}$, the spokes $u_i x_i$ and a adjoint principal cycle $\bar{C} = \{x_i, x_{k+i}, x_{2k+i}, \cdots, x_{3k-1}\}$, where $i = 1, 2, \cdots, 3k-1$ and addition is taken modulo $\{3k-1\}$. A drawing of $P[3k-1, k]$ is shown in Figure 2.

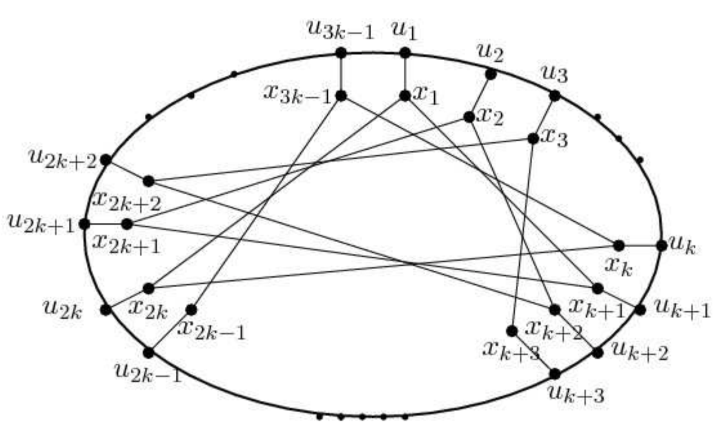

Figure 2

Let ϕ be a good drawing. In order to get an upper bound for the crossing number of $P[3k-1,k]$, we have shown a good drawing of $P[3k-1,k]$ in Figure 3.

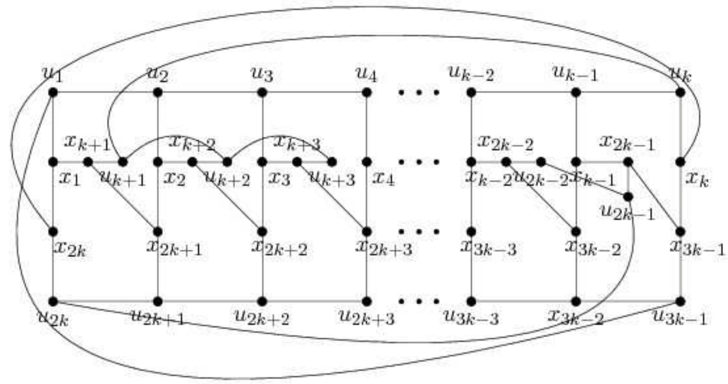

Figure 3: a good drawing of $P[3k-1,k]$

Figure 3 sets the upper bounded equal to $k+1$. To show that $cr(P[3k-1,k]) \geq k$, in the drawing of Figure 3, we note that by deleting the not crossed edges of $x_i x_{i+k}$ (where $k+1 \leq i \leq 2k-1$), and wipe away these 2-degree vertices $\{x_{k+1}, x_{k+2}, \cdots, x_{2k-1}, x_{2k+1}, \cdots, x_{3k-1}\}$. Considering the spoke $x_{2k}u_{2k}$ is clean or crossed, we now consider the following two cases.

Case 1 First, we consider that the spoke $x_{2k}u_{2k}$ is clean. Then contract the spoke $x_{2k}u_{2k}$ to the vertex u_{2k}, we obtain the graph G_k is shown in Figure 4, such that $G_k \supseteq (S_3 \times P_{k-1} - \Gamma)$. Obviously, $P[3k-1,k]$ contains G_k as a subgraph. By Proposition 1.1 and Proposition 1.3, we have $cr(P[3k-1,k]) \geq cr(G_k)$. Thus, in order to get a lower bound for the crossing number of $P[3k-1,k]$, we can simply consider the crossing number of the graph G_k as shown in Figure 4.

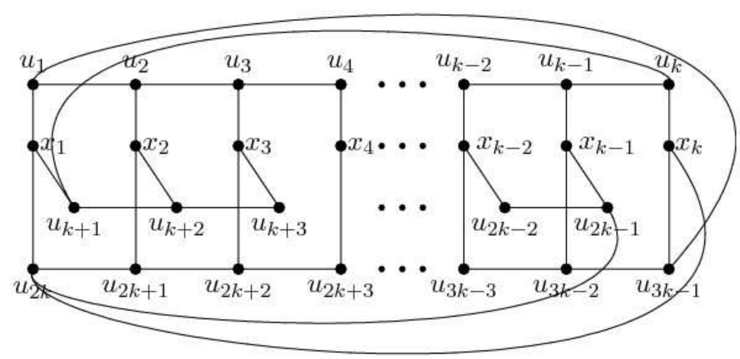

Figure 4: A good drawing of G_k

Lemma 3.1 $cr(P[3k-1,k]) \geq k \ \ (k \geq 3)$.

Proof For $k=1$, $P[2,1]$ is a planar graph. For $k=2$, from above $cr(P[5,2])=2$. Now we consider the case for $k \geq 3$.

For $k=3$, from [6] we have $cr(P[8,3]) = 4 \geq 3$. The theorem is true for $k=3$. Now suppose that for $k>3$. In order to prove that $cr(P[3k-1,k]) \geq k$, we only should to prove that $cr(G_k) \geq k$ for $k > 3$. We assume that t is the least value of the crossing number k for which $cr(G_t) \leq t-1$, t is greater than 3. We also note that the deletion of the vertex x_i and the edges incident to it (for values of i between 2 and $k-1$) from G_k yields a graph homeomorph to G_{k-1}. Therefore, since G_t contains G_{t-1} as a subgraph, we have $cr(G_t) \geq cr(G_{t-1}) \geq t-1$ by minimality of t. Thus, we only need to show that $cr(G_t) \neq t-1$. By assuming, for contradiction that it is

$$cr(G_t) = t-1. \qquad (1).$$

We divide the problem into three cases to prove that $cr(G_t) \geq t$.

Case 1 The spoke $x_{2k}u_{2k}$ is not crossed.

Case 1.1 First we consider an optimal drawing D of G_t and assume that a star edge of $\{x_iu_i, x_iu_{i+t}, x_iu_{i+2t-1}\}$ for $2 \leq i \leq t-1$, makes a positive contribution to the crossing number of G_t. In this case, when we delete the hub $x_i(2 \leq i \leq t-1)$ we get an induced drawing D_1 of homeomorph of G_{t-1} such that

$$\begin{aligned} t-1 &= cr(G_t) \geq cr_{D_1}(G_{t-1}) + 1 \\ &\geq (t-1) + 1, \end{aligned}$$

by the inductive hypothesis, a contradiction.

Case 1.2 Now we consider an optimal drawing D of G_t and assume that the star edge of $\{x_1u_1, x_1u_{k+1}, x_1u_{2k}\}$ or $\{x_ku_k, x_ku_{3k-1}, x_ku_{2k}\}$ makes a positive contribution to the crossing number of G_t. In this case, we delete the hub $x_i(x = 1$ or $k)$ because there is no crossing in the $\{3t-1\}$ principal cycle C. So the edge $\{u_{2k}u_{2k+1}\}$ or $\{u_1u_{3k-1}\}$ is clean. Then contract the edge $\{u_{2k}u_{2k+1}\}$ or $\{u_1u_{3k-1}\}$. Following the same arguments presented in Case 1.1, we get an induced drawing D_2 of homeomorph of G_{t-1} such that

$$\begin{aligned} t-1 &= cr(G_t) \geq cr_{D_2}(G_{t-1}) + 1 \\ &\geq (t-1) + 1, \end{aligned}$$

by the inductive hypothesis, a contradiction.

Case 1.3 Thus, we can assume that all the $\{t-1\}$ crossings of G_t are self-intersections of the $\{3t-1\}$ principal cycle C made up of the edges u_iu_{i+1} for $1 \leq i \leq 3t-1$ addition modulo $\{3t-1\}$.

Therefore, there exists an optimal drawing D_3 of G_t such that in D_3 the edges of the stars do not contribute to the crossing number. We divide the problem in to the following different subcases.

Subcase 1.3.1 If there is an edge e in u_iu_{i+1} in D_3 which is crossed twice and more, then deleting e together with the two other edges at distance t from e along C. As it is the edges $\{u_iu_{i+1}, u_{k+i}u_{k+i+1}, u_{2k+i-1}u_{2k+i}\}$, then we get a subgraph homeomorphic to $(S_3 \times P_{t-1} - \Gamma)$. Therefore, by Lemma 2.1 above.

$$\begin{aligned} t-1 &= cr(G_t) \geq cr(S_3 \times P_{t-1} - \Gamma) + 2 \\ &= (t-2) + 2 = t, \text{ a contradition.} \end{aligned}$$

Subcase 1.3.2 If there are two edges in D_3 at distance t or $t-1$ from each other which are crossed but do not cross each other, then repeating the same procedure as in case 1. So we have

$$\begin{aligned} t-1 &= cr(G_k) \geq cr(S_3 \times P_{t-1} - \Gamma) + 2 \\ &= (t-2) + 2 = t, \text{ acontradiction.} \end{aligned}$$

We can therefore assume hereafter that in D_3 there is no edge which is crossed twice and no two edges at distant t from each other giving a contribution of two to the crossing. As it is, now we consider the case there is an edge e in D_3 which is crossed once. There remains to show that if in D_3:

(i) There are no two edges at a distance t which not intersected, or

(ii) There are two edges at a distance t from each other which are pairwise intersecting, then in both cases we get a contradiction.

Let us first assume that no two edges at a distance t or $t-1$ from each other can be found such that they are both intersected. We divide the $\{3t-1\}$ principal cycle C into two $\{t\}$-sectors and one $\{t-1\}$-sector such that the number of crossed edges in each sector is p, q and r, respectively. Since in Sector 1 there are p crossed edges which cannot be matched to crossed edges in Sector 2 and 3. Hence in each of Sector 2 and 3 there are p edges which are not intersected. Similarly for q and r. Thus, the number of uncrossed edges is at least $2(p+q+r)$. However, the total number of edges≥(numbers of crossed edges)+(number of uncrossed edges) is,

$$\begin{aligned} 3t-1 &\geq (p+q+r) + 2(p+q+r) \\ &= 3(p+q+r). \end{aligned}$$

Let $p+q+r = x$. This implies that $3t - 3x \geq 1$. As t is the least value of k for the crossing number, then $t \leq x$. Thus $3t - 3x \geq 1$, a contradiction.

We now assume that there are two edges in D_3 at distant t or $t-1$ from each other that at intersect each other. That is, if $u_iu_{i+1}(1 \leq i \leq t$ addition taken modulo t) is intersected by an edge e, then $e \in \{u_{i+t}u_{i+t+1}, u_{i+2t-1}u_{i+2t}\}$. Without loss of generality, we assume that $e = \{u_{i+t}u_{i+t+1}\}$ and consequently, that the edge $u_{i+2t-1}u_{i+2t}$ is not intersected.

We consider the subgraph H induced by $S(x_i) \cup S(x_{i+1}) \setminus \{x_iu_{i+t}, x_{i+1}u_{i+t+1}\}$ (shown in Figure 5(a)). This is a 6-circuit none of whose edges is intersected, with the sole exception of u_iu_{i+1} which is intersect once by $u_{i+t}u_{i+t+1}$. ,Thus, H is planarly embedded and without loss of generality, we let $u_{i+1} \in Int(H)$ and $u_{i+t+1} \in Ext(H)$(since if these vertices are both in Int(H) or in Ext(H), then the edges u_iu_{i+1} is crossed an even number of times). Therefore, we have the subgraph shown in the drawing of Figure5(b). Now x_{i+2} either lies in $Int(H)$ or $Ext(H)$ and none of the edges of S_{i+2} can be crossed. Also, the edges of the subgraph in Figure 4(b) cannot be crossed(apart from the crossing shown), giving us the required contradiction. Therefore, $cr(G_k) \neq t-1$. So, formula (1) does not hold. So, we have shown that $cr(G_k) \geq k$, hence

$$cr(P[3k-1,k]) \geq cr(G_k) \geq k \quad (k \geq 3). \tag{2}$$

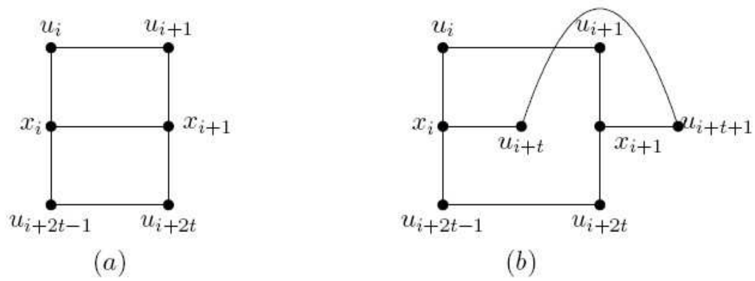

Figure 5

Case 2 Now we consider that the spoke $x_{2k}u_{2k}$ is crossed. Using the analogous arguments presented in Case 1. Then contract the spoke $x_{2k}u_{2k}$ to the vertex u_{2k}. We obtain the graph G_k is same shown in Figure 4, such that $G_k \supseteq (S_3 \times P_{k-1} - \Gamma)$. By Proposition 1.2 and Proposition 1.3, then we have $cr(P[3k-1,k]) \geq cr(G_k)+1$. We can follow the same conclusion $cr(G_k) \geq k$ presented in Case 1. Hence

$$cr(P[3k-1,k]) \geq k+1. \tag{3}$$

As a result, from all the above cases, combine with formula (2) and (3), we have shown that $cr(P[3k-1,k]) \geq k$. □

Theorem 3.1 $k \leq cr(P[3k-1,k]) \leq k+1 \quad (k \geq 3)$.

Proof A good drawing of $P[3k-1,k]$ in Fig.3 shows that $cr(P[3k-1,k]) \leq k+1$ for $k \geq 3$. This together with Lemma 3.1 immediately indicate that

$$k \leq cr(P[3k-1,k] \leq k+1 \quad (k \geq 3).$$ □

We end this paper by presenting the following conjecture.

Conjecture $cr(P[3k-1,k]) = k+1 \quad (k \geq 3)$.

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