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Magic Graphoidal on Join of Two Graphs

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Abstract: B. D. Acharya and E. Sampathkumar [1] defined graphoidal cover as partition of edge set of G into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of G. Let $G = \{V, E\}$ be a graph and let ψ be a graphoidal cover of G. Define $f : V \cup E \rightarrow \{1, 2, \ldots, p + q\}$ such that for every path $P = (v_0v_1v_2 \ldots v_n)$ in ψ with

$$f^*(P) = f(v_0) + f(v_n) + \sum_1^n f(v_{i-1}v_i) = k,$$

a constant, where f^* is the induced labeling on ψ. Then, we say that G admits ψ - magic graphoidal total labeling of G. A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ- magic graphoidal total labeling. In this paper, we proved that Wheel $W_n = C_{n-1} + K_1, K_2 + mK_1, K_2' + mk_1$, Fan $P_n + K_1$, Double Fan $P_n + 2K_1$ and Parachute $W_{n,2} = P_{2,n-2}$ are magic graphoidal.

Key Words: Graphoidal cover, magic graphoidal, graphoidal constant.

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§1. Introduction

By a graph we mean a finite simple and undirected graph. The vertex set and edge set of a graph G denoted are by $V(G)$ and $E(G)$ respectively. Wheel $W_n = C_{n-1} + K_1$ is a wheel, $K_2 + mK_1$ is a graph obtained by joining m isolated vertices to each end of K_2, a graph of path of length $1, \overline{K}_2 + mK_1$ is a graph obtained by joining m isolated vertices to each end of $\overline{K}_2, P_n + K_1$ is a fan, $P_n + 2K_1$ is a double fan and $W_{n,2} = P_{2,n-2}$ is Parachute. Terms and notations not used here are as in [3].

§2. Preliminaries

Let $G = \{V, E\}$ be a graph with p vertices and q edges. A graphoidal cover ψ of G is a collection of (open) paths such that

(i) every edge is in exactly one path of ψ

(ii) every vertex is an interval vertex of almost one path in ψ.

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We define $\gamma(G) = \min_{\psi \in \zeta} |\psi|$, where ζ is the collection of graphoidal covers ψ of G and γ is graphoidal covering number of G.

Let ψ be a graphoidal cover of G. Then we say that G admits ψ-magic graphoidal total labeling of G if there exists a bijection $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that for every path $P = (v_0 v_1 v_2 \ldots v_n)$ in ψ, then $f^*(P) = f(v_0) + f(v_n) + \sum_1^n f(v_{i-1}v_i) = k$, a constant, where f^* is the induced labeling of ψ. A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ-magic graphoidal total labeling. In this paper, we proved that Wheel $W_n = C_{n-1} + K_1, K_2 + mK_1, K_2' + mk_1$, Fan $P_n + K_1$, Double Fan $P_n + 2K_1$ and Parachute $W_{n,2} = P_{2,n-2}$ are magic graphoidal.

Result 2.1([11]) *Let $G = (p, q)$ be a simple graph. If every vertex of G is an internal vertex in ψ, then $\gamma(G) = q - p$.*

Result 2.2([11]) *If every vertex v of a simple graph G, where degree is more than one, i.e., $d(v) > 1$, is an internal vertex of ψ is minimum graphoidal cover of G and $\gamma(G) = q - p + n$, where n is the number of vertices having degree one.*

Result 2.3([11]) *Let G be (p, q) a simple graph. Then $\gamma(G) = q - p + t$, where t is the number of vertices which are not internal.*

Result 2.4([11]) *For any k-regular graph $G, k \geq 3, \gamma(G) = q - p$.*

Result 2.5([11]) *For any graph G with $\delta \geq 3, \gamma(G) = q - p$.*

§3. Magic Graphoidal on Join of Two Graphs

Theorem 3.1 *The graph $K_2 + mK_1$ is magic graphoidal.*

Proof Let $G = K_2 + mK_1$ with $V(G) = \{u, v, u_i, 1 \leq i \leq m\}$ and $E(G) = \{(uv)\} \cup \{(uu_i)(vu_i), 1 \leq i \leq m\}$. Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(u) = 2m+2; \quad f(v) = 2m+3; \quad f(uv) = 2m+1,$$
$$f(uu_i) = i,\ 1 \leq i \leq m,$$
$$f(vu_i) = 2m+1-i,\ 1 \leq i \leq m.$$

Let $\psi = \{(uv), [(uu_iv) : 1 \leq i \leq m]\}$. Then

$$\begin{aligned} f^*[(uv)] &= f(u) + f(v) + f(uv) \\ &= 2m+2+2m+3+2m+1 \\ &= 6m+6. \end{aligned} \tag{1}$$

For integers $1 \leq i \leq m$,

$$\begin{aligned} f^*[(uu_iv)] &= f(u) + f(v) + f(uu_i) + f(u_iv) \\ &= 2m + 2 + 2m + 3 + i + 2m + 1 - i \\ &= 6m + 6. \end{aligned} \tag{2}$$

From (1) and (2), we conclude that ψ is minimum magic graphoidal cover. Hence, K_2+mK_1 is magic graphoidal. $\square$

For example, the magic graphoidal cover of $K_2 + 7K_1$ is shown in Figure 1,

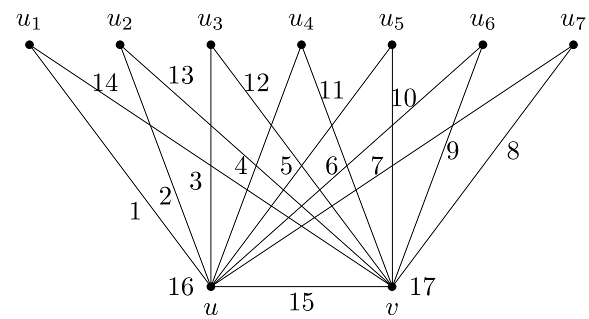

Figure 1 $K_2 + 7K_1$

where, $\psi = \{(uv), (uu_1v), (uu_2v), (uu_3v), (uu_4v), (uu_5v), (uu_6v), (uu_7v)\}, \gamma = 8$ and $K = 48$.

Theorem 3.2 *Parachute $W_{n,2} = P_{2,n-2}$ is magic graphoidal.*

Proof Let $G = W_{n,2}$ with $V(G) = \{u_i : 1 \leq i \leq n], v\}$ and $E(G) = \{(u_iu_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_1u_n)\} \cup \{(vu_i) : 1 \leq i \leq 2\}$.

Case 1 n is odd.

Let $\psi = \left\{\left(u_nvu_1u_2 \ldots u_{\frac{n+1}{2}}\right), \left(u_1u_nu_{n-1}u_{n-2} \ldots u_{\frac{n+1}{2}}\right)\right\}$.

Subcase 1.1 $n = 1(mod\ 4)$.

Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(u_1) = 4, \quad f(u_n) = 1, \quad f(u_1u_n) = 5,$$

$$f(vu_1) = 6, \quad f(vu_n) = 2, \quad f\left(u_{\frac{n+1}{2}}\right) = p+q$$

$$f(u_iu_{i+1}) = \begin{cases} 2i+5 & \text{if } i = 1(mod\ 2), 1 \leq i \leq \dfrac{n-1}{2} \\ 2i+6 & \text{if } i = 0(mod\ 2), 1 \leq i \leq \dfrac{n-1}{2} \end{cases}$$

$$f(u_{n+1-i}u_{n-i}) = \begin{cases} 2i+6 & \text{if } i = 1(mod\ 2), 1 \leq i \leq \dfrac{n-1}{2} \\ 2i+5 & \text{if } i = 0(mod\ 2), 1 \leq i \leq \dfrac{n-1}{2} \end{cases}$$

Then

$$\begin{aligned}
f^*\left[\left(u_n v u_1 u_2 \dots u_{\frac{n+1}{2}}\right)\right] &= f(u_n) + f\left(u_{\frac{n+1}{2}}\right) + f(u_n v) \\
&\quad + f(vu_1) + f(u_1u_2) + \cdots + f\left(u_{\frac{n-1}{2}} u_{\frac{n+1}{2}}\right) \\
&= 1 + p + q + 2 + 6 + \sum_{i=1,3}^{\frac{n+1}{2}-2} (2i+5) + \sum_{i=2,4}^{\frac{n+1}{2}-1} (2i+6) \\
&= p + q + 9 + 11\left(\frac{n-1}{4}\right) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \qquad (3)
\end{aligned}$$

$$\begin{aligned}
f^*\left[\left(u_1 u_n u_{n-1} u_{n-2} \dots u_{\frac{n+1}{2}}\right)\right] &= f(u_1) + f\left(u_{\frac{n+1}{2}}\right) + f(u_1 u_n) + f(u_n u_{n-1}) \\
&\quad + f(u_{n-1}u_{n-2}) + \cdots + f\left(u_{\frac{n-1}{2}} u_{\frac{n+1}{2}}\right) \\
&= 4 + p + q + 5 + \sum_{i=1,3}^{\frac{n+1}{2}-2} (2i+6) + \sum_{i=2,4}^{\frac{n+1}{2}-1} (2i+5) \\
&= p + q + 9 + 11\left(\frac{n-1}{4}\right) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \qquad (4)
\end{aligned}$$

From (3) and (4), we conclude that ψ is minimum magic graphoidal cover. Hence, $W_{n,2}$ is magic graphoidal. For example, the magic graphoidal cover of $W_{5,2}$ is shown in Figure 2.

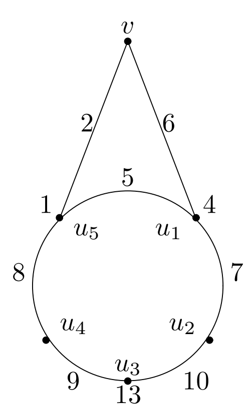

Figure 2 $W_{5.2}$

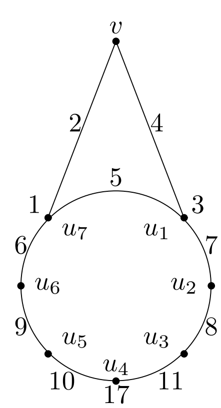

Figure 3 $W_{7.2}$

Subcase 1.2 $n = 3(mod\ 4)$.

Define $f : V \cup E \to \{1, 2, \dots, p+q\}$ by

$$\begin{aligned}
&f(u_1) = 3; \quad f(u_n) = 1; \quad f(vu_n) = 2, \\
&f(vu_1) = 4, \quad f(u_1u_n) = 5; \quad f\left(u_{\frac{n+1}{2}}\right) = p + q,
\end{aligned}$$

$$f(u_iu_{i+1}) = \begin{cases} 2i+5 & \text{if } i = 1(mod\ 2), 1 \le i \le \dfrac{n-1}{2} \\ 2i+4 & \text{if } i = 0(mod\ 2), 1 \le i \le \dfrac{n-1}{2} \end{cases}$$

$$f(u_{n+1-i}u_{n-i}) = \begin{cases} 2i+4 & \text{if } i = 1(mod\ 2), 1 \le i \le \dfrac{n-1}{2} \\ 2i+5 & \text{if } i = 0(mod\ 2), 1 \le i \le \dfrac{n-1}{2} \end{cases}$$

Then

$$\begin{aligned} f^*\left[\left(u_nvu_1u_2\dots u_{\frac{n+1}{2}}\right)\right] &= f(u_n) + f\left(u_{\frac{n+1}{2}}\right) + f(u_nv) \\ &\quad + f(vu_1) + f(u_1u_2) + \cdots + f\left(u_{\frac{n-1}{2}}u_{\frac{n+1}{2}}\right) \\ &= 1 + p + q + 2 + 4 + \sum_{i=1,3}^{\frac{n+1}{2}-1}(2i+5) + \sum_{i=2,4}^{\frac{n+1}{2}-2}(2i+4) \\ &= p + q + 3 + 9\left(\frac{n+1}{4}\right) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \end{aligned} \tag{5}$$

$$\begin{aligned} f^*\left[\left(u_1u_nu_{n-1}u_{n-2}\dots u_{\frac{n+1}{2}}\right)\right] &= f(u_1) + f\left(u_{\frac{n+1}{2}}\right) + f(u_1u_n) + f(u_nu_{n-1}) \\ &\quad + f(u_{n-1}u_{n-2}) + \cdots + f\left(u_{\frac{n-1}{2}}u_{\frac{n+1}{2}}\right) \\ &= 3 + p + q + 5 + \sum_{i=1,3}^{\frac{n+1}{2}-1}(2i+4) + \sum_{i=2,4}^{\frac{n+1}{2}-2}(2i+5) \\ &= p + q + 3 + 9\left(\frac{n+1}{4}\right) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \end{aligned} \tag{6}$$

From (5) and (6), we conclude that ψ is minimum magic graphoidal cover. Hence, $W_{n,2}$ is magic graphoidal. For example, the magic graphoidal cover of $W_{7,2}$ is shown in Figure 3.

Case 2 n is even.

Let $\psi = \left\{\left(u_nvu_1u_2\dots u_{\frac{n}{2}}\right), \left(u_1u_nu_{n-1}u_{n-2}\dots u_{\frac{n}{2}}\right)\right\}$.

Subcase 2.1 $n = 2(mod\ 4)$.

Define $f : V \cup E \to \{1, 2, \dots, p+q\}$ by

$$f(u_1) = 2;\quad f(vu_1) = 7;\quad f(u_n) = 1,$$
$$f(u_1u_n) = 4;\quad f(vu_n) = 3;\quad f(u_nu_{n-1}) = 5,$$
$$f\left(u_{\frac{n}{2}}\right) = p + q,$$

$$f(u_iu_{i+1}) = \begin{cases} 2i+6 & \text{if } i = 1(mod\ 2), 1 \le i < \dfrac{n}{2} \\ 2i+7 & \text{if } i = 0(mod\ 2), 1 \le i < \dfrac{n}{2} \end{cases},$$

$$f(u_{n+1-i}u_{n-i}) = \begin{cases} 2i+4 & \text{if } i = 1(mod\ 2), 3 \le i \le \dfrac{n}{2} \\ 2i+5 & \text{if } i = 0(mod\ 2), 2 \le i \le \dfrac{n}{2} \end{cases}.$$

Then,

$$\begin{aligned}
f^*\left[\left(u_n v u_1 u_2 \ldots u_{\frac{n}{2}}\right)\right] &= f(u_n) + f\left(u_{\frac{n}{2}}\right) + f(u_n v) \\
&\quad + f(vu_1) + f(u_1u_2) + \cdots + f\left(u_{\frac{n}{2}-1}u_{\frac{n}{2}}\right) \\
&= 1 + p + q + 3 + 7 + \sum_{i=1,3}^{\frac{n}{2}-2}(2i+6) + \sum_{i=2,4}^{\frac{n}{2}-1}(2i+7) \\
&= p + q + 9 + 11 + \frac{13}{2}\left(\frac{n}{2}-1\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) \qquad (7) \\
f^*\left[\left(u_1 u_n u_{n-1} u_{n-2} \ldots u_{\frac{n}{2}}\right)\right] &= f(u_1) + f\left(u_{\frac{n}{2}}\right) + f(u_1u_n) + f(u_nu_{n-1}) \\
&\quad + f(u_{n-1}u_{n-2}) + \cdots + f\left(u_{\frac{n}{2}+1}u_{\frac{n}{2}}\right) \\
&= 2 + p + q + 4 + 5 + \sum_{i=2,4}^{\frac{n}{2}-1}(2i+5) + \sum_{i=1,3}^{\frac{n}{2}}(2i+4) \\
&= p + q + 11 + \frac{13}{2}\left(\frac{n}{2}-1\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) \qquad (8)
\end{aligned}$$

From (7) and (8), we conclude that ψ is minimum magic graphoidal cover. Hence, $W_{n,2}$ is magic graphoidal. For example, the magic graphoidal cover of $W_{6,2}$ is shown in Figure 4.

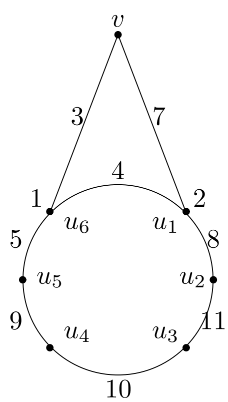

Figure 4 $W_{6.2}$

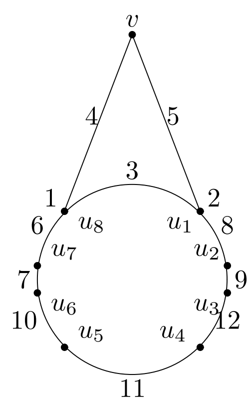

Figure 5 $W_{8.2}$

Subcase 2.2 $n = 0(mod\ 4)$.

Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(u_1) = 2; \quad f(u_n) = 1; f\left(u_{\frac{n}{2}}\right) = p+q,$$

$$f(u_nu_{n-1}) = 6; \quad f(vu_1) = 5; \quad f(vu_n) = 4; \quad f(u_1u_n) = 3,$$

$$f(u_iu_{i+1}) = \begin{cases} 2i+6 & \text{if } i = 1(mod\ 2), 1 \le i \le \frac{n}{2}-1 \\ 2i+5 & \text{if } i = 0(mod\ 2), 1 \le i \le \frac{n}{2}-1 \end{cases},$$

$$f(u_{n+1-i}u_{n-i}) = \begin{cases} 2i+4 & \text{if } i = 1(mod\ 2), 3 \le i \le \frac{n}{2}-1 \\ 2i+3 & \text{if } i = 0(mod\ 2), 2 \le i \le \frac{n}{2} \end{cases}.$$

Then

$$
\begin{aligned}
f^*\left[\left(u_n v u_1 u_2 \ldots u_{\frac{n}{2}}\right)\right] &= f(u_n) + f\left(u_{\frac{n}{2}}\right) + f(u_n v) \\
&\quad + f(vu_1) + f(u_1u_2) + \cdots + f\left(u_{\frac{n}{2}-1}u_{\frac{n}{2}}\right) \\
&= 1 + p + q + 4 + 7 + \sum_{i=1,3}^{\frac{n}{2}-1}(2i+6) + \sum_{i=2,4}^{\frac{n}{2}-2}(2i+5) \\
&= p + q + 5 + 11\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) \qquad (9) \\
f^*\left[\left(u_1 u_n u_{n-1} u_{n-2} \ldots u_{\frac{n}{2}}\right)\right] &= f(u_1) + f\left(u_{\frac{n}{2}}\right) + f(u_1u_n) + f(u_nu_{n-1}) \\
&\quad + f(u_{n-1}u_{n-2}) + \cdots + f\left(u_{\frac{n}{2}+1}u_{\frac{n}{2}}\right) \\
&= 2 + p + q + 3 + 6 + \sum_{i=2,4}^{\frac{n}{2}}(2i+3) + \sum_{i=1,3}^{\frac{n}{2}-1}(2i+4) \\
&= p + q + 5 + 11\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) \qquad (10)
\end{aligned}
$$

From (9) and (10), we conclude that ψ is minimum magic graphoidal cover. Hence, $W_{n,2}$ is magic graphoidal. For example, the magic graphoidal cover of $W_{8,2}$ is shown in Figure 5. Hence, Parachute admits magic graphoidal. □

Theorem 3.3 *A Fan $P_n + K_1$ is magic graphoidal for $n \equiv 0(mod 2)$.*

Proof If $n = 2$, a fan becomes K_3. If $n = 4$, a labeling on $P_4 + K_1$ is shown in Figure 6,

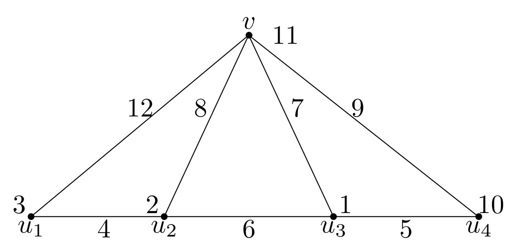

Figure 6 $P_4 + K_1$

where, $\psi = \{(u_1v), (u_1u_2v), (u_2u_3v), (u_3u_4v)\}, \gamma = 4$ and $K = 26$.

If $n > 4$, let $G = P_n + K_1$ with $V(G) = \{v, u_i : 1 \le i \le n\}$ and $E(G) = \{(vu_i) : 1 \le i \le n\} \cup \{(u_iu_{i+1}) : 1 \le i \le n-1\}$ Let $\psi = \{(u_1v), (u_iu_{i+1}v) : 1 \le i \le n-1\}$. Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(v) = p + q - 1, \quad f(u_n) = p + q - 2, \quad f(u_1) = 3\left(\frac{n-6}{2}\right) + 6,$$

$$f(u_i) = f(u_1) + (i-1),\ i = 2, 3, \quad f(vu_1) = p + q, \quad f(u_iu_{i+1}) = 7 - 2i, \quad 1 \le i \le 3$$

$$f(u_{n-i}) = f(u_1) + 2 + i,\ 1 \le i \le \frac{n}{2} - 1, \quad f(u_{3+i}) = f(u_1) - i \text{ if } n > 6, 1 \le i \le \frac{n}{2} - 3,$$

$$f(u_{n+1-i}u_{n-i}) = n - 2i,\ 1 \le i \le \frac{n}{2} - 1, \quad f(u_{3+i}u_{4+i}) = 5 + 2i,\ 1 \le i \le \frac{n}{2} - 3,$$

$$f(vu_{i+1}) = p+q-6+i,\ 1 \le i \le 3, \quad f(vu_{n+1-i}) = f(u_1) + \frac{n+2}{2} + i,\ 1 \le i \le n-4.$$

Then

$$\begin{aligned} f*[(u_1v)] &= f(u_1)+f(v)+f(u_1v) \\ &= \frac{3}{2}(n-6)+6+3n-1+3n = 7n+\frac{n}{2}-4. \end{aligned} \tag{11}$$

For $2 \le i \le 3$,

$$\begin{aligned} f^*[(u_iu_{i+1}v)] &= f(u_i)+f(v)+f(u_iu_{i+1})+f(u_{i+1}v) \\ &= 3\left(\frac{n-6}{2}\right)+6+(i-1)+3n-1+7-2i+3n-6+i = 7n+\frac{n}{2}-4. \end{aligned} \tag{12}$$

For $4 \le i \le \frac{n}{2}-3$,

$$\begin{aligned} f^*[(u_iu_{i+1}v)] &= f(u_i)+f(v)+f(u_iu_{i+1})+f(u_{i+1}v) \\ &= 3\left(\frac{n-6}{2}\right)+6-(i-3)+3n-1+5+2(i-3) \\ &\quad +3\left(\frac{n-6}{2}\right)+6+\frac{n+2}{2}+n-i = 7n+\frac{n}{2}-4. \end{aligned} \tag{13}$$

For $\frac{n}{2}-3 < i \le n-1$,

$$\begin{aligned} f*[(u_iu_{i+1}v)] &= f(u_i)+f(v)+f(u_iu_{i+1})+f(u_{i+1}v) \\ &= 3\left(\frac{n-6}{2}\right)+6+2+n-i+3n-1+n-2(n-i) \\ &\quad +3\left(\frac{n-6}{2}\right)+6+\frac{n+2}{2}+n-i = 7n+\frac{n}{2}-4. \end{aligned} \tag{14}$$

From (11), (12), (13) and (14), we conclude that ψ is minimum magic graphoidal cover. Hence, a Fan $P_n + K_1$, (n-even, $n > 4$) is magic graphoidal. □

For example, the magic graphoidal cover of $P_{10} + K_1$ is shown in Figure 7,

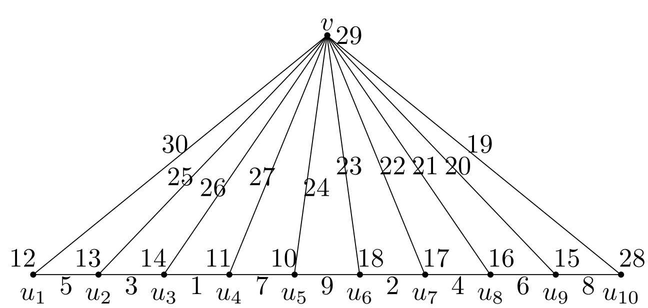

Figure 7 $P_{10} + K_1$

where, $\psi = \{(u_1v), (u_1u_2v), (u_2u_3v), (u_3u_4v), (u_4u_5v), (u_5u_6v), (u_6u_7v), (u_7u_8v), (u_8u_9v), (u_9u_{10}v)\}$, $\gamma = 10$ and $K = 71$.

Theorem 3.4 *A double Fan $P_n + 2K_1$is magic graphoidal.*

Proof Let $G = P_n + 2K_1$ with $V(G) = \{v, w\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(G) = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(vu_i) \cup (wu_i) : 1 \leq i \leq n\}$. The discussion is divided into to cases following.

Case 1 $\quad n$ is odd.

In this case, define $f : V \cup E \to \{1, 2, \ldots, 4n+1\}$ by

$$f(w) = 4n+1;\ \ f(u_i v) = i,\ 1 \leq i \leq n;\ \ f(u_i w) = 2n+1-i,\ 1 \leq i \leq n,$$
$$f(u_2 i) = 2n+i,\ 1 \leq i \leq \frac{n-1}{2}; f(v) = \frac{5n+1}{2}$$
$$f(u_{2i-1}) = \frac{5n+1}{2} + i,\ 1 \leq i \leq \frac{n+1}{2};\quad f(u_{n+1-i} u_{n-i}) = 3n+1+i,\ 1 \leq i \leq n-1.$$

Let $\psi = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(vu_i w) : 1 \leq i \leq n\}$. Then, for $1 \leq i \leq n-1, i \equiv 1(mod\, 2)$,

$$\begin{aligned} f^*[(u_i u_{i+1})] &= f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) \\ &= \frac{5n+1}{2} + \frac{i+1}{2} + 2n + \frac{i+1}{2} + 3n + 1 + n - i = \frac{17n+5}{2}; \end{aligned} \tag{15}$$

for $1 \leq i \leq n-1, i \equiv 0(mod\ 2)$.

$$\begin{aligned} f^*[(u_i u_{i+1})] &= f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) \\ &= 2n + \frac{i}{2} + \frac{5n+1}{2} + \frac{i+2}{2} + 3n + 1 + n - i = \frac{17n+5}{2}; \end{aligned} \tag{16}$$

for $1 \leq i \leq n$,

$$\begin{aligned} f[(vu_i w)] &= f(v) + f(w) + f(vu_i) + f(u_i w) \\ &= \frac{5n+1}{2} + \frac{i+1}{2} + 2n + \frac{i+1}{2} + 3n + 1 + n - i = \frac{17n+5}{2} \end{aligned} \tag{17}$$

From (15), (16) and (17), we conclude that ψ is minimum magic graphoidal cover. Hence, a double Fan $P_n + 2K_1$ (n-odd) is magic graphoidal. For example, the magic graphoidal cover of $P_5 + 2K_1$ is shown in Figure 8 with $\psi = \{(vu_1 w), (vu_2 w), (vu_3 w), (vu_4 w), (vu_5 w), (u_1 u_2), (u_2 u_3), (u_3 u_4), (u_4 u_5)\}, \gamma = 9$ and $K = 45$.

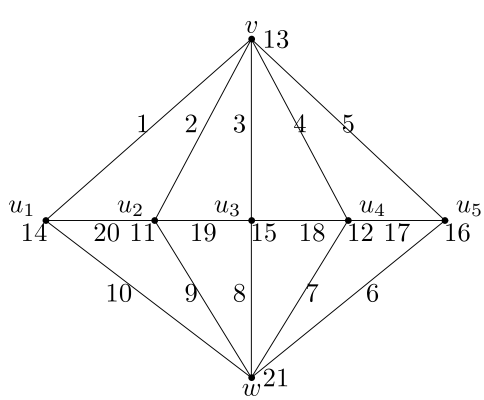

Figure 8 $\quad P_5 + 2K_1$

Case 2 n is even.

In this case, define $f : V \cup E \to \{1, 2, \ldots, 4n+1\}$ by

$$f(w) = 4n+1, \quad f(u_iv) = i,\ 1 \le i \le n, \quad f(u_iw) = 2n+1-i,\ 1 \le i \le n$$
$$f(u_2i) = 2n+i,\ 1 \le i \le \frac{n}{2}, \quad f(v) = \frac{5n}{2}+1$$
$$f(u_{2i-1}) = \frac{5n}{2}+1+i,\ 1 \le i \le \frac{n}{2}, \quad f(u_{n+1-i}u_{n-i}) = 3n+1+i,\ 1 \le i \le n-1.$$

Let $\psi = \{[(u_iu_{i+1}) : 1 \le i \le n-1], [(vu_iw) : 1 \le i \le n]\}$. Then, for $1 \le i \le n-1, i \equiv 1(mod\ 2)$,

$$\begin{aligned} f^*[(u_iu_{i+1})] &= f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) \\ &= \frac{5n}{2} + 1 + \frac{i+1}{2} + 2n + \frac{i+1}{2} + 3n + 1 + n - i = \frac{17n+6}{2}; \end{aligned} \tag{18}$$

for $1 \le i \le n-1, i \equiv 0(mod\ 2)$

$$\begin{aligned} f^*[(u_iu_{i+1})] &= f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) \\ &= 2n + \frac{i}{2} + \frac{5n}{2} + 1 + \frac{i+2}{2} + 3n + 1 + n - i = \frac{17n+6}{2}; \end{aligned} \tag{19}$$

for $1 \le i \le n$,

$$\begin{aligned} f^*[(vu_iw)] &= f(v) + f(w) + f(vu_i) + f(u_iw) \\ &= \frac{5n}{2} + 1 + 4n + 1 + i + 2n + 1 - i = \frac{17n+6}{2} \end{aligned} \tag{20}$$

From (18), (19) and (20), we conclude that ψ is minimum magic graphoidal cover. Hence, a double Fan $P_n + 2K_1$ (n-even) is magic graphoidal. For example, the magic graphoidal cover of $P_6 + 2K_1$ is shown in Figure 9,

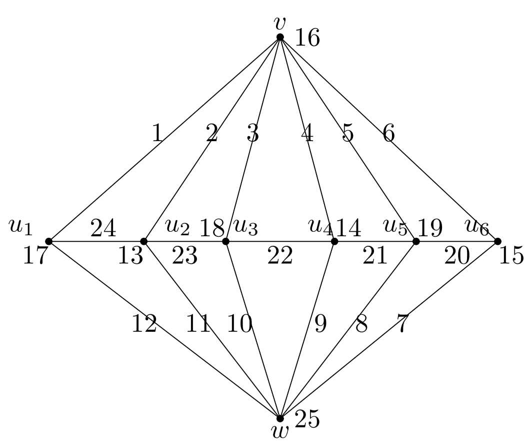

Figure 9 $P_6 + 2K_1$

where, $\psi = \{(vu_1w), (vu_2w), (vu_3w), (vu_4w), (vu_5w), (vu_6w), (u_1u_2), (u_2u_3), (u_3u_4), (u_4u_5),$ $(u_5u_6)\}$, $\gamma = 11$ and $K = 54$. □

Theorem 3.5 *A wheel $W_n = C_{n-1} + K_1$ (n- even) is magic graphoidal.*

Proof Let $G = W_n$ with $V(G) = \{v, u_i : 1 \le i \le n-1\}$ and $E(G) = \{(u_i u_{i+1}) : 1 \le i \le n-2\} \cup \{(u_1 u_{n-1})\} \cup \{(vu_i) : 1 \le i \le n-1\}$. Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(u_1) = n-2; \quad f(u_{i+1}) = 2i-1, \ 1 \le i \le \frac{n}{2},$$
$$f\left(u_{\frac{n}{2}+1+i}\right) = 2i, \ 1 \le i \le \frac{n}{2} - 2, \quad f(v) = p+q,$$
$$f(u_i u_{i+1}) = 3\frac{n}{2} - i, \ 1 \le i \le \frac{n}{2}; \quad f(vu_i) = 3n-2-i, \ 1 \le i \le n-1,$$
$$f\left(u_{\frac{n}{2}+i} u_{\frac{n}{2}+1+i}\right) = 2n-1-i, \ 1 \le i \le \frac{n}{2} - 1, \text{ where } u_n = u_1.$$

Let $\psi = \{[(vu_i u_{i+1}) : 1 \le i \le n-2], (vu_{n-1}u_1)\}$. Then, for $1 \le i \le \frac{n}{2}$,

$$\begin{aligned} f^*[(vu_i u_{i+1})] &= f(v) + f(u_{i+1}) + f(vu_i) + f(u_i u_{i+1}) \\ &= 3n-2+2i-1+3n-2-i+\frac{3}{2}n - i = 7n + \frac{n}{2} - 5; \end{aligned} \tag{21}$$

for $\frac{n}{2} + 1 \le i \le n-2$,

$$\begin{aligned} f^*[(vu_i u_{i+1})] &= f(v) + f(u_{i+1}) + f(vu_i) + f(u_i u_{i+1}) \\ &= 3n-2+2\left(i - \frac{n}{2}\right) + 3n-2-i+2n-1-\left(i - \frac{n}{2}\right) = 7n + \frac{n}{2} - 5 \end{aligned} \tag{22}$$

$$\begin{aligned} f^*[(vu_{n-1}u_1)] &= f(v) + f(u_1) + f(vu_{n-1}) + f(u_{n-1}u_1) \\ &= 3n-2+n-2+3n-2-(n-1)+2n-1-\left(\frac{n}{2} - 1\right) = 7n + \frac{n}{2} - 5. \end{aligned} \tag{23}$$

From (21), (22) and (23), we conclude that ψ is minimum magic graphoidal cover. Hence, a wheel W_n (n-even) is magic graphoidal. □

For example, the magic graphoidal cover of W_8 is shown in Figure 10,

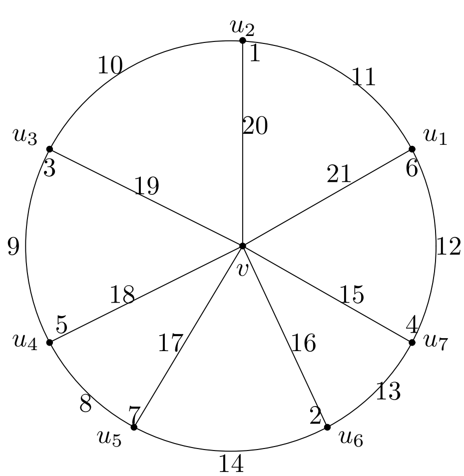

Figure 10 W_8

where, $\psi = (vu_1u_2), (vu_2u_3), (vu_3u_4), (vu_4u_5), (vu_5u_6), (vu_6u_7), (vu_7u_1), \gamma = 7$ and $K = 55$.

Theorem 3.6 *The graph $\overline{K}_2 + mK_1$ is magic graphoidal.*

Proof Let $G = \overline{K}_2 + mK_1$ with $V(G) = \{u_1, u_2, [v_i : 1 \leq i \leq m]\}$ and $E(G) = \{(u_1v_i) : 1 \leq i \leq m\} \cup \{(u_2v_i) : 1 \leq i \leq m\}$. Define $f : V \cup E \to \{1, 2, \ldots, 3m+2\}$ by

$$f(u_1) = 2m+1; \quad f(u_1v_i) = i, \ 1 \leq i \leq m,$$
$$f(u_2) = 2m+2; \quad f(u_2v_i) = 2m+1+i, \ 1 \leq i \leq m-1.$$

Let $\psi = \{(u_1v_iu_2) : 1 \leq i \leq m\}$. Then,

$$\begin{aligned} f^*[(u_1v_iu_2)] &= f(u_1) + f(u_2) + f(u_1v_i) + f(v_iu_2) \\ &= 2m+1+2m+2+i+2m+1-i = 6m+4. \end{aligned}$$

Thus, $f^*[(u_1v_iu_2)]$ is independent of i, depends only on m. So it is a constant. Therefore, $\overline{K}_2 + mK_1$ admits a ψ-magic total labeling. Hence, $\overline{K}_2 + mK_1$ is magic graphoidal. □

For example, the magic graphoidal cover of $\overline{K}_2 + 6K_1$ is shown in Figure 11,

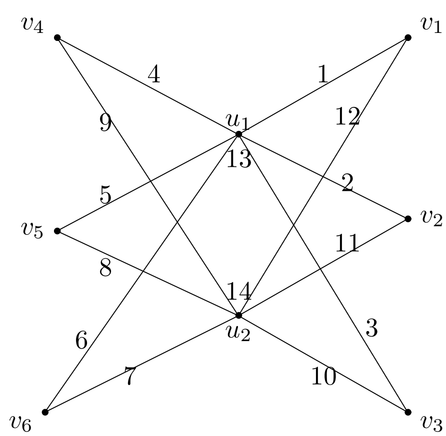

Figure 11 $\overline{K}_2 + 6K_1$

where, $\psi = \{(u_1v_1u_2), (u_1v_2u_2), (u_1v_3u_2), (u_1v_4u_2), (u_1v_5u_2), (u_1v_6u_2)\}, \gamma = 6, K = 40.$

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