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Further Results on Global Connected Domination Number of Graphs

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Abstract: A subset S of vertices in a graph $G=(V,E)$ is a dominating set if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set S of a connected graph G is called a connected dominating set if the induced subgraph $< S >$ is connected. A set S is called a global dominating set of G if S is a dominating set of both G and $\overline{G}$. A subset S of vertices of a graph G is called a global connected dominating set if S is both a global dominating and a connected dominating set. The global connected domination number is the minimum cardinality of a global connected dominating set of G and is denoted by $\gamma_{gc}(G)$. In this paper we obtained the upper bound for the sum of global connected domination number and chromatic number and characterize the corresponding extremal graphs.

Key Words: Global domination number, chromatic number.

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§1. Introduction

Graphs discussed in this paper are simple, finite and undirected graphs. A subset S of vertices in a graph $G=(V,E)$ is a dominating set if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set S of a connected graph G is called a connected dominating set if the induced subgraph $\langle S\rangle$ is connected. A set S is called a global dominating set of G if S is a dominating set of both G and $\overline{G}$. A subset S of vertices of a graph G is called a global connected dominating set if S is both a global dominating and a connected dominating set. The global connected domination number is the minimum cardinality of a global connected dominating set of G and is denoted by $\gamma_{gc}(G)$. Note that any global connected dominating set of a graph G has to be connected in G (but not necessarily in $\overline{G}$). Here global connected domination number γ_{gc} is well defined for any connected graph. For a cycle C_n of order $n\geq 6$, $\gamma_g(C_n)=\lceil n/3\rceil$ while $\gamma_{gc}(C_n)=n-2$ and $\gamma_g(K_n)=1$, while $\gamma_{gc}(K_n)=n$. The chromatic number $\chi(G)$ is defined as the minimum number of colors required to color all the vertices such that adjacent vertices do not receive the same color.

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Notation 1.1 $K_n(P_k)$ is the graph obtained from K_n by attaching the end vertex of P_k to any one vertices of K_n. $K_n(mP_k)$ is the graph obtained from K_n by attaching the end vertices of m copies of P_k to any one vertices of K_n.

Some preliminary results on global connected domination number of graphs are listed in the following.

Theorem 1.2([1]) *Let G be a graph of order $n \geq 2$. Then*

(1) $2 \leq \gamma_{gc}(G) \leq n$,

(2) $\gamma_{gc}(G) = n$ if and only if $G \cong K_n$.

Corollary 1.3([1]) *For all positive integers p and q $\gamma_{gc}(K_{p,q}) = 2$.*

Theorem 1.4([1], Brooke's Theorem) *If G is a connected simple graph and is neither a complete graph nor an odd cycle then $\chi(G) \leq \Delta(G)$.*

Theorem 1.5([1]) *For any graph G of order $n \geq 3$, $\gamma_{gc}(G) = n - 1$ if and only if $G \cong K_n - e$, where e is an edge of K_n.*

Corollary 1.6([1]) *For all $n \geq 4$ $\gamma_{gc}(C_n) = n - 2$.*

§2. Main Result

Theorem 2.1 *For any connected graph G, $\gamma_{gc}(G) + \chi(G) < 2n - 1$.*

Proof By Theorem 1.2, for any graph G of order $n \geq 2$, $\gamma_{gc}(G) \leq n$.

By Theorem 1.4, $\chi(G) \leq \Delta(G)$. Therefore, $\gamma_{gc}(G) + \chi(G) \leq n + \Delta = n + (n-1) = 2n - 1$. Hence $\gamma_{gc}(G) + \chi(G) \leq 2n - 1$. Let $\gamma_{gc}(G) + \chi(G) = 2n - 1$. It is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n - 1$ (or) $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n$.

Case 1 Let $\gamma_{gc}(G) = n$ and $\chi(G) = n - 1$. Since $\chi(G) = n - 1$, G contains a clique K on $n - 1$ vertices or does not contain a clique K on $n - 1$ vertices. Suppose G contains a clique K on $n - 1$ vertices. Let v be the vertex not in K_{n-1}. Since G is connected, the vertex v_1 is adjacent to some vertex u_i of K_{n-1}. Then $\{v_1, u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$, therefore $K \cong K_1$ which is a contradiction. Hence no graph exist. If G does not contain the clique K on $n - 1$ vertices, then it can be verified that no graph exists.

Case 2 Let $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n$. Since $\gamma_{gc}(G) = n - 1$ by Theorem 1.5, $G \cong K_n - e$ But for $K_n - e$, $\chi(G) = n - 1$ which is a contradiction. Hence no graph exists. Hence, $\gamma_{gc}(G) + \chi(G) < 2n - 1$. □

Theorem 2.2 *For any connected graph G, for $n \geq 3$ $\gamma_{gc}(G) + \chi(G) = 2n - 2$ if and only if $G \cong K_n - e$, where e is an any edge of K_n.*

Proof Assume that $\gamma_{gc}(G) + \chi(G) = 2n - 2$. This is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n - 2$ (or) $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n - 1$ (or) $\gamma_{gc}(G) = n - 2$ and $\chi(G) = n$.

Case 1 Let $\gamma_{gc}(G) = n$ and$\chi(G) = n - 2$. Since $\chi(G) = n - 2$, G contains a clique K on $n - 2$ vertices or does not contain a clique K on $n - 2$ vertices . Suppose G contains a clique K on $n-2$ vertices. Let $S = \{v_1, v_2\} \subset V - K$. Then the induced sub graph $< S >$ has the following possible cases, $< S >= K_2$ and $\overline{K_2}$.

Subcase 1.1 Let $\langle S \rangle = K_2$. Since G is connected there exist a vertex u_i of K_{n-2} which is adjacent to anyone of v_1, v_2. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, u_i\}$ forms a global connected dominating set in G. So that $\gamma_{gc}(G) = 2$ which is a contradiction.

Subcase 1.2 Let $\langle S \rangle = \overline{K_2}$. Since G is connected, let both the vertices of $\overline{K_2}$ be adjacent to vertex u_i for some i in $K_n - 2$. Let $\{v_1, v_2\}$ be the vertices of $\overline{K_2}$. Then anyone of the vertices of $\overline{K_2}$ and u_i forms a global connected dominating set in G. Then, $\{v_1, u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$, which is a contradiction. If both the vertices of $\overline{K_2}$ are adjacent two distinct vertices of K_{n-2} say u_i and u_j for $i \neq j$ in K_{n-2}. Then $\{u_i, u_j\}$ forms a global connected dominating set. Hence $\gamma_{gc}(G) = 2$, Which is a contradiction. If G does not contain the clique K on $n-2$ vertices, then it can be verified that no graph exists.

Case 2 Let $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n - 1$. Since $\gamma_{gc}(G) = n - 1$, by Theorem 1.5, $G \cong K_n - e$. But for $K_n - e$, $\chi(G) = n - 1$. Hence $G \cong K_n - e$ for $n \geq 3$. If G does not contain the clique K on $n - 1$ vertices, then it can be verified that no graph exists.

Case 3 Let $\gamma_{gc}(G) = n - 3$ and $\chi(G) = n$. Since $\chi(G) = n$. Then $G \cong K_n$. But for K_n, $\gamma_{gc}(G) = n$, which is a contradiction.

Conversely if G is anyone of the graph $K_n - e$, for $n \geq 3$ then it can be verified that $\gamma_{gc}(G) + \chi(G) = 2n - 2$. □

Theorem 2.3 *For any connected graph G, $\gamma_{gc}(G) + \chi(G) = 2n - 3$ if and only if $G \cong K_3(P_2), K_n - \{e_1, e_2\}$, where e_1, e_2 are edges in the cycle of graph, $n \geq 5$.*

Proof Assume that $\gamma_{gc}(G) + \chi(G) = 2n - 3$. This is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n - 3$ (or) $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n - 2$ (or) $\gamma_{gc}(G) = n - 2$ and $\chi(G) = n - 1$ (or) $\gamma_{gc}(G) = n - 3$ and $\chi(G) = n$.

Case 1 Let $\gamma_{gc}(G) = n$ and $\chi(G) = n - 3$. Since $\gamma_{gc}(G) = n - 3$, G contains a clique K on $n - 3$ vertices or does not contain a clique K on $n - 3$ vertices. Suppose G contains a clique K on $n - 3$ vertices. Let $S = \{v_1, v_2, v_3\} \subset V - K$. Then the induced sub graph $\langle S \rangle$ has the following possible cases $\langle S \rangle = K_3$, $\overline{K_3}$, $K_2 \cup K_1$, P_3.

Subcase 1.1 Let $\langle S \rangle = K_3$. Since G is connected, these exist a vertex u_i of K_{n-3} which is adjacent to anyone $\{v_1, v_2, v_3\}$. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, u_i\}$ is a global connected dominating set. Hence $\gamma_{gc}(G) = 2$ which is a contradiction.

Subcase 1.2 Let $\langle S \rangle = \overline{K_3}$. Since G is connected, there exist a vertex u_i of K_{n-3} which is adjacent to all the vertices of $\overline{K_3}$. Let v_1, v_2, v_3 be the vertices of $\overline{K_3}$. Then anyone of the vertices of $\overline{K_3}$ and u_i forms a global connected dominating set in G. Without loss of generality $\{v_1, u_i\}$

forms a global connected dominating set in G. Then $\gamma_{gc}(G) = 2$, which is a contradiction. If two vertices of $\overline{K_3}$ are adjacent to u_i and third vertex adjacent to u_j for some $i \neq j$. Then $\{v_1, u_i, u_j\}$ forms a global connected dominating set in G. Then $\gamma_{gc}(G) = 3$, which is a contradiction. If three vertices of $\overline{K_3}$ are adjacent to three distinct vertices of K_{n-3} say u_i, u_j, u_k for some $i \neq j \neq k$. Then $\{v_1, u_i, u_j, u_k\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 4$ which is a contradiction.

Subcase 1.3 Let $\langle S\rangle = K_2 \cup K_1$. Since G is connected, there exist a vertex u_i of K_{n-3} which is adjacent to anyone of $\{v_1, v_2\}$ and v_3. Then $\{v_1, u_i\}$ is a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$, which is a contradiction. Let there exist a vertex u_i of K_{n-3} be adjacent to anyone of $\{v_1, v_2\}$ and u_j for some $i \neq j$ in K_{n-3} adjacent to v_3. Without loss of generality, let v_1 be adjacent to u_i. Then $\{v_1, u_i, u_j\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 3$ which is a contradiction.

Subcase 1.4 Let $\langle S\rangle = P_3$. Let $\{v_1, v_2, v_3\}$ be the vertices of P_3. Since G is connected there exist a vertex u_i of K_{n-3} which is adjacent to anyone of the pendant vertices v_1 or v_3. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, v_2, u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 3$ which is a contradiction. On increasing the degree of u_i, which is a contradiction. If G does not contain the clique K on $n - 3$ vertices, then it can be verified that no graph exists.

Case 2 Let $\gamma_{gc}(G) = n-1$ and $\chi(G) = n-2$. Since $\gamma_{gc}(G) = n-1$ by Theorem 1.5, $G \cong K_n - e$. But for $K_n - e$, $\chi(G) = n - 1$, which is a contradiction. Hence no graph exists. If G does not contain the clique K on $n - 3$ vertices, then it can be verified that no graph exists.

Case 3 Let $\gamma_{gc}(G) = n - 2$ and $\chi(G) = n - 1$. Since $\chi(G) = n - 1$, G contains a clique K on $n-1$ vertices. Let v be the vertex not in K_{n-1}. Since G is connected, the vertex v is adjacent to vertex u_i of K_{n-1}. Then $\{v_1, u_i\}$ is a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$. Then $K \cong K_3$. Then $G \cong K_3(P_2)$. On increasing the degree of v $G \cong K_n - \{e_1, e_2\}$ for $n \geq 5$ and e_1, e_2 is an edge in outside the cycle of graph. If G does not contain the clique K on $n - 1$ vertices, then it can be verified that no graph exists.

Case 4. Let $\gamma_{gc}(G) = n - 3$ and $\chi(G) = n$. Since $\chi(G) = n$ then $G \cong K_n$. But for K_n $\gamma_{gc}(G) = n$, then it is a contradiction.

Conversely if G is anyone of the graph $K_3(P_2)$, $K_n - \{e_1, e_2\}$ for $n \geq 5$, then it can be verified that $\gamma_{gc}(G) + \chi(G) = 2n - 3$. □

Theorem 2.4 *For any connected graph G $\gamma_{gc}(G) + \chi(G) = 2n - 4$ if and only if $G \cong P_4, C_4, K_2(2P_2), K_4(P_2), K_n - \{e_1, e_2, e_3\}$ wheree_1, e_2, e_3 are edges in the cycle of graph of order $n \geq 6$*

Proof Let $\gamma_{gc}(G) + \chi(G) = 2n - 4$. This is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n - 4$ (or) $\gamma_{gc}(G) = n-1$ and $\chi(G) = n-3$ (or) $\gamma_{gc}(G) = n-2$ and $\chi(G) = n-2$ (or) $\gamma_{gc}(G) = n-3$ and $\chi(G) = n - 1$ (or) $\gamma_{gc}(G) = n - 4$ and $\gamma(G) = n$.

Case 1 Let $\gamma_{gc}(G) = n$ and $\chi(G) = n - 4$. Since $\chi(G) = n - 4$, G contains a clique K on $n - 4$ vertices or does not contain a clique K on $n - 4$ vertices. Suppose G contains a clique K on

$n-4$ vertices. Let $S = \{v_1, v_2, v_3, v_4\} \subset V - K$. Then the induced subgraph $< S >$ has the following possible cases.$< S >= K_4, \overline{K_4}, K_3 \cup K_1, K_4 - e, K_2 \cup K_2, K_2 \cup \overline{K_2}, K_3(P_2)$ it can be verified that all the above cases no graph exist.

If G does not contain the clique K on $n-4$ vertices, then it can be verified that no graph exists.

Case 2 Let $\gamma_{gc}(G) = n-1$ and $\chi(G) = n-3$. Since $\gamma_{gc}(G) = n-1$ by Theorem 1.5, $G \cong K_n - e$. But for $K_n - e$ $\chi(G) = n-1$ which is a contradiction.

If G does not contain the clique K on $n-3$ vertices, then it can be verified that no graph exists.

Case 3 Let $\gamma_{gc}(G) = n-2$ and $\chi(G) = n-2$. Since $\chi(G) = n-2$, then G contains a clique K on $n-2$ vertices or does not contain a clique K on $n-2$ vertices. Suppose G contains a clique K on $n-2$ vertices. Let $S = \{v_1, v_2\} \subset V - K$. Then the induced subgraph $< S >$ has the following possible cases,$< S >= K_2$ and $\overline{K_2}$.

Subcase 3.1 Let $\langle S \rangle = K_2$. Since G is connected there exist a vertex u_i of K_{n-2} which is adjacent to anyone of $\{v_1, v_2\}$. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, u_i\}$ is a global connected dominating set in G hence $\gamma_{gc}(G) = 2$. Then $K \cong K_2$. Let $\{v_1, v_2\}$ be the vertices of K_2. If $d(v_1) = 1$, $d(v_2) = 2$ Then $G \cong P_4$. If $d(v_1) = 2$ $d(v_2) = 2$ then $G \cong C_4$.

Subcase 3.2 Let $\langle S \rangle = \overline{K_2}$. Since G is connected. Let both the vertices of $\overline{K_2}$ be adjacent to vertex u_i for some i in K_{n-2}. Let $\{v_1, v_2\}$ be the vertices of $\overline{K_2}$. Then anyone of the vertices of $\overline{K_2}$ and u_i forms a global connected dominating set in G. Without loss of generality v_1 and u_i forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$ so that $K \cong K_2$. Then $G \cong K_2(2P_2)$. If two vertices of $\overline{K_2}$ are adjacent two distinct vertices K_{n-2} say u_i and u_j for $i \neq j$ .Then $\{v_1, u_i, u_j\}$ forms global connected dominating set in G. Hence $\gamma_{gc}(G) = 3$ so that $K \cong K_2$. Then $G \cong K_3(P_2, P_2, 0)$.

If G does not contain the clique K on $n-2$ vertices, then it can be verified that no graph exists.

Case 4 Let $\gamma_{gc}(G) = n-3$ and $\chi(G) = n-1$. Since $\chi(G) = n-1$, G contains a clique K on $n-1$ vertices. Let v be the vertex not in K_{n-1}. Since G is connected the vertex v is adjacent to vertex u_i of K_{n-1}. Then $\{v_1, u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$ so that $K \cong K_4$. Then $G \cong K_4(P_2)$. On increasing the degree $G \cong K_n - \{e_1, e_2, e_3\}$ where e_1, e_2, e_3 are edges in the cycle of the graph of order $n \geq 6$. If G does not contain the clique K on $n-1$ vertices, then it can be verified that no graph exists.

Case 5 Let $\gamma_{gc}(G) = n-4$ and $\chi(G) = n$. Since $\chi(G) = n$. Then $G \cong K_n$. But for K_n, $\gamma_{gc}(Kn) = n$, Which is a contradiction.

Conversely if G is anyone of the graph $P_4, C_4, K_2(2P_2), K_4(P_2), K_n - 3e$ for $n \geq 6$ then it can be verified that $\gamma_{gc}(G) + \chi(G) = 2n - 4$. □

The authors obtained graphs for which $\gamma_{gc}(G) + \chi(G) = 2n-5, 2n-6, 2n-7$, which will be reported later.

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