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# Chebyshev Polynomials and Spanning Tree Formulas

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**Abstract**: The number of spanning trees in graphs (networks) is an important invariant, it is also an important measure of reliability of a network. In this paper we derive simple formulas of the complexity, number of spanning trees, of some new graphs generated by a new operation, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

**Key Words**: Number of spanning trees, Chebyshev polynomials, Kirchhoff matrix .

**AMS(2010)**: 33C45

## §1. Introduction

In this work we deal with simple and finite undirected graphs $G = (V, E)$, where $V$ is the vertex set and $E$ is the edge set. For a graph $G$, a spanning tree in $G$ is a tree which has the same vertex set as $G$. The number of spanning trees in $G$, also called, the complexity of the graph, denoted by $\tau(G)$, is a well-studied quantity (for long time). A classical result of Kirchhoff in [1] can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = \{v_1, v_2, \cdots, v_n\}$, then the Kirchhoff matrix $H$ defined as an $n \times n$ characteristic matrix $H = D - A$, where $D$ is the diagonal matrix of the degrees of $G$ and $A$ is the adjacency matrix of $G$, $H = [a_{ij}]$ defined as follows:

$(i)$ $a_{ij} = -1$ if $v_i$ and $v_j$ are adjacent and $i \neq j$;
$(ii)$ $a_{ij}$ equals to the degree of vertex $v_i$ if $i = j$;
$(iii)$ $a_{ij} = 0$, otherwise.

All of co-factors of $H$ are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ denote the eignvalues of $H$ matrix of a $p$ point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [2] shown that $\tau(G) = \dfrac{1}{p}\prod_{k=1}^{p-1}\mu_k$. The formula for the number of spanning trees in a $d$-regular graph $G$ can be expressed as $\tau(G) = \dfrac{1}{p}\prod_{k=1}^{p-1}(d - \lambda_k)$, where $\lambda_0 = d, \lambda_1, \lambda_2, \cdots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exists simple formulas that make it much easier to calculate and determine the number of corresponding

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spanning trees especially when these numbers are very large. One of the first such result is due to Cayley [3] who showed that complete graph on $n$ vertices, $K_n$ has $n^{n-2}$ spanning trees that he showed $\tau(K_n) = n^{n-2},\ n \geq 2$. Another result, $\tau(K_{p,q}) = p^{q-1}q^{p-1},\ p \geq 1, q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing $p$ and $q$ vertices, respectively. It is well known, as in e.g., [4,5]. Another result is due to Sedlacek [6] who derived a formula for the wheel $W_{n+1}$ on $n+1$ vertices, he showed that

$$\tau(W_{n+1}) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$$

for $n \geq 3$. Sedlacek [7] also later derived a formula for the number of spanning trees in a Mobius ladder $M_n$,

$$\tau(M_n) = \frac{n}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + 2\right]$$

for $n \geq 2$. Another class of graphs for which an explicit formula has been derived is based on a prism. Boesch, et .al. [8,9].

## §2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, Yuanping, et. al. [10].

Let $A_n(x)$ be an $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \cdots & \cdots \\ -1 & 2x & -1 & \ddots & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & -1 \\ \cdots & \ddots & 0 & -1 & 2x \end{pmatrix},$$

where all other elements are zeros. Further we recall that the Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x). \tag{1}$$

and the Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)}. \tag{2}$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \tag{3}$$

and it can then be shown from this recursion that by expanding $\det A_n(x)$ one gets

$$U_n(x) = \det(A_n(x)), \quad n \geq 1. \tag{4}$$

Furthermore by using standard methods for solving the recursion (3), one obtains the explicit formula:

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right], \quad n \geq 1, \tag{5}$$

where the identity is true for all complex $x$ (except at $x = \pm 1$ where the function can be taken as the limit). The definition of $U_n(x)$ easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x) = 2^{n-1}\prod_{j=1}^{n-1}\left(x - \cos\frac{j\pi}{n}\right). \tag{6}$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1}U_{n-1}(x). \tag{7}$$

These two results yield another formula for $U_n(x)$ following,

$$U_{n-1}^2(x) = 4^{n-1}\prod_{j=1}^{n-1}\left(x^2 - \cos^2\frac{j\pi}{n}\right). \tag{8}$$

Finally, simple manipulation of the above formula yields the following, which also will be extremely useful to us latter:

$$U_{n-1}^2\left(\sqrt{\frac{x+2}{4}}\right) = \prod_{j=1}^{n-1}\left(x - 2\cos\frac{2j\pi}{n}\right). \tag{9}$$

Furthermore, one can shows that

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}\left[1 - T_{2n}\right] = \frac{1}{2(1-x^2)}\left[1 - T_n\left(2x^2-1\right)\right]. \tag{10}$$

and

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right]. \tag{11}$$

## §3. **Results**

In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from given set of graphs.

**Definition** 3.1 *If $G_1$ and $G_2$ are vertex-disjoint graphs of the same number of vertices. Then the symmetric join, $G_1 * G_2$ is the graph of $G_1 + G_2$, in which each vertex of $G_1$ is adjacent to every vertex of $G_2$ except the paired ones. See Fig.*1 *for details.*

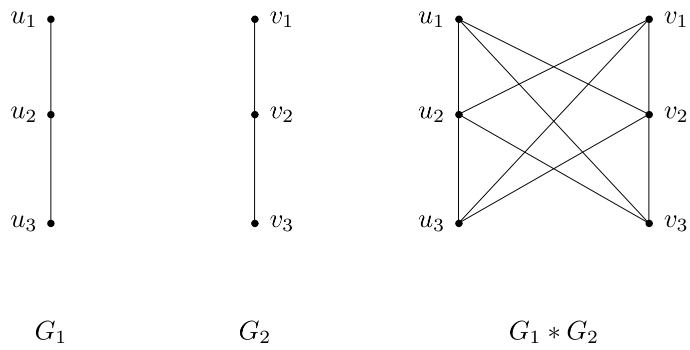


**Fig.**1

Now, we can introduce the following lemma.

**Lemma** 3.2 $\tau(G) = \dfrac{1}{p^2}\det(pI - \overline{D} + \overline{A})$, *where* $\overline{A}, \overline{D}$ *are the adjacency and degree matrices of* $\overline{G}$, *the complement of* $G$, *respectively, and* $I$ *is the* $p \times p$ *unit matrix.*

*Proof* Straightforward by properties of determinants, matrices and matrix-tree theorem.□

The advantage of these formulas in Lemma 3.2 is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

**Lemma** 3.3 *Let* $A \in F^{n\times n}, B \in F^{n\times m}, C \in F^{m\times n}$ *and* $D \in F^{m\times m}$. *If* $D$ *is non-singular, then*

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^{nm}\det\left(A - BD^{-1}C\right)\det D.$$

*Proof* It is easy to see that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

Thus

$$\begin{aligned} \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det\begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}\det\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \\ &= (-1^{nm})\det\left(A - BD^{-1}C\right)\det D. \end{aligned}$$ □

This formula gives some sort of symmetry in some matrices which facilitates our calculation of determinants.

**Theorem** 3.4 *For any integer $n \geq 3$,*

$$\tau(P_n * P_n) = \frac{n-1}{2^{2n} n \sqrt{(n^2-4)\left((n+2)^2-4\right)}} \times \left[\left(n+\sqrt{n^2-4}\right)^n - \left(n-\sqrt{n^2-4}\right)^n\right] \times \left[\left(n+2+\sqrt{(n+2)^2-4}\right)^n - \left(n+2-\sqrt{(n+2)^2-4}\right)^n\right].$$

*Proof* Applying Lemma 3.2, we get that

$$\tau(P_n * P_n) = \frac{1}{(2n)^2} \det\left(2n \times I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n+1 & 0 & 1 & \cdots & 1 & & & & & \\ 0 & n+2 & 1 & \ddots & \vdots & & & & & \\ 1 & 0 & \ddots & \ddots & 1 & & & I & & \\ \vdots & \ddots & \ddots & n+2 & 0 & & & & & \\ 1 & \cdots & 1 & 0 & n+1 & & & & & \\ & & & & & n+1 & 0 & 1 & \ddots & 1 \\ & & & & & 0 & n+2 & 0 & \ddots & \vdots \\ & & I & & & 1 & 0 & \ddots & \ddots & \vdots \\ & & & & & \vdots & \ddots & \ddots & \ddots & 1 \\ & & & & & \vdots & \ddots & \ddots & n+2 & 0 \\ & & & & & 1 & \cdots & 1 & 0 & n+1 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n & 0 & 1 & \cdots & 1 \\ 0 & n+1 & 0 & \ddots & \vdots \\ 1 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n+1 & 0 \\ 1 & \ddots & 1 & 0 & n \end{vmatrix} \times \begin{vmatrix} n+2 & 0 & 1 & \cdots & 1 \\ 0 & n+3 & 0 & \ddots & \vdots \\ 1 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n+3 & 0 \\ 1 & \cdots & 1 & 0 & n+2 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \frac{2n-2}{n-2}$$

$$\times \begin{vmatrix} n-1 & -1 & 0 & \cdots & 0 \\ -1 & n & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & n & -1 \\ 0 & \cdots & 0 & -1 & n-1 \end{vmatrix} \times 2 \begin{vmatrix} n+1 & -1 & 0 & \cdots & 0 \\ -1 & n+2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & n+2 & -1 \\ 0 & \cdots & 0 & -1 & n+1 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \frac{2n-2}{n-2} \times (n-2)$$

$$\times \begin{vmatrix} n & -1 & 0 & \cdots & \cdots \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ \vdots & \cdots & 0 & -1 & n \end{vmatrix}_{(n-1)\times(n-1)} \times 2n \begin{vmatrix} n+2 & -1 & 0 & \cdots & \cdots \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ \vdots & \cdots & 0 & -1 & n+2 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \frac{1}{(2n)^2} \times \frac{2n-2}{n-2} \left[(n-2)U_{n-1}\left(\frac{n}{2}\right)\right] \times \left[2nU_{n-1}\left(\frac{n+2}{2}\right)\right]$$

$$= \frac{1}{(2n)^2} \times \frac{2(n-1)}{n-2} \times (n-2) \times \frac{1}{2^n\sqrt{n^2-4}} \left[\left(n+\sqrt{n^2-4}\right)^n - \left(n-\sqrt{n^2-4}\right)^n\right]$$

$$\times \frac{2n}{2^n\sqrt{(n+2)^2+4^2}} \left[\left(n+2+\sqrt{(n-2)^2-4}\right)^n - \left(n+2-\sqrt{(n+2)^2-4}\right)^n\right]$$

$$= \frac{n-1}{2^{2n}n\sqrt{(n^2-4)((n+2)^2-4)}} \times \left[\left(n+\sqrt{n^2-4}\right)^n - \left(n-\sqrt{n^2-4}\right)^n\right]$$

$$\times \left[\left(n+2+\sqrt{(n-2)^2-4}\right)^n - \left(n+2-\sqrt{(n+2)^2-4}\right)^n\right]. \qquad \square$$

**Theorem** 3.5 *For any integer $n \geq 3$,*

$$\tau(N_n * N_n) = n^{n-2}(n-1)(n-2)^{n-1}.$$

*Proof* Applying Lemma 3.2, we have

$$\tau(N_n * N_n) = \frac{1}{(2n)^2}\det\left(2n \times I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n & 1 & \cdots & 1 & & & & \\ 1 & \ddots & \ddots & \vdots & & & I & \\ \vdots & \ddots & \ddots & 1 & & & & \\ 1 & \cdots & 1 & n & & & & \\ & & & & n & 1 & \cdots & 1 \\ & & I & & 1 & \ddots & \ddots & \vdots \\ & & & & \vdots & \ddots & \ddots & 1 \\ & & & & 1 & \cdots & 1 & n \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n+1 & 1 & 1 & \cdots & 1 \\ 1 & n+1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \cdots & \cdots & n+1 & 1 \\ 1 & \cdots & 1 & 1 & n+1 \end{vmatrix} \times \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 \\ 1 & n-1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n-1 & 1 \\ 1 & \cdots & 1 & 1 & n-1 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \prod_{j=1}^{n} ((n+1) + \omega_j + \omega_j^2 + \cdots + \omega_j^{n-1}) \times \prod_{j=1}^{n} ((n-1) + \omega_j + \omega_j^2 + \cdots + \omega_j^{n-1})$$

$$= \frac{1}{(2n)^2} ((n+1) + 1 + \cdots + 1) \times \prod_{j=1, \omega_j \neq 1}^{n} ((n+1) + \underbrace{\omega_j + \omega_j^2 + \cdots + \omega_j^{n-1}}_{=-1})$$

$$\times ((n-1) + 1 + 1 + \cdots + 1) \times \prod_{j=1, \omega_j \neq 1}^{n} ((n-1) + \underbrace{\omega_j + \omega_j^2 + \cdots + \omega_j^{n-1}}_{=-1})$$

$$= \frac{1}{(2n)^2} (n+1+n-1) \times (n+1-1)^{n-1} \times (n-1+n-1) \times (n-1-1)^{n-1}$$

$$= \frac{1}{(2n)^2} \times 2n^n \times 2(n-1)(n-2)^{n-1} = n^{n-2}(n-1)(n-2)^{n-1}. \qquad \square$$

**Theorem** 3.6 *For any integer $n \geq 3$,*

$$\tau(N_n * P_n) = \frac{n-1}{2^n n \sqrt{n^4 - 8n^2 + 8n}} \times \left[ \left( n^2 - 2 + \sqrt{n^4 - 8n^2 + 8n} \right)^n - \left( n^2 - 2 - \sqrt{n^4 - 8n^2 + 8n} \right)^n \right].$$

*Proof* Applying Lemma 3.2, we get that

$$\tau(N_n * P_n) = \frac{1}{(2n)^2} \det \left( 2n \times I - \overline{D} + \overline{A} \right)$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n & 1 & 1 & \cdots & 1 & & & & & \\ 1 & n & 1 & \ddots & \vdots & & & & & \\ 1 & 1 & \ddots & \ddots & 1 & & & I & & \\ \vdots & \ddots & \ddots & n & 1 & & & & & \\ 1 & \cdots & 1 & 1 & n & & & & & \\ & & & & & n+1 & 0 & 1 & \cdots & 1 \\ & & & & & 0 & n+2 & 0 & \ddots & \vdots \\ & & I & & & 1 & 0 & \ddots & \ddots & 1 \\ & & & & & \vdots & \ddots & \ddots & n+2 & 0 \\ & & & & & 1 & \cdots & 1 & 0 & n+1 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n^2+2n-3 & 2n-1 & 3n-2 & \cdots & \cdots & 3n-2 \\ 2n-1 & n^2+3n-4 & 2n-1 & 3n-2 & \ddots & \vdots \\ 3n-2 & 2n-1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 3n-2 \\ \vdots & \ddots & \ddots & \ddots & n^2+3n-4 & 2n-1 \\ 3n-2 & \cdots & \cdots & 3n-2 & 2n-1 & n^2+2n-3 \end{vmatrix}$$

$$= \frac{(n-1)^n}{(2n)^2} \times \begin{vmatrix} n+3 & \frac{2n-1}{n-1} & \frac{3n-2}{n-1} & \cdots & \cdots & \frac{3n-2}{n-1} \\ \frac{2n-1}{n-1} & n+4 & \ddots & \ddots & \ddots & \vdots \\ \frac{3n-2}{n-1} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{3n-2}{n-1} \\ \vdots & \ddots & \ddots & \ddots & n+4 & \frac{2n-1}{n-1} \\ \frac{3n-2}{n-1} & \cdots & \cdots & \frac{3n-2}{n-1} & \frac{2n-1}{n-1} & n+3 \end{vmatrix}.$$

Straight forward induction by using properties of determinants. We obtain that

$$\tau(N_n * P_n) = \frac{(n-1)^{n+1}}{(2n-3)n^2} \times \begin{vmatrix} \frac{n^2-2}{n-1} & 0 & 1 & \cdots & \cdots & 1 \\ 0 & \frac{n^2-2}{n-1}+1 & 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \frac{n^2-2}{n-1}+1 & 0 \\ 1 & \cdots & \cdots & 1 & 0 & \frac{n^2-2}{n-1} \end{vmatrix}$$

$$= \frac{(n-1)^{n+1}}{(2n-3)n^2} \times \frac{2n-3}{n-2} \begin{vmatrix} \frac{n^2-2}{n-1}-1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & \frac{n^2-2}{n-1} & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{n^2-2}{n-1} & -1 \\ 1 & \cdots & \cdots & 0 & -1 & \frac{n^2-2}{n-1}-1 \end{vmatrix}$$

$$= \frac{(n-1)^{n+1}}{(2n-3)n^2} \times \frac{2n-3}{n-2} \times \left(\frac{n^2-2}{n-1} - 2\right) \begin{vmatrix} \frac{n^2-2}{n-1} - 1 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & \frac{n^2-2}{n-1} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \frac{(n-1)^{n+1}}{(n-2)n^2} \times \left(\frac{n^2-2}{n-1} - 2\right) \times U_{n-1}\left(\frac{n^2-2}{2(n-1)}\right) = \frac{(n-1)^n}{n} \times U_{n-1}\left(\frac{n^2-2}{2(n-1)}\right)$$

$$= \frac{n-1}{2^n n\sqrt{n^4-8n^2+8n}} \times \left[\left(n^2-2+\sqrt{n^4-8n^2+8n}\right)^n - \left(n^2-2-\sqrt{n^4-8n^2+8n}\right)^n\right]. \square$$

**Lemma** 3.7 *Let $B_n(x)$ be an $n \times n,\ n \geq 3$ matrix such that*

$$B_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 0 & x \end{pmatrix}$$

with $x \geq 4$. Then

$$\det(B_n(x)) = \frac{2(x+n-3)}{x-3}\left[T_n\left(\frac{x-1}{2} - 1\right)\right].$$

*Proof* Straightforward induction using properties of determinants and above mentioned definitions of Chebyshev polynomial of the first and second kind. □

**Theorem** 3.8 *For any integer $n \geq 3$,*

$$\tau(N_n * C_n) = \frac{(n-1)^{n+1}}{2^n n^2(n-2)}$$
$$\times \left[\left(\frac{n^2-2}{n-1} + \sqrt{\left(\frac{n^2-2}{n-1}\right)^2 - 4}\right)^n + \left(\frac{n^2-2}{n-1} - \sqrt{\left(\frac{n^2-2}{n-1}\right)^2 - 4}\right)^n - 2^{n+1}\right].$$

*Proof* Applying Lemma 3.2, we know that

$$\tau(N_n * C_n) = \frac{1}{(2n)^2}\det\left(2n \times I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n & 1 & 1 & \cdots & 1 & & & & & \\ 1 & n & 1 & \ddots & \vdots & & & & & \\ 1 & 1 & \ddots & \ddots & 1 & & & I & & \\ \vdots & \ddots & \ddots & n & 1 & & & & & \\ 1 & \ddots & 1 & 1 & n & & & & & \\ & & & & & n+2 & 0 & 1 & \cdots & 0 \\ & & & & & 0 & n+2 & 0 & \ddots & \vdots \\ & & & I & & 1 & 0 & \vdots & \ddots & 1 \\ & & & & & \vdots & \ddots & \ddots & n+2 & 0 \\ & & & & & 0 & \cdots & 1 & 0 & n+2 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n^2+3n-4 & 2n-1 & 3n-2 & \cdots & 3n-2 & 2n-1 \\ 2n-1 & \ddots & \ddots & \ddots & \ddots & 3n-2 \\ 3n-2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 3n-2 \\ 3n-2 & \ddots & \ddots & \ddots & \ddots & 2n-1 \\ 2n-1 & 3n-2 & \cdots & \cdots & 2n-1 & n^2+3n-4 \end{vmatrix}$$

$$= \frac{(n-1)^n}{(2n)^2} \times \begin{vmatrix} n+4 & \frac{2n-1}{n-1} & \frac{3n-2}{n-1} & \cdots & \frac{3n-2}{2n-1} & \frac{2n-1}{n-1} \\ \frac{2n-1}{n-1} & n+4 & \ddots & \ddots & \ddots & \frac{3n-2}{n-1} \\ \frac{3n-2}{n-1} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{3n-2}{n-1} \\ \frac{3n-2}{n-1} & \ddots & \ddots & \ddots & n+4 & \frac{2n-1}{n-1} \\ \frac{2n-1}{n-1} & \frac{3n-2}{n-1} & \cdots & \frac{3n-2}{n-1} & \frac{2n-1}{n-1} & n+4 \end{vmatrix}.$$

Straight forward induction by using properties of determinants. We obtain that

$$\tau(N_n * C_n) = \frac{(n-1)^n}{(2n-3)n^2} \times \begin{vmatrix} \frac{n^2-2}{n-1}+1 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{n^2-2}{n-1}+1 & 0 & 1 & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \frac{n^2-2}{n-1}+1 & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{n^2-2}{n-1}+1 \end{vmatrix}.$$

Applying Lemma 3.7, we get that

$$\tau(N_n * C_n) = \frac{(n-1)^{n+1}}{(2n-3)n^2} \times \frac{2(n-3)}{n-2}\left[T_n\left(\frac{n^2-2}{2(n-1)}\right)-1\right]$$
$$= \frac{(n-1)^{n+1}}{2^n n^2(n-2)}$$
$$\times\left[\left(\frac{n^2-2}{n-1}+\sqrt{\left(\frac{n^2-2}{n-1}\right)^2-4}\right)^n+\left(\frac{n^2-2}{n-1}-\sqrt{\left(\frac{n^2-2}{n-1}\right)^2-4}\right)^n-2^{n+1}\right]. \qquad \square$$

**theorem** 3.9 *For any integer $n \geq 3$,*

$$\begin{aligned}\tau(C_n * C_n) &= \frac{n-1}{2^n n^2(n-2)}\left[\left(n+\sqrt{n^2-4}\right)^n+\left(n-\sqrt{n^2-4}\right)^n-2^{n+1}\right]\\ &\quad \times\left[\left(n+2+\sqrt{(n+2)^2-4}\right)^n+\left(n+2-\sqrt{(n+2)^2-4}\right)^n-2^{n+1}\right].\end{aligned}$$

*Proof* applying Lemma 3.2, we get that

$$\tau(C_n * C_n) = \frac{1}{(2n)^2}\det\left(2n \times I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{(2n)^2} \times \left|\begin{array}{cc} \begin{array}{cccccc} n+2 & 0 & 1 & \cdots & 1 & 0 \\ 0 & n+2 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & n+2 & 0 \\ 0 & 1 & \cdots & 1 & 0 & n+2 \end{array} & I \\ I & \begin{array}{cccccc} n+2 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & n+2 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & n+2 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & n+2 \end{array} \end{array}\right|$$

$$= \frac{1}{(2n)^2} \times \left|\begin{array}{cccccc} n+1 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & n+1 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & n+1 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & n+1 \end{array}\right| \times \left|\begin{array}{cccccc} n+3 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & n+3 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & n+3 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & n+3 \end{array}\right|.$$

Applying Lemma 3.7, we get that

$$\begin{aligned}\tau(C_n * C_n) &= \frac{1}{(2n)^2} \times \frac{4(n-1)}{n-2}\left[T_n\left(\frac{n}{2}\right)-1\right] \times 4\left[T_n\left(\frac{n+2}{2}\right)-1\right] \\ &= \frac{n-1}{2^n n^2(n-2)}\left[\left(n+\sqrt{n^2-4}\right)^n+\left(n-\sqrt{n^2-4}\right)^n-2^{n+1}\right] \\ &\quad \times\left[\left(n+2+\sqrt{(n+2)^2-4}\right)^n+\left(n+2-\sqrt{(n+2)^2-4}\right)^n-2^{n+1}\right]. \ \square\end{aligned}$$

## §4. Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and their proofs.

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