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# An Equation Related to $\theta$-Centralizers in Semiprime Gamma Rings

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**Abstract**: Let $M$ be a 2-torsion free semiprime $\Gamma$-ring satisfying a certain assumption and $\theta$ be an endomorphism on $M$. Let $T : M \to M$ be an additive mapping such that

$$2T(a\alpha b\beta a) \quad = T(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha\theta(b)\beta T(a) \tag{1}$$

holds for all pairs $a, b \in M$, and $\alpha, \beta \in \Gamma$. Then we prove that $T$ is a $\theta$-centralizer.

**Key Words**: Semiprime $\Gamma$-ring, left centralizer, centralizer, Jordan centralizer, left $\theta$-centralizer, $\theta$-centralizer, Jordan $\theta$-centralizer.

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## §1. Introduction

Let $M$ and $\Gamma$ be additive Abelian groups. If there exists a mapping $(x, \alpha, y) \to x\alpha y$ of $M \times \Gamma \times M \to M$, which satisfies the conditions

$(i)$ $x\alpha y \in M$;
$(ii)$ $(x + y)\alpha z$=$x\alpha z$+$y\alpha z$, $x(\alpha + \beta)z$=$x\alpha z$+$x\beta z$, $x\alpha(y + z)$=$x\alpha y$+$x\alpha z$;
$(iii)$ $(x\alpha y)\beta z$=$x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

then $M$ is called a $\Gamma$-ring.

Every ring $M$ is a $\Gamma$-ring with $M$=$\Gamma$. However a $\Gamma$-ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa[13]. Bernes[1] weakened slightly the conditions in the definition of $\Gamma$-ring in the sense of Nobusawa.

Let $M$ be a $\Gamma$-ring. Then an additive subgroup $U$ of $M$ is called a left (right) ideal of $M$ if $M\Gamma U \subset U(U\Gamma M \subset U)$. If $U$ is both a left and a right ideal , then we say $U$ is an ideal of $M$. Suppose again that $M$ is a $\Gamma$-ring. Then $M$ is said to be a 2-torsion free if $2x$=0 implies $x$=0 for all $x \in M$. An ideal $P_1$ of a $\Gamma$-ring $M$ is said to be prime if for any ideals $A$ and $B$ of $M$,

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$A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal $P_2$ of a $\Gamma$-ring $M$ is said to be semiprime if for any ideal $U$ of $M$, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A $\Gamma$-ring $M$ is said to be prime if $a\Gamma M\Gamma b$=(0) with $a, b \in M$, implies $a$=0 or $b$=0 and semiprime if $a\Gamma M\Gamma a$=(0) with $a \in M$ implies $a$=0. Furthermore, $M$ is said to be commutative $\Gamma$-ring if $x\alpha y$=$y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover,the set $Z(M)$ =$\{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma, y \in M\}$ is called the centre of the $\Gamma$-ring $M$.

If $M$ is a $\Gamma$-ring, then $[x, y]_\alpha$=$x\alpha y - y\alpha x$ is known as the commutator of $x$ and $y$ with respect to $\alpha$, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta$$
$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$$

for all $x, y.z \in M$ and $\alpha, \beta \in \Gamma$. We consider the following assumption:

*(A) $x\alpha y\beta z$=$x\beta y\alpha z$, for all $x, y, z \in M$, and $\alpha, \beta \in \Gamma$.*

According to the assumption (A), the above two identities reduce to

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$$
$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta,$$

which we extensively used. An additive mapping $T : M \to M$ is a left(right) centralizer if

$$T(x\alpha y) = T(x)\alpha y(T(x\alpha y) = x\alpha T(y))$$

holds for all $x, y \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T(x) = a\alpha x$ is a left centralizer and $T(x) = x\alpha a$ is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers. An additive mapping $D : M \to M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$, and $\alpha \in \Gamma$ and is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$ for all $x \in M$ and $\alpha \in \Gamma$. An additive mapping $T : M \to M$ is Jordan left(right) centralizer if

$$T(x\alpha x) = T(x)\alpha x(T(x\alpha x) = x\alpha T(x))$$

for all $x \in M$, and $\alpha \in \Gamma$.

Every left centralizer is a Jordan left centralizer but the converse is not ingeneral true.

An additive mappings $T : M \to M$ is called a Jordan centralizer if $T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Bernes[1], Luh [8] and Kyuno[7] studied the structure of $\Gamma$-rings and obtained various generalizations of corresponding parts in ring theory. Borut Zalar [15] worked on centralizers of semiprime rings and proved that Jordan centralizers and centralizers of this rings coincide. Joso Vukman[12, 13, 14] developed some remarkable results using centralizers on prime and semiprime rings. Vukman and Irena [11] proved that if $R$ is a 2-tortion free semiprime ring and $T : R \to R$ is an additive mapping such that $2T(xyx) = T(x)yx + xyx$ holds for all $x, y \in R$, then $T$ is a centralizer. Y.Ceven [2] worked on Jordan left derivations on completely prime $\Gamma$-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime $\Gamma$-ring that makes the $\Gamma$-ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime $\Gamma$-ring is a left derivation on it.

In [3], M.F. Hoque and A.C Paul have proved that every Jordan centralizer of a 2-torsion free semiprime $\Gamma$-ring is a centralizer. There they also gave an example of a Jordan centralizer which is not a centralizer.

In [4], M.F. Hoque and A.C Paul have proved that if $M$ is a 2-torsion free semiprime $\Gamma$-ring satisfying the assumption (A) and if $T : M \to M$ is an additive mapping such that

$$T(x\alpha y\beta x) \quad = x\alpha T(y)\beta x$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then $T$ is a centralizer. Also, they have proved that $T$ is a centralizer if $M$ contains a multiplicative identity 1.

In [5], M.F. Hoque and A.C Paul have proved that if $M$ be a 2-torsion free semiprime $\Gamma$-ring satisfying the assumption (A) and let $T : M \to M$ be an additive mapping such that

$$2T(a\alpha b\beta a) \quad = T(a)\alpha b\beta a + a\alpha b\beta T(a)$$

holds for all pairs $a, b \in M$, and $\alpha, \beta \in \Gamma$. Then $T$ is a centralizer.

In [10], Z.Ullah and M.A.Chaudhary have proved that every Jordan $\theta$-centralizer of a 2-torsion free semiprime $\Gamma$-ring is a $\theta$-centralizer.

In [6] M.F. Hoque and A.C Paul have given an example of a Jordan $\theta$-centralizer which is not a $\theta$-centralizer and another two examples which was ensure that $\theta$-centralizer and a Jordan $\theta$-centralizer exist in $\Gamma$-ring. There they also have proved that if $M$ be a 2-torsion free semiprime $\Gamma$-ring satisfying a certain assumption and $\theta$ be an endomorphism of $M$. Let $T : M \to M$ be an additive mapping such that

$$T(x\alpha y\beta x) \quad = \theta(x)\alpha T(y)\beta\theta(x)$$

holds for all $x, y \in M$, and $\alpha, \beta \in \Gamma$. Then $T$ is a $\theta$-centralizer.

In this paper we study certain results using the concept of $\theta$-centralizer on semiprime gamma ring.

## §2. The $\theta$-Centralizers of Semiprime Gamma Rings

In this section we have given the following definitions:

Let $M$ be a 2-torsion free semiprime $\Gamma$-ring and let $\theta$ be an endomorphism of $M$. An additive mapping $T : M \to M$ is a left(right) $\theta$-centralizer if $T(x\alpha y) = T(x)\alpha\theta(y)(T(x\alpha y) = \theta(x)\alpha T(y))$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. If $T$ is a left and a right $\theta$-centralizer, then it is natural to call $T$ a $\theta$-centralizer.

Let $M$ be a $\Gamma$-ring and let $a \in M$ and $\alpha \in \Gamma$ be fixed element. Let $\theta : M \to M$ be an endomorphism. Define a mapping $T : M \to M$ by $T(x)a\alpha\theta(x)$. Then it is clear that $T$ is a left $\theta$-centralizer. If $T(x) = \theta(x)\alpha a$ is defined, then $T$ is a right $\theta$-centralizer.

An additive mapping $T: M \to M$ is Jordan left(right) $\theta$-centralizer if

$$T(x\alpha x) = T(x)\alpha\theta(x)(T(x\alpha x) = \theta(x)\alpha T(x))$$

holds for all $x \in M$ and $\alpha \in \Gamma$. It is obivous that every left $\theta$-centralizer is a Jordan left $\theta$-centralizer but in general Jordan left $\theta$-centralizer is not a left $\theta$-centralizer.

Let $M$ be a $\Gamma$-ring and let $\theta$ be an endomorphism on $M$. An additive mapping $T: M \to M$ is called a Jordan $\theta$-centralizer if $T(x\alpha y + y\alpha x) = T(x)\alpha\theta(y) + \theta(y)\alpha T(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$. It is clear that every $\theta$-centralizer is a Jordan $\theta$-centralizer but the converse is not in general a $\theta$- centralizer.

An additive mapping $D: M \to M$ is called a $(\theta, \theta)$-derivation if $D(x\alpha y) = D(x)\alpha\theta(y) + \theta(x)\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$ and is called a Jordan $(\theta, \theta)$-derivation if $D(x, x) = D(x)\alpha\theta(x) + \theta(x)\alpha D(x)$ holds for all $x \in M$ and $\alpha \in \Gamma$.

For proving our main results, we need the following Lemmas:

**Lemma 2.1**([4]) *Suppose $M$ is a semiprime $\Gamma$-ring satisfying the assumption (A). Suppose that the relation $x\alpha a\beta y + y\alpha a\beta z = 0$ holds for all $a \in M$, some $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then $(x + z)\alpha a\beta y = 0$ is satisfied for all $a \in M$ and $\alpha, \beta \in \Gamma$.*

**Lemma 2.2** *Let $M$ be a 2-torsion free semiprime $\Gamma$-ring satisfying the assumption (A) and $\theta$ be an endomorphism of $M$. Suppose that $T: M \to M$ is an additive mapping such that*

$$2T(a\alpha b\beta a) \quad = T(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha\theta(b)\beta T(a)$$

*holds for all pairs $a, b \in M$ and $\alpha, \beta \in \Gamma$. Then $2T(a\gamma a) = T(a)\gamma\theta(a) + \theta(a)\gamma T(a)$.*

*Proof* Putting $a + c$ for $a$ in (1)(linearization), we have

$$\begin{aligned} 2T(a\alpha b\beta c + c\alpha b\beta a) \quad &= \quad T(a)\alpha\theta(b)\beta\theta(c) + T(c)\alpha\theta(b)\beta\theta(a) \\ &\quad +\theta(c)\alpha\theta(b)\beta T(a) + \theta(a)\alpha\theta(b)\beta T(c) \end{aligned} \tag{2}$$

Putting $c = a\gamma a$ in (2), we have

$$\begin{aligned} 2T(a\alpha b\beta a\gamma a + a\gamma a\alpha b\beta a) \quad &= \quad T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) + T(a\gamma a)\alpha\theta(b)\beta\theta(a) \\ &\quad +\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a) + \theta(a)\alpha\theta(b)\beta T(a\gamma a) \end{aligned} \tag{3}$$

Replacing $b$ by $a\gamma b + b\gamma a$ in (1), we have

$$\begin{aligned} 2T(a\alpha a\gamma b\beta a + a\alpha b\gamma a\beta a) \quad &= \quad T(a)\alpha\theta(a)\gamma\theta(b)\beta\theta(a) + T(a)\alpha\theta(b)\gamma\theta(a)\beta\theta(a) \\ &\quad +\theta(a)\alpha\theta(a)\gamma\theta(b)\beta T(a) + \theta(a)\alpha\theta(b)\gamma\theta(a)\beta T(a) \end{aligned} \tag{4}$$

Subtracting (4) from (3), using assumption (A), gives

$$(T(a\gamma a) - T(a)\gamma\theta(a))\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha(b)\beta(T(a\gamma a) - \theta(a)\gamma T(a)) \quad = 0.$$

Taking $\theta(x) = T(a\gamma a) - T(a)\gamma\theta(a)$, $y = a$, $c = b$ and $\theta(z) = T(a\gamma a) - \theta(a)\gamma T(a)$. Then the above relation becomes $\theta(x)\alpha\theta(c)\beta\theta(y) + \theta(y)\alpha\theta(c)\beta\theta(z) = 0$. Thus using Lemma 2.1, we get $(\theta(x) + \theta(z))\alpha\theta(c)\beta\theta(y) = 0$. Hence

$$(2T(a\gamma a) - T(a)\gamma\theta(a) - \theta(a)\gamma T(a))\alpha\theta(b)\beta\theta(a) \quad = 0.$$

If we take

$$A(a) \quad = 2T(a\gamma a) - T(a)\gamma\theta(a) - \theta(a)\gamma T(a),$$

then the above relation becomes

$$A(a)\alpha\theta(b)\beta\theta(a) \quad = 0$$

Using the assumption (A), we obtain

$$A(a)\beta\theta(b)\alpha\theta(a) \quad = 0 \tag{5}$$

Replacing $b$ by $a\alpha b\gamma A(a)$ in (5), we have

$$A(a)\beta\theta(a)\alpha\theta(b)\gamma A(a)\alpha\theta(a) \quad = 0$$

Again using the assumption (A), we have

$$A(a)\alpha\theta(a)\beta\theta(b)\gamma A(a)\alpha\theta(a) \quad = 0$$

By the semiprimeness of $M$, we have

$$A(a)\alpha\theta(a) = 0 \tag{6}$$

Similarly, if we multiplying (5) from the left by $\theta(a)\alpha$ and from the right side by $\gamma A(a)$, we obtain

$$\theta(a)\alpha A(a)\beta\theta(b)\alpha\theta(a)\gamma A(a) \quad = 0$$

Using the assumption (A),

$$\theta(a)\alpha A(a)\beta\theta(b)\gamma\theta(a)\alpha A(a) \quad = 0$$

and by the semiprimeness, we obtain

$$\theta(a)\alpha A(a) \quad = 0 \tag{7}$$

Replacing $a$ by $a+b$ in (6)(linearization), we have

$$A(a)\alpha\theta(b) + A(b)\alpha\theta(a) + B(\theta(a),\theta(b))\alpha\theta(a) + B(\theta(a),\theta(b))\alpha\theta(b) = 0,$$

where

$$B(\theta(a),\theta(b)) \quad = 2T(a\gamma b + b\gamma a) - T(a)\gamma\theta(b) - T(b)\gamma\theta(a) - \theta(a)\gamma T(b) - \theta(b)\gamma T(a)$$

Replacing $a$ by $-a$ in the above relation and comparing these relation, and by using the 2-torsion freeness of $M$, we arrive at

$$A(a)\alpha\theta(b) + B(\theta(a),\theta(b))\alpha\theta(a) \quad = 0 \tag{8}$$

Right multiplication of the above relation by $\beta A(a)$ along with (7) gives

$$A(a)\alpha\theta(b)\beta A(a)+B(\theta(a),\theta(b))\alpha\theta(a)\beta A(a) \quad = 0$$

Since $\theta(a)\beta A(a)=0$, for all $\beta\in\Gamma$, we have

$$B(\theta(a),\theta(b))\alpha\theta(a)\beta A(a) \quad = 0$$

This implies that

$$A(a)\alpha\theta(b)\beta A(a) \quad = 0$$

By semiprimeness, we have

$$A(a) \quad = 0$$

Thus we have

$$2T(a\gamma a)=T(a)\gamma\theta(a)+\theta(a)\gamma T(a). \qquad \square$$

**Lemma** 2.3 *Let $M$ be a 2-torsion free semiprime $\Gamma$-ring satisfying the assumption (A) and $\theta$ be an endomorphism of $M$. Let $T:M\to M$ be an additive mapping such that*

$$2T(a\alpha b\beta a) \quad = T(a)\alpha\theta(b)\beta\theta(a)+\theta(a)\alpha\theta(b)\beta T(a)$$

*holds for all pairs $a,b\in M$ and $\alpha,\beta\in\Gamma$. Then*

$$[T(a),\theta(a)]_{\alpha} \quad = 0 \tag{9}$$

*Proof* Replacing $a$ by $a+b$ in relation (9)(linearization) gives

$$2T(a\gamma b+b\gamma a) \quad = T(a)\gamma\theta(b)+T(b)\gamma\theta(a)+\theta(a)\gamma T(b)+\theta(b)\gamma T(a) \tag{10}$$

Replacing $b$ with $2a\alpha b\beta a$ in (11) and use (1), we obtain

$$\begin{aligned} 4T(a\gamma a\alpha b\beta a+a\alpha b\beta a\gamma a) \quad &= \quad 2T(a)\gamma\theta(a)\alpha\theta(b)\beta\theta(a)+2T(a\alpha b\beta a)\gamma\theta(a) \\ &\quad +2\theta(a)\gamma T(a\alpha b\beta a)+2\theta(a)\alpha\theta(b)\beta\theta(a)\gamma T(a) \\ &= \quad 2T(a)\gamma\theta(a)\alpha\theta(b)\beta\theta(a)+T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) \\ &\quad +\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a)+\theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a) \\ &\quad +\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)+2\theta(a)\alpha\theta(b)\beta\theta(a)\gamma T(a) \end{aligned}$$

$$\begin{aligned} 4T(a\gamma a\alpha b\beta a+a\alpha b\beta a\gamma a) \quad &= \quad 2T(a)\gamma\theta(a)\alpha\theta(b)\beta\theta(a)+T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) \\ &\quad +\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a)+\theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a) \\ &\quad +\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a) \\ &\quad +2\theta(a)\alpha\theta(b)\beta\theta(a)\gamma T(a) \end{aligned} \tag{11}$$

Comparing (4) and (12), we arrive at

$$\begin{aligned} T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) &+ \theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a) \\ &- \theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a) - \theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a) = 0 \end{aligned} \tag{12}$$

Putting $b\gamma a$ for $b$ in the above relation, we have

$$\begin{aligned} &T(a)\alpha\theta(b)\gamma\theta(a)\beta\theta(a)\gamma\theta(a) + \theta(a)\gamma\theta(a)\alpha\theta(b)\gamma\theta(a)\beta T(a) \\ &\quad -\theta(a)\alpha\theta(b)\gamma\theta(a)\beta T(a)\gamma\theta(a) - \theta(a)\gamma T(a)\alpha\theta(b)\gamma\theta(a)\beta\theta(a) = 0 \end{aligned} \tag{13}$$

Right multiplication of (13) by $\gamma\theta(a)$ gives

$$\begin{aligned} &T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a)\gamma\theta(a) + \theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a) \\ &\quad -\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a)\gamma\theta(a) - \theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) = 0 \end{aligned} \tag{14}$$

Subtracting (14) from (15) and using assumption (A), we get

$$\theta(a)\gamma\theta(a)\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha - \theta(a)\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0 \tag{15}$$

The substitution $T(a)\alpha b$ for b in (16), we have

$$\theta(a)\gamma\theta(a)\gamma T(a)\alpha\theta(b)\beta[T(a),\theta(a)]_\alpha - \theta(a)\gamma T(a)\alpha\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0 \tag{16}$$

Left multiplication of (16) by $T(a)\alpha$ gives

$$T(a)\alpha\theta(a)\gamma\theta(a)\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha - T(a)\alpha\theta(a)\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0 \tag{17}$$

Subtracting (17) from (18), we arrive at

$$[T(a),\theta(a)\gamma\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha - [T(a),\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0$$

In the above relation let

$$\theta(x) = [T(a),\theta(a)\gamma\theta(a)]_\alpha,\ \theta(y) = [T(a),\theta(a)]_\alpha,\ \theta(z) = -[T(a),\theta(a)]_\alpha\gamma\theta(a)$$

and $c = b$. Then we have

$$\theta(x)\gamma\theta(c)\beta\theta(y) + \theta(y)\gamma\theta(c)\beta\theta(z) \quad = 0$$

Thus from Lemma 2.1, we have

$$\begin{aligned} &(\theta(x) + \theta(z))\gamma\theta(c)\beta\theta(y) = 0 \\ &\Rightarrow ([T(a),\theta(a)\gamma\theta(a)]_\alpha - [T(a),\theta(a)]_\alpha\gamma\theta(a))\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha = 0 \end{aligned}$$

This implies that

$$\begin{aligned} &([T(a),\theta(a)]_\alpha\gamma\theta(a) + \theta(a)\gamma[T(a),\theta(a)]_\alpha - [T(a),\theta(a)]_\alpha\gamma\theta(a))\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha = 0 \\ &\Rightarrow \theta(a)\gamma[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha = 0 \end{aligned}$$

Putting $b = b\alpha a$ in the above relation, we have

$$\theta(a)\gamma[T(a),\theta(a)]_\alpha\gamma\theta(b)\alpha\theta(a)\beta[T(a),\theta(a)]_\alpha = 0$$
$$\Rightarrow \theta(a)\gamma[T(a),\theta(a)]_\alpha\alpha\theta(b)\beta\theta(a)\gamma[T(a),\theta(a)]_\alpha = 0$$

using the assumption (A). By the semiprimeness of $M$, we obtain

$$\theta(a)\gamma[T(a),\theta(a)]_\alpha \quad = 0 \tag{18}$$

Putting $a\gamma b$ for $b$ in the relation (13), we obtain

$$T(a)\alpha\theta(a)\gamma\theta(b)\beta\theta(a)\gamma\theta(a) + \theta(a)\gamma\theta(a)\alpha\theta(a)\gamma\theta(b)\beta T(a)$$
$$-\theta(a)\alpha\theta(a)\gamma\theta(b)\beta T(a)\gamma\theta(a) - \theta(a)\gamma T(a)\alpha\theta(a)\gamma\theta(b)\beta\theta(a) = 0 \tag{19}$$

Left multiplication of (13) by $\theta(a)\gamma$, we have

$$\theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) + \theta(a)\gamma\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)$$
$$-\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a) - \theta(a)\gamma\theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a) = 0 \tag{20}$$

Subtracting (21) from (20), and using assumption (A), we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta\theta(a)\gamma\theta(a) - \theta(a)\gamma[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta\theta(a) \quad = 0$$

Using (19) in the above relation, we obtain

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta\theta(a)\gamma\theta(a) \quad = 0 \tag{21}$$

Putting $b\alpha T(a)$ for $b$ in (22), we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\alpha T(a)\beta\theta(a)\gamma\theta(a) \quad = 0 \tag{22}$$

Right multiplication of (22) by $\alpha T(a)$ gives

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta\theta(a)\gamma\theta(a)\alpha T(a) \quad = 0 \tag{23}$$

Subtracting (24) from (23) and using assumption (A), we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)\gamma\theta(a)]_\alpha \quad = 0$$

The above relation can be rewritten and using (19), we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0$$

Putting $a\alpha b$ for $b$ in the above relation, we obtain

$$[T(a),\theta(a)]_\alpha\gamma\theta(a)\alpha\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0$$

By semiprimeness of $M$, we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0 \tag{24}$$

Replacing $a$ by $a+b$ in (19) and then using (19) gives

$$\theta(a)\gamma[T(a),\theta(b)]_\alpha+\theta(a)\gamma[T(b),\theta(a)]_\alpha+\theta(a)\gamma[T(b),\theta(b)]_\alpha$$
$$+\theta(b)\gamma[T(a),\theta(a)]_\alpha+\theta(b)\gamma[T(a),\theta(b)]_\alpha+\theta(b)\gamma[T(b),\theta(a)]_\alpha=0$$

Replacing $a$ by $-a$ in the above relation and comparing the relation so obtained with the above relation, we have

$$\theta(a)\gamma[T(a),\theta(b)]_\alpha+\theta(a)\gamma[T(b),\theta(a)]_\alpha+\theta(b)\gamma[T(a),\theta(a)]_\alpha \quad =0 \tag{25}$$

Left multiplication of (26) by $[T(a),\theta(a)]_\alpha\beta$ and then use (25), we have

$$[T(a),\theta(a)]_\alpha\beta\theta(b)\gamma[T(a),\theta(a)]_\alpha \quad =0$$

By semiprimeness of $M$, we have

$$[T(a),\theta(a)]_\alpha \quad =0$$

Hence the relation (10) follows. □

**Theorem** 2.1 *Let $M$ be a 2-torsion free semiprime $\Gamma$-ring satisfying the assumption (A) and and $\theta$ be an endomorphism of $M$. Let $T:M\to M$ be an additive mapping such that $2T(a\alpha b\beta a)=T(a)\alpha\theta(b)\beta\theta(a)+\theta(a)\alpha\theta(b)\beta T(a)$ holds for all pairs $a,b\in M$ and $\alpha,\beta\in\Gamma$. Then $T$ is a $\theta$-centralizer.*

*Proof* The relation (9) in Lemma 2.2 and the relation (10) in Lemma 2.3 give

$$T(a\alpha a)=T(a)\alpha\theta(a) \quad \text{and} \quad T(a\alpha a)=\theta(a)\alpha T(a)$$

since $M$ is a 2-torsion free. Hence $T$ is a left and also a right Jordan $\theta$-centralizers. By Theorem 2.1 in [3], it follows that $T$ is a left and also a right $\theta$-centralizer which completes the proof of the theorem. □

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