International J.Math. Combin. Vol.4(2011), 54-58

Random Walk on a Finitely Generated Monoid

Driss Gretete

(Ecole nationale des sciences appliquées, Université Ibn Tofail, Kenitra, Maroc)

E-mail: drissgretete@hotmail.com

Abstract: We study the stability of the waiting time of arrival at first point of length n on a finitely generated monoid. As an example we show that the asymptotic behavior $\psi(n)$ of the average waiting time of arrival at the first element of length n on a monogenic monoid is $\psi(n) \asymp n\ln(n)$, and that of a finitely generated free monoid of at least two generators is $\psi(n) \asymp n$.

Key Words: Random walk, monoid, group, probability, Markov chain

AMS(2010): 11B39, 33C05

§1. Introduction

The study of random walks on finitely generated groups and locally compact groups has identified an invariant for these groups, which is the asymptotic behavior $\phi(n)$ of probabilities of return to the origin (for details see [1], [5], [11]-[12]). In 1959 Kesten (see [7]) showed that $\phi(n)$ decays like $\exp(-n)$ if and only if the group G is non amenable. Later Varoupolos proved that $\exp(-n)$ is a lower bound of $\phi(n)$, that is $\phi(n) \succeq \exp(-n)$.

For a simple random walk on a discrete subgroup G of a connected Lie group, three and only three behaviors may occur (see [1], [3]-[4], [6], [12]-[13]):

1. the group G is non amenable and $\phi(n) \asymp \exp(-n)$,
2. the group G is virtually nilpotent of volume growth $V(n) \asymp n^d$, in this case $\phi(n) \asymp n^{-d/2}$,
3. the group G is virtually polycyclic of exponential volume growth, in this case $\phi(n) \asymp \exp(-n^{1/3})$.

For a large class of solvable groups (see [2], [5], [8]-[10]) the random walk decays like $\exp(-n^{1/3})$.

In the sequel a monoid is a set with an associative internal composition law and has a neutral element denoted by e. In this paper we are interested in the case of monoids and we try to find an invariant in terms of random walks, which does not involve the concept of symmetry.

Let M be a finitely generated monoid, S be a finite minimal generating subset of M. We define for all x of M the length of x by

$$\ell_S(x) = min\{k \in \mathbb{N}; x \in S^k\}$$

[1]Received December 15, 2010. Accepted November 21, 2011.

where $S^k = \{x_1 x_2 \cdots x_k | x_i \in S\}$ if $k \neq 0$ and $S^0 = \{e\}$. For a positive integer n we consider $\Omega = (S^n)^{\mathbb{N}}$.
Let $X_i : \Omega \to G$ the i-th canonical projection.

We denote by $A|B$ the event A such that B is realized. We define a probability P on Ω, such that

$$\forall k \in \mathbb{N} \forall g \in M, P(X_{k+1} = g | X_k = g) = \frac{\ell_S(g)}{\operatorname{card}(S^n)},$$

$$\forall g \in M, \forall s \in S, P(X_{k+1} = gs | X_k = g) = \left(1 - \frac{\ell_S(g)}{\operatorname{card}(S^n)}\right) \frac{1}{\operatorname{card}(S)},$$

and $P(X_{k+1} = l | X_k = g) = 0$ in the other cases.

For two real valued functions f, g defined on a discrete subset of $]0, +\infty[$, we define the relation $f \preceq g$ by

$$\exists \alpha, \beta \in]0, +\infty[, \forall x \in]0, +\infty[; \overline{f}(x) \leq \alpha \overline{g}(\beta x) + \alpha,$$

where $\overline{f}$ and $\overline{g}$ are the linear interpolations of f and g. When $f \preceq g$ and $g \preceq f$, we write $f \asymp g$. The asymptotic behavior of f is the equivalence class of f for the relation $\asymp$.

We define the random variable $U_n = \operatorname{card}\{k; X_k \in S^{n-1}\}$, which is the waiting time of arrival at the first element of length n in M. When $U_n = +\infty$ from a certain rank, we say that the random walk on the monoid M is slow. When it is not slow, we are interested in asymptotic behavior of $\psi(n) = E(U_n | X_0 = e)$ when n tends to infinity, which represents the average waiting time of arrival at the first element of length n, starting from the origin. The case of finite monoid is a model on which the random walk is slow since for such a monoid

$$\forall n > \max\{\ell_S(x); x \in M\}; U_n = +\infty.$$

§2. Stability of Asymptotic Behavior of the Average Waiting Time $\phi(n)$

In this section we show that the asymptotic behavior of $\phi(n)$ is independent of the generating set S, which allow us to construct an invariant of the monoid M

Proposition 1 *If S and S' are two minimum generating sets of M, then $\phi_S(n) \asymp \phi_{S'}(n)$.*

Proof Let X'_k be the k-th canonical projection on $\Omega' = (S'^n)^{\mathbb{N}}$ and $U'_n = \operatorname{card}\{i; X'_i \in S'^{n-1}\}$. There exists a positive integer p such that $S \subset S'^p$. By an induction on i, one gets

$$\{i, X_i \in S^{n-1}\} \cap \{X_0 = e\} \subset \{i; X'_i \in S'^{np-1}\} \cap \{X'_0 = e\}.$$

Hence

$$E(U_n | X_0 = e) \leq E(U'_{np} | X'_0 = e).$$

It follows that $\phi(n) \leq \phi(np)$ where $\phi_S(n) \preceq \phi_{S'}(n)$, and exchanging the roles of S and S' we obtain the result. □

§3. **Infinite Monogenic Monoids**

In this section we prove the following result.

Theorem 3.1 *If M is an infinite monogenic monoid, then the average waiting time on M satisfies $\psi(n) \asymp n\ln(n)$.*

Proof The monoid M is monogenic, then there exists $a \in M$ such that $S = \{a\}$ is a minimal generating subset of M. We can write $U_n = \text{card}\{k; \ell_S(X_k) < n\}$. For $k \in \{0, \ldots, n-1\}$,let $u(n,k) = E(U_n|\ell_S(X_0) = k)$, so $\psi(n) = u(n,0)$. Then for all $k \in \{0, \ldots, n-1\}$,

$$\begin{aligned} u(n,k) &= E(U_n|\ell_S(X_0) = k) \\ &= E(U_n|\ell_S(X_0) = k, \ell_S(X_1) = k)P(\ell_S(X_1) = k|\ell_S(X_1) = k) + \\ &\quad E(U_n|\ell_S(X_0) = k, \ell_S(X_1) = k+1)P(\ell_S(X_1) = k+1|\ell_S(X_0) = k) \\ &= \frac{k}{n}u(n,k) + \frac{n-k}{n}u(n,k+1) + 1. \end{aligned}$$

Hence $u(n,k) = u(n,k+1) + \frac{n}{n-k}$ so, $u(n,0) = \sum_0^{n-1} \frac{n}{n-k} \asymp n\ln(n)$ and it follows that $\psi(n) \asymp n\ln(n)$. □

§4. **Free Monoids**

For a free monoid, we have the following result.

Theorem 4.1 *Let M be a free monoid with generators p, $p > 1$. Then the average waiting time of the visit of the n-th ring is $\psi(n) \asymp n$.*

Proof We consider a minimal generating subset $S = \{x_1, \ldots, x_p\}$ of M. Keeping the notations introduced in the preceding section, we have

$$u(n,k) = 1 + \frac{k}{p^n}u(n,k) + (1 - \frac{k}{p^n})u(n,k+1).$$

Therefore

$$u(n,k) - u(n,k+1) = \frac{1}{1 - \frac{k}{p^n}}.$$

Hence $u(n,0) = \sum_{k=0}^{n-1} \frac{1}{1-\frac{k}{p^n}}$, and we obtain $n \le \psi(n) \le n\frac{p^n}{p^n - n + 1}$, and the result follows. □

§5. **Lower and Upper bounds of $\psi(n)$**

We have the following property about the lower bound $\psi(n)$.

Proposition 2 *For any finitely generated monoid M, $\psi(n) \ge n$.*

Proof We have $\psi(n) = E(U_n | X_0 = e)$. When $X_0 = e$ is realized, then $X_1 = X_0$ or $X_1 \in S$ and by induction $X_0, X_1, \ldots, X_{n-1}$ are realized. Consequently, $\{0, 1, \ldots, n-1\} \subset \{i, \ell_S(X_i) < n\}$. So we obtain the lower bound. $\square$

Proposition 3 *For any finitely generated monoid M, for non slow random walk, $\psi(n) \preceq n\ln(n)$.*

Proof We have

$$u(n,k) \leq \frac{k}{\mathrm{card}(S^n)} u(n,k) + \left(1 - \frac{k}{\mathrm{card}(S^n)}\right) u(n,k+1) + 1$$

and since the random walk on M is not slow then for any n, we have $S^n \subsetneq S^{n+1}$. So for all positive integer n, $\mathrm{card}(S^n) \geq n$, then

$$u(n,0) \leq \sum_{k=0}^{n-1} \frac{\mathrm{card} S^n}{\mathrm{card} S^n - k} \leq \sum_{k=0}^{n-1} \frac{n}{n-k} \leq n(1 + \ln(n)). \qquad \square$$

§6. Questions

Several questions arise with respect to the profile $\psi(n)$ following.

1. Is there an asymptotic behavior of $\psi(n)$ between n and $n\ln(n)$?

2. For a monoid M, G is the group obtained by symmetrization of M, what is the relationship between the asymptotic behavior of $\psi(n)$ and the probability of return on G?

3. For a non amenable monoid, have we $\psi(n) \asymp n$?

4. What is the asymptotic behavior of $\psi(n)$ in the case of a monoid with polynomial growth of degree d?

References

[1] Alexopoulos G., A lower estimate for central probabilities on polycyclic groups, *Canad. J. Math.* 44 (1992), 897-910. MR1186471

[2] Bendikov A. Pittet Ch., Roman Sauer, Spectral distribution and $L2$-isoperimetric profile of Laplace operators on groups, *Arxiv preprint:0901.0271*, 2009 - arxiv.org.

[3] Coulhon Th., Grigorýan A, On diagonal lower bounds for heat kernels on non-compact manifolds and Markov chains, *Duke Math. J.* 89 (1997), 133-199. (MR1458975)

[4] Coulhon Th., Grigorýan A., Pittet, Ch., A geometric approach to on-diagonal heat kernel lower bounds on groups, *Ann. Inst. Fourier* 51 (2001), 1763-1827. (MR1871289)

[5] Gretete D., Stabilité du comportement des marches aléatoires sur un groupe localement compact, *Ann. Inst. Henri Poincaré Probab. Stat.* 44(2008), no. 1, 129-142 (French, with English and French summaries).

[6] Hebisch W., On heat kernels on Lie groups, *Math. Zeit.* 210 (1992), 593-605. (MR1175724)

[7] Kesten H., Full Banach mean values on countable groups, *Math. Scand.* 7 (1959) 146-156.

[8] Mustapha S., *C.R. Acad.Sci. Paris*, Ser.I 340(2005).

[9] Mustapha S.,Lower estimates for random walks on a class of amenable p-adic groups, *Journal URL*, Vol. 14 , no 51, pp. 1513-1531(2009).

[10] Pittet C. and Saloff-Coste L. *Random walks on finite rank Solvable Groups*, May 2002.

[11] Pittet C. and Saloff-Coste L. On the Stability of the behavior of Random Walks on Groups, *J. Geometric Analysis*, 10, 713-737, (2000).

[12] Varopoulos N., Saloff-Coste L., Coulhon Th., Analysis and geomety on groups, *Cambridge Tracts in Mathematics* 102, Cambridge University Press, 1993. (MR1218884)

[14] Woess Wolfgang, Random walks on infinite graphs and groups *Cambridge Tracts in Mathematics* 138, Cambridge University Press, 2000.