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# On the Series Expansion of the Ramanujan Cubic Continued Fraction

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**Abstract**: If the Ramanujan cubic continued fraction (or its reciprocal) is expanded as a power series, the sign of the coefficients is periodic with period 3. We give the combinatorial interpretations for the coefficients from which the result follows immediate. We also derive some interesting identities involving coefficients.

**Key Words**: Ramanujan cubic continued fraction, Jacobi's triple product identity, partitions, color partitions.

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## §1. Introduction

As usual for any complex number $a$, we define

$$(a)_0 := 1,$$
$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

where $q$ is any complex number with $|q| < 1$. We also define

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a_1, a_2, a_3, ..., a_n; q)_\infty := \prod_{i=1}^{n}(a_i;q)_\infty$$

and

$$(q^{r\pm};q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where $s$ and $r$ are positive integers with $r < s$. One of the most celebrated $q$-series identity is the following Jacobi's triple product identity:

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \left(\frac{-q}{z};q^2\right)_\infty \left(-qz;q^2\right)_\infty \left(q^2;q^2\right)_\infty, \qquad |z| < 1. \tag{1.1}$$

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Ramanujan [13], [6] expressed the above identity in the following form:

$$f(a,b): = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$$
$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab|<1. \tag{1.2}$$

Further, Ramanujan defined the following particular case of $f(a,b)$:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

Ramanujan also define

$$\chi(q) := (-q;q^2)_\infty.$$

The famous Rogers-Ramanujan continued fraction is defined as

$$R(q) := 1 + \frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots}.$$

B. Richmond and G. Szekeres [15] examined asymptotically the power series co-efficients of $R(q)$, in particular if

$$R(q) := \sum_{n=0}^{\infty} r_n q^n,$$

they have proved that for $n$ sufficiently large

$$r_{5n},\ r_{5n+1} > 0$$

and

$$r_{5n+2},\ r_{5n+3},\ r_{5n+4} < 0.$$

A similar result was also shown for the coefficients of $R^{-1}(q)$. In examining Ramanujan's lost notebook, G. E. Andrews discovered some relevant formulae. He [4] then established a combinatorial interpretation of these formulae. M. D. Hirschhorn [11] later gave a simple proofs of these identities using only quintuple product identity.

On page 229 of his notebook [12], Ramanujan recorded interesting continued fraction $H(q)$ defined by

$$H(q) := \frac{1}{1+q+}\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+\ldots}$$
$$= \frac{f(-q,-q^7)}{f(-q^3,-q^5)}. \tag{1.5}$$

Without any knowledge of Ramanujan's work, Gordon [9] and Göllnitz [10] rediscovered and proved (1.5). Richmond and Szekeres [15], Andrews and D. M. Bressoud [5], K. Alladi and B. Gordon [1], Hirschhorn [12], and S- D. Chen and S- S. Huang [8] have studied the periodicity of signs of Taylor series coefficients of the expansion of $H(q)$.

Recently S. H. Chan and H. Yesilyurt [7] shown the periodicity of large number of quotients of certain infinite products. For example, they have deduced Corollary 2.2 below from their general result.

On page 366 of his lost notebook [14] Ramanajan investigated another beautiful continued fraction $G(q)$ defined by

$$\begin{aligned} G(q): \quad &= \frac{1}{1_{+}}\,\frac{q+q^2}{1}{}_{+}\,\frac{q^2+q^4}{1}{}_{+}\,\frac{q^3+q^6}{1}{}_{+\ldots} \\ &= \frac{f(-q,-q^5)}{f(-q^3,-q^3)} \end{aligned}$$

and claimed that there are many results of $G(q)$ which are analogous to results of $R(q)$. Motivated by Ramanujan's claim and the above mentioned works on $R(q)$ and $H(q)$, in this paper, we give the combinatorial interpretation of the co-efficient in the series expansion of $G(q)$ and its reciprocal.

We conclude this introduction by letting

$$G(q) = \sum_{n=0}^{\infty} a_n q^n \tag{1.6}$$

and

$$M(q) = \frac{1}{G(q)} = \sum_{n=0}^{\infty} b_n q^n. \tag{1.7}$$

## §2. Combinatorial Interpretations of $a_n$ and $b_n$

**Lemma** 2.1 *We have*

$$G(q) = \frac{1}{\varphi(-q^3)} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n}. \tag{2.1}$$

*Proof* We have [6, p. 345]

$$2G(q) = \frac{q^{-\frac{1}{3}}}{\varphi(-q^3)}\left[\varphi(-q^3) - \varphi(-q^{\frac{1}{3}})\right].$$

Upon using (1.3) in the above, we obtain

$$2G(q) = \frac{1}{q^{\frac{1}{3}}\varphi(-q^3)}\left[\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{3}}\right]$$

Converting the second series in the right hand side of the above into sum of three series, we deduce that

$$2G(q) = \frac{1}{\varphi(-q^3)}\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n} - \sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+4n+1}\right].$$

Now replacing $n$ by $-(n+1)$ in the second series, we obtain

$$G(q) = \frac{1}{\varphi(-q^3)}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n}. \qquad \square$$

**Theorem** 2.1 *Let $a_n$ be as defined in (1.6). Then*

$$\sum_{n=0}^{\infty} a_{3n}q^n = \frac{1}{(q;q^2)_\infty (q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+2n}, \tag{2.2}$$

$$\sum_{n=0}^{\infty} a_{3n+1}q^n = \frac{-1}{(q;q^2)_\infty (q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+4n} \tag{2.3}$$

*and*

$$\sum_{n=0}^{\infty} a_{3n+2}q^n = \frac{-q}{(q;q^2)_\infty (q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+8n}. \tag{2.4}$$

*Proof* Recall that the operator $U_3$ [3, p. 161], operating on a power series (1.6) is defined by

$$U_3G(q) := \sum_{n=0}^{\infty} a_{3n}q^n = \frac{1}{3}\sum_{j=0}^{2} G(t^j q^{\frac{1}{3}}),$$

where $t = e^{\frac{2\pi i}{3}}$. Hence for $0 \le k \le 2$, by Lemma 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{3n+k}q^n &= U_3\left[q^{-k}G(q)\right] \\
&= \frac{1}{3}\sum_{j=0}^{2}(t^j q^{\frac{1}{3}})^{-k} G(t^j q^{\frac{1}{3}}) \\
&= \frac{1}{3\varphi(-q)}\sum_{n=-\infty}^{\infty}(-1)^n \sum_{j=0}^{2} t^{(3n^2+2n-k)j} q^{\frac{3n^2+2n-k}{3}}.
\end{aligned}$$

Now $3n^2 + 2n - k \equiv 0 \pmod 3$ for $2n \equiv k \pmod 3$. It therefore follows from the above, that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{3n+k}q^n &= \frac{1}{\varphi(-q)}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3(6n+4k)^2+4(6n+4k)-4k}{12}} \\
&= \frac{1}{\varphi(-q)}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+12nk+2n+4k^2+k}.
\end{aligned} \tag{2.5}$$

$\square$

The identities (2.2)-(2.4) now follows by setting $k = 0, 1, 2$ respectively in (2.5). In the case (2.3) the index of the summation needs to be changed by replacing $n$ by $n-1$ and then $n$ to $-n$, in the case (2.4) the index of the summation need to be changed $n$ by $n-1$.

**Theorem** 2.2 *Let $P_s(n)$ denote the number of partitions of $n$ with parts not congruent to 0 (mod 18) and each odd parts having two colours except parts congruent to $\pm s \equiv$ (mod 18). Then*

$$a_{3n} = P_7(n), \tag{2.6}$$

$$a_{3n+1} = -P_5(n) \tag{2.7}$$

*and*

$$a_{3n+2} = -P_1(n-1). \tag{2.8}$$

*Proof* From (1.1), we have

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2} z^n = \left(\frac{q^9}{z}; q^{18}\right)_\infty \left(zq^9; q^{18}\right)_\infty \left(q^{18}; q^{18}\right)_\infty . \tag{2.9}$$

Using (2.9) with $z = q^2$ in (2.2), we have

$$\sum_{n=0}^{\infty} a_{3n} q^n = \frac{(q^7, q^{11}, q^{18}; q^{18})_\infty}{(q; q^2)_\infty (q; q)_\infty}.$$

Clearly right side of the above identity is the generating function for $P_7(n)$. Thus, we have

$$\sum_{n=0}^{\infty} a_{3n} q^n = \sum_{n=0}^{\infty} P_7(n) q^n,$$

which implies (2.6). Similarly by taking $z = q^4$ and $z = q^8$ in (2.9), using them in (2.3) and (2.4), we obtain (2.7) and (2.8) respectively. □

**Example** 2.1 By using Maple we have been able to find the following series expansion for $G(q)$

$$\begin{aligned} G(q) \;\; = \;\; & 1 - q + 2q^3 - 2q^4 - q^5 + 4q^6 - 4q^7 - q^8 + 8q^9 - 8q^{10} - 2q^{11} \\ & + 14q^{12} - 14q^{13} - 4q^{14} + 24q^{15} - 23q^{16} - 6q^{17} + 40q^{18} - 38q^{19} \\ & - 10q^{20} + 63q^{21} - 60q^{22} - 16q^{23} + 98q^{24} - 92q^{25} - 24q^{26} + 150q^{27} \\ & - 140q^{128} - 36q^{29} + 224q^{30} + ... \qquad . \end{aligned}$$

The following table verifies the case $n = 4$ in the Theorem 2.2.

| $P_7(4)$=14 $=a_{12}$ | $P_5(4)$ =+14 $=-a_{13}$ | $p_1(3)=4$ $=-a_{14}$ |
|---|---|---|
| $4$=$3_r+1_r$=$3_r+1_g$=$3_g+1_g$= $3_g+1_r$=2+2=$2+1_r+1_r$= $2+1_r+1_g$=$2+1_g+1_g$= $1_r+1_r+1_r+1_r$=$1_r+1_r+1_r+1_g$= $1_r+1_r+1_g+1_g$=$1_r+1_g+1_g$ $+1_g=1_g+1_g+1_g+1_g$ | $4$=$3_r+1_r$=$3_r+1_g$=$3_g+1_g$= $3_g+1_r$=2+2=$2+1_r+1_r$= $2+1_r+1_g$=$2+1_g+1_g$= $1_r+1_r+1_r+1_r$=$1_r+1_r+1_r$ $+1_g=1_r+1_r+1_g+1_g$=$1_r+1_g$ $+1_g+1_g$=$1_g+1_g+1_g+1_g$. | $3_r$=$3_g=2+1$ =1+1+1 |

**Corollary** 2.1 *With $a_n$ given by (1.8), $a_2=0$. The remaining $a_n$ satisfy, for $n\geq 0$,*

$$a_{3n}>0,\ a_{3n+1}<0,\ a_{3n+2}\leq 0.$$

*Proof* This follows directly from Theorem 2.2. □

**Lemma** 2.2 *We have*

$$M(q)=\frac{1}{\psi(q^3)}\left[\sum_{n=-\infty}^{\infty}q^{6n^2-n}+\sum_{n=-\infty}^{\infty}q^{6n^2-5n+1}\right]. \tag{2.10}$$

*Proof* We have [6, p. 345]

$$\frac{1}{G(q)}=\frac{1}{\psi(q^3)}\left[\psi(q^{1/3})-q^{1/3}\psi(q^3)\right]$$

Upon using (1.4) in the above, we obtain

$$M(q)=\frac{1}{\psi(q^3)}\left[\sum_{n=-\infty}^{\infty}q^{\frac{2n^2-n}{3}}-\sum_{n=-\infty}^{\infty}q^{\frac{18n^2-9n+1}{3}}\right].$$

Converting the first series in the right hand side of the above into sum of three series and replacing $n$ by $-n$ in the second series, we obtain

$$M(q)=\frac{1}{\psi(q^3)}\left[\sum_{n=-\infty}^{\infty}q^{6n^2-n}+\sum_{n=-\infty}^{\infty}q^{6n^2-5n+1}\right].$$ □

**Theorem** 2.3 *Let $b_n$ be as defined in (1.7). Then*

$$\sum_{n=0}^{\infty}b_{3n}q^n=\frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\left[\sum_{n=-\infty}^{\infty}q^{18n^2+n}+q^4\sum_{n=-\infty}^{\infty}q^{18n^2+17n}\right], \tag{2.11}$$

$$\sum_{n=0}^{\infty}b_{3n+1}q^n=\frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\left[\sum_{n=-\infty}^{\infty}q^{18n^2+5n}+q^2\sum_{n=-\infty}^{\infty}q^{18n^2+13n}\right] \tag{2.12}$$

*and*

$$\sum_{n=0}^{\infty} b_{3n+2}q^n = \frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\left[\sum_{n=-\infty}^{\infty} q^{18n^2+7n} + q\sum_{n=-\infty}^{\infty} q^{18n^2+11n}\right]. \tag{2.13}$$

*Proof* Following similar steps used in the proof of Theorem 2.1, for (2.10) we have for $0 \le k \le 2$,

$$\sum_{n=0}^{\infty} b_{3n+k}q^n = \frac{1}{3}\frac{1}{\psi(q)}\sum_{j=0}^{2}\left[\sum_{n=-\infty}^{\infty} t^{(6n^2-n-k)j} q^{\frac{6n^2-n-k}{3}} + \sum_{n=-\infty}^{\infty} t^{(6n^2-5n+1-k)j} q^{\frac{6n^2-5n+1-k}{3}}\right].$$

Now $6n^2-n-k \equiv 0 \pmod 3$ for $n \equiv -k \pmod 3$. And for the second summation $6n^2-5n+1-k \equiv 0 \pmod 3$ for $2n \equiv 1-k \pmod 3$. It therefore follows from the above that

$$\begin{aligned}\sum_{n=0}^{\infty} b_{3n+k}q^n &= \frac{1}{\psi(q)}\left[\sum_{n=-\infty}^{\infty} q^{\frac{6(3n-k)^2-(3n-k)-k}{3}} + \sum_{n=-\infty}^{\infty} q^{\frac{6(6n-4k+4)^2-10(6n-4k+4)+4-4k}{12}}\right]\\ &= \frac{1}{\psi(q)}\left[\sum_{n=-\infty}^{\infty} q^{18n^2-12nk-n+2k^2} + \sum_{n=-\infty}^{\infty} q^{18n^2-24nk+19n+8k^2-13k+5}\right].\end{aligned} \tag{2.14}$$

□

The identities (2.11)- (2.13) now follow by setting $k = 0, 1, 2$ respectively in (2.14). In the case (2.11) the index of the summation needs to be changed by replacing $n$ to $-n$ in the first summation and in the second summation by $n$ to $n+1$ and then by replacing $n$ to $-n$. For the case (2.12), the index of the summation in the second summation is to be changed by replacing $n$ to $-n$. In the case (2.13) the index of the summation in the first and second series are to be changed by replacing $n$ to $n+1$.

**Theorem** 2.4 *Let $P_s(n)$ denote the number of partitions of $n$ into part such that odd parts are not congruent to $\pm s$ (mod 36) and the even part congruent to $\pm 4, \pm 8, \pm 12, \pm 16$ (mod 36). Then*

$$b_{3n} = (-1)^n\left[P_{17}(n) + P_1(n-4)\right], \tag{2.15}$$

$$b_{3n+1} = (-1)^n\left[P_{13}(n) + P_5(n-2)\right] \tag{2.16}$$

*and*

$$b_{3n+2} = (-1)^n\left[P_{11}(n) - P_7(n-1)\right]. \tag{2.17}$$

*Proof* It is easy to see that

$$\frac{(-q;-q)_\infty}{(q^2;q^2)_\infty^2} = \frac{1}{(q;q^2)_\infty(q^4;q^4)_\infty}, \tag{2.18}$$

Replacing $q$ to $q^2$ in (2.9) and then setting $z = -q$, $z = -q^{17}$, $-q^5$, $-q^{13}$, $-q^7$, $-q^{11}$ respectively, we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{18n^2+n} = (q^{17\pm}, q^{36}; q^{36})_\infty,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+17n} = (q^{1\pm}, q^{36}; q^{36})_\infty,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+5n} = (q^{13\pm}, q^{36}; q^{36})_\infty,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+13n} = (q^{5\pm}, q^{36}; q^{36})_\infty,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+7n} = (q^{11\pm}, q^{36}; q^{36})_\infty$$

and

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+11n} = (q^{7\pm}, q^{36}; q^{36})_\infty.$$

Now changing $q$ to $-q$ in (2.11), (2.12) and (2.13) and then using (2.18) and the above, we deduce that

$$\sum_{n=0}^{\infty}(-1)^n b_{3n} q^n = \frac{(q^{17\pm}, q^{36}; q^{36})_\infty + q^4(q^{1\pm}, q^{36}; q^{36})_\infty}{(q;q^2)_\infty (q^4;q^4)_\infty}, \tag{2.19}$$

$$\sum_{n=0}^{\infty}(-1)^n b_{3n+1} q^n = \frac{(q^{13\pm}, q^{36}; q^{36})_\infty + q^2(q^{5\pm}, q^{36}; q^{36})_\infty}{(q;q^2)_\infty (q^4;q^4)_\infty} \tag{2.20}$$

and

$$\sum_{n=0}^{\infty}(-1)^n b_{3n+2} q^n = \frac{(q^{11\pm}, q^{36}; q^{36})_\infty - q(q^{7\pm}, q^{36}; q^{36})_\infty}{(q;q^2)_\infty (q^4;q^4)_\infty}. \tag{2.21}$$

Now (2.15), (2.16) and (2.17) follow from (2.19), (2.20) and (2.21) respectively. □

**Example** 2.2 By using Maple we have been able to find the following series expansion for $M(q)$:

$$\begin{aligned} M(q) \;=\; & 1 + q + q^2 - q^3 - q^4 + q^6 + 2q^7 - 2q^9 - 3q^{10} - q^{11} + 4q^{12} + 4q^{13} + q^{14} \\ & -4q^{15} - 6q^{16} - q^{17} + 5q^{18} + 8q^{19} + q^{20} - 8q^{21} - 10q^{22} - 2q^{23} + 11q^{24} \\ & +14q^{25} + 4q^{26} - 14q^{27} - 19q^{28} - 4q^{29} + 17q^{30} \cdots . \end{aligned}$$

The following table verifies the case $n = 5$ in Theorem 2.4:

| $(-1)^5[P_{17}(5) + P_1(1)]$ =−4=$a_{15}$ | | $(-1)^5[P_{13}(5) + P_5(3)]$ = −6 = $a_{16}$ | | $(-1)^5[P_{11}(5) - P_7(4)]$ = −1 = $a_{17}$ | |
|---|---|---|---|---|---|
| $P_{17}(5)$ | $P_1(1)$ | $P_{13}(5)$ | $P_5(3)$ | $P_{11}(5)$ | $P_7(4)$ |
| 5 | | 5 | 3 | 5 | 4 |
| =4+1 | | =4+1 | =1+1+1 | =4+1 | =3+1 |
| =3+1+1 | | =3+1+1 | | =3+1+1 | =1+1+1+1 |
| =1+1+1+1+1 | | 1+1+1+1+1 | | =1+1+1+1+1 | |

**Corollary** 2.2([7]) *With $b_n$ given by (1.7), $b_5 = 0$ and $b_8 = 0$. The remaining $b_n$ satisfy for $n \geq 0$,*

$$b_{6n} > 0, \qquad b_{6n+1} > 0, \qquad b_{6n+2} > 0$$
$$b_{6n+3} < 0, \qquad b_{6n+4} < 0, \qquad b_{6n+5} < 0.$$

*Proof* The result clearly follows from Theorem 2.4. □

## §3. Further Identities of $G(q)$ and $M(q)$

Following the notation in [17], we define

$$L(q) : = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{f(-q,-q^5)}{\psi(-q)} \tag{3.1}$$

and

$$N(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{f(-q^3,-q^3)}{\psi(-q)}. \tag{3.2}$$

The two identities on the right of (3.1) and (3.2) are the cubic identities due to G. E. Andrews [2] and L. J. Slater [16] respectively. Andrews [2] shown that

$$q^{1/3}G(q) = q^{1/3}\frac{L(q)}{N(q)}. \tag{3.3}$$

We note that

$$L(q) = \frac{f_6^2}{f_3 f_4}, \qquad N(q) = \frac{f_3^2 f_2}{f_1 f_4 f_6}$$

and

$$f(q) = \frac{f_2^3}{f_1 f_4} \tag{3.4}$$

where $f_n := (q^n;q^n)_\infty$.

**Lemma** 3.1([17]) *We have*

$$L(-q)N(q) - L(q)N(-q) = 2q\frac{f_2 f_{12}^4}{f_4^3 f_6^2} \tag{3.5}$$

*and*

$$L(-q)N(q) + L(q)N(-q) = 2\frac{f_4}{f_2}. \tag{3.6}$$

**Theorem** 3.1 *We have*

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{(q^2,q^4,q^6;q^6)_\infty^2}{(q^3,q^3,q^6;q^6)_\infty^2} = \frac{\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-n}\right]^2}{\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}\right]^2}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} a_{2n+1}q^n = \frac{(q,q^5,q^6;q^6)_\infty^2}{(q^3,q^3,q^6;q^6)_\infty^2} = \frac{\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}\right]^2}{\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}\right]^2}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} b_{2n}q^n = \frac{(q^2,q^4,q^6;q^6)_\infty^2}{(q,q^3,q^3,q^5,q^6,q^6;q^6)_\infty} = \frac{\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-n}\right]^2}{\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}\right]\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}\right]} \tag{3.9}$$

*and*

$$\sum_{n=0}^{\infty} b_{2n+1}q^n = \frac{(q,q^5,q^6;q^6)_\infty}{(q^3,q^3,q^6;q^6)_\infty} = \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}}. \tag{3.10}$$

*Proof* We have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n}q^{2n} &= \frac{1}{2}\left[G(q)+G(-q)\right] = \frac{1}{2}\left[\frac{L(q)}{N(q)}+\frac{L(-q)}{M(-q)}\right]\\ &= \frac{1}{2}\frac{L(q)M(-q)+L(-q)N(q)}{N(q)M(-q)},\end{aligned}$$

which on employing (3.6) and (3.4) yields

$$\sum_{n=0}^{\infty} a_{2n}q^{2n} = \frac{(q^4,q^8,q^{12};q^{12})_\infty^2}{(q^6,q^6,q^{12};q^{12})_\infty^2}.$$

Changing $q$ to $q^{1/2}$ in the above, we obtain the first equality of (3.7). The second equality follows by appealing to (1.1). Similarly, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n+1}q^{2n} &= \frac{1}{2q}\left[G(q)-G(-q)\right] = \frac{1}{2q}\left[\frac{L(q)}{N(q)}-\frac{L(-q)}{M(-q)}\right]\\ &= \frac{1}{2q}\frac{L(q)M(-q)-L(-q)N(q)}{N(q)M(-q)}.\end{aligned}$$

Now employing (3.5), (3.4) and (1.1) gives

$$\sum_{n=0}^{\infty} a_{2n+1}q^{2n} = -\frac{(q^2,q^{10},q^{12};q^{12})_\infty^2}{(q^6,q^6,q^{12};q^{12})_\infty^2}.$$

The first equality in result (3.8) now follows by changing $q$ to $q^{1/2}$ in the above and the second equality follows by employing (1.1). By similar arguments, one can derive (3.9) and (3.10). □

**Theorem** 3.2 *For* $|q| < 1$

$$\frac{\sum_{n=0}^{\infty} a_{2n} q^n}{\sum_{n=0}^{\infty} a_{2n+1} q^n} = -\frac{\sum_{n=0}^{\infty} b_{2n} q^n}{\sum_{n=0}^{\infty} b_{2n+1} q^n}.$$

*Proof* Follows from Theorem 3.1. □

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