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On The Isoperimetric Number of Line Graphs

Ersin Aslan

(Turgutlu Vocational Training School , Celal Bayar University, Turgutlu-Manisa, Turkey)

Alpay Kirlangic

(Department of Mathematics, Ege University, 35100 Bornova-Izmir, Turkey)

E-mail: ersin.aslan@bayar.edu.tr, alpay.kirlangic@ege.edu.tr

Abstract: The *isoperimetric number* of a graph G, denoted $i(G)$, was introduced in 1987 by Mohar [8]. Given a graph G and a subset of X of its vertices, let $\partial(X)$ denote the edge boundary of X: i.e. the set of edges which connect vertices in X with vertices not in X. The isoperimetric number of G defined as $i(G) = min_{1\leq|X|\leq\frac{|V(G)|}{2}} \frac{|\partial(X)|}{|X|}$. This paper obtains some results about the isoperimetric number of graphs obtained from graph operations are given.

Key Words: Isoperimetric number, line graph, graph operations.

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§1. Introduction

The *isoperimetric number* of a graph G, denoted $i(G)$, was introduced in 1987 by Mohar [8]. Given a graph G and a subset of X of its vertices , let $\partial(X)$ denote the edge boundary of X, i.e. the set of edges which connect vertices in X with vertices not in X. The isoperimetric number of G defined as

$$i(G) = min_{1\leq|X|\leq\frac{|V(G)|}{2}} \frac{|\partial(X)|}{|X|}. \tag{1.1}$$

Clearly, $i(G)$ can be defined in a more symmetric form as

$$i(G) = min\frac{|E(X,Y)|}{min\{|X|,|Y|\}}, \tag{1.2}$$

where the minimum runs over all partitions of $V(G) = X \cup Y$ into non empty subsets X and Y , and $E(X,Y) = \partial X = \partial Y$ are the edges between X and Y.

The importance of *i(G)* lies in various interesting interpretations of this number [8]:

(1) From (1.2) it is evident that, in trying to determine *i(G)*, we have to find a small edge-cut *E(X,Y)* separating as large a subset X (assume $|X| \leq |Y|$) as possible from the remaining larger part Y. So, it is evident that *i(G)* can serve as measure of *connectivity* of graphs. It seems that there might be possible applications in problems concerning connected networks and

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the ways to "destroy" them by removing a large portion of the network by cutting only a few edges.

(2) The problem of the partitioning *V(G)* into two equally sized subsets (to within one element) in such a way that the number of the edges in the cut is minimal, is known as the *bisection width* problem. It is important in VLSI design and some other practical applications. Clearly, it is related to isoperimetric number.

In Section 2 known results on the isoperimetric number and some definitions are given. Section 3 gives some results about the isoperimetric number of graphs obtained from graph operations.

§2. Basic Results

In this section we will review some of the known results.

Theorem 2.1([8]) *Let m, n be the positive integers. The isoperimetric number of some common graphs are as follows,*

(1) *The complete graph K_n has $i(K_n) = \lceil \frac{n}{2} \rceil$;*

(2) *The cycle C_n has $i(C_n) = \frac{2}{\lfloor \frac{n}{2} \rfloor}$;*

(3) *The path P_n on n vertices has $i(P_n) = \frac{1}{\lfloor \frac{n}{2} \rfloor}$;*

(4) *The isoperimetric numbers of complete bipartite graphs $K_{m,n}$ are respectively*

$$i(K_{m,n}) = \begin{cases} \dfrac{mn}{m+n} & if\ m\ and\ n\ are\ even, \\ \dfrac{mn+1}{m+n} & if\ m\ and\ n\ are\ odd, \\ \dfrac{mn}{m+n-1} & if\ m+n\ is\ odd, \end{cases}$$

it can be shortened to $i(K_{m,n}) = \lceil mn/2 \rceil / \lfloor (m+n)/2 \rfloor$.

Theorem 2.2([8]) *Some of the theorems that Mohar state are below,*

(1) *$i(G) = 0$ if and only if G is disconnected;*

(2) *If G is k-edge-connected then $i(G) \geq 2k/|V(G)|$;*

(3) *If δ is the minimal degree of vertices in G then $i(G) \leq \delta$;*

(4) *If $e = uv$ is an edge of G and $|V(G)| \geq 4$ then*

$$i(G) \leq \frac{deg(u) + deg(v) - 2}{2};$$

(5) *If Δ is the maximum vertex degree in G then $i(G) \leq (\Delta - 2) + 2/\lfloor |V(G)|/2 \rfloor$. If G has cycle with a most half the vertices of G then $i(G) \leq \Delta - 2$.*

Now we will give some definitions.

Definition 1.1([7]) *The line graph of G denoted $L(G)$, is the intersection graph $\Omega(x)$. Thus the points of $L(G)$ are the lines of G with two points of $L(G)$ adjacent whenever the corresponding lines of G are. If $x = uv$ is a line of G, then the degree of x in $L(G)$ is clearly $deg(u)+deg(v)-2$.*

Definition 1.2([7]) *A subset S of $V(G)$ such that every edge of G has at least one end in S called a covering set of G. The number of vertices in a minimum covering set of G is the covering number of G is denoted by $\alpha(G)$.*

§3. Isoperimetric Number of Line Graphs

Firstly, we can say the following observation for the isoperimetric number of line graphs of graphs P_n and C_n.

- P_n – Let P_n be a path graph with n vertices. Then $i(L(P_n)) = i(P_{n-1})$.
- C_n – Let C_n be a cycle graph with n vertices. Then $i(L(C_n)) = i(C_n)$.
- If G is a graph with $\alpha(G) = 1$, then $i(L(G)) = \lceil \frac{n}{2} \rceil$.

Next we consider some of the operations on graphs. We start with the complement of a graph and give the definition its.

Definition 3.1([7]) *The complement of $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ defined by the edge $e \in E(\overline{G})$ if only if $e \notin E(G)$, where $e = uv, u, v, \in V(G)$.*

Theorem 3.1 *Let $\overline{L(P_n)}$ be a complement of line graph of path graph with $n-1$ vertices then*

$$i(\overline{L(P_n)}) = \begin{cases} \frac{n}{2} - 2, & if\ n\ is\ even, \\ \frac{(\frac{n-1}{2} - 1)^2}{\frac{n-1}{2}}, & if\ n\ is\ odd, \end{cases}$$

Proof First, we prove this result for odd n. Let $X \subseteq V(\overline{L(P_n)})$ where $|X| \leq \frac{n-1}{2}$ and let $V(\overline{L(P_n)}) = 1, 2, \ldots, n-1$. Assume that $X \subseteq V(\overline{L(P_n)})$ and $V(\overline{L(P_n)}) = V(G_1) \cup V(G_2)$ where $V(G_1) = 1, 3, \ldots, n-2$ and $V(G_2) = 2, 4, \ldots, n-1$. $(\overline{L(P_n)})$ contains two complete graphs $K_{\frac{n-1}{2}}$ formed by the vertices of $V(G_1)$ and $V(G_2)$ respectively.

Case 1 Assume that $|X| \subseteq V(G_1)$ and $|X \cap V(G_2)| = 0$ or $|X| \subseteq V(G_2)$ and $|X \cap V(G_1)| = 0$ where $|X| < \frac{n-1}{2}$. If $|X| = r$ then $|\partial(X)| \geq r(\frac{n-1}{2} - 2)$. Therefore

$$\frac{|\partial(X)|}{|X|} \geq \frac{r(\frac{n-1}{2} - 2)}{r} = (\frac{n-1}{2} - 2).$$

Case 2 Suppose that $|X| \subseteq V(G_1)$, $|X \cap V(G_2)| = 0$ and $|X| = \frac{n-1}{2}$ or $|X| \subseteq V(G_2)$, $|X \cap V(G_1)| = 0$ and $|X| = \frac{n-1}{2}$. Since $|X| = r$ then $|\partial(X)| = (\frac{n-1}{2} - 2)(\frac{n-1}{2} - 1) +$

$(\frac{n-1}{2}-1)=(\frac{n-1}{2}-1)^2$. Therefore

$$\frac{|\partial(X)|}{|X|}=\frac{(\frac{n-1}{2}-1)^2}{\frac{n-1}{2}}.$$

Case 3 Let $X\subseteq V(\overline{L(P_n)})$ where $|X\cap V(G_1)|=a$ and $|X\cap V(G_2)|=b$ for $a\neq 0$ and $b\neq 0$. We have two cases according to $(a+b)$.

Subcase 1 Let $a+b<\frac{n-1}{2}$. In this case a vertices are connected to G_2 with $(\frac{n-1}{2}-2-b)$ edges and connected $(|V(G_1)|-a)$ with $(\frac{n-1}{2}-a)$ edges. Similarly b vertices are connected to G_1 with $(\frac{n-1}{2}-2-a)$ edges and connected $(|V(G_2)|-b)$ with $(\frac{n-1}{2}-a)$ edges. Hence

$$\begin{aligned}|\partial(X)| &\geq a(\frac{n-1}{2}-2-b)+b(\frac{n-1}{2}-2-a)+a(\frac{n-1}{2}-a)+b(\frac{n-1}{2}-a)\\ &= (a+b)(n-1-a-b-2).\end{aligned}$$

Therefore

$$\frac{|\partial(X)|}{|X|}\geq\frac{(a+b)(n-1-a-b-2)}{a+b}\geq n-1-(a+b)-2\geq\frac{n-1}{2}-1.$$

Subcase 2 If $a+b=\frac{n-1}{2}$ then

$$\begin{aligned}|\partial(X)| &\geq a(\frac{n-1}{2}-2-b)+b(\frac{n-1}{2}-2-a)+a(\frac{n-1}{2}-a)+b(\frac{n-1}{2}-a)+1+1\\ &= (a+b)(n-1-a-b-2)+2.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{|\partial(X)|}{|X|} &\geq \frac{(a+b)(n-1-a-b-2)+2}{a+b}\\ &\geq n-1-(a+b)-\frac{2}{a+b}\\ &\geq \frac{n-1}{2}-2+\frac{2}{n-1}.\end{aligned}$$

Combining Cases $1-3$, the proof is completed for odd n.

For the case of n being even, the proof is very similar to that of odd n. □

Theorem 3.2 *Let $\overline{L(C_n)}$ be a complement of line graph of cycle graph with n vertices then*

$$i(\overline{L(C_n)})=\begin{cases}\frac{n}{2}-2, & if\ n\ is\ even,\\ \frac{n+1}{2}-2, & if\ n\ is\ odd.\end{cases}$$

Proof The proof is similar to that of Theorem 3.1. □

We consider now the isoperimetric number of the join of two graphs and give the definition of join operation.

Definition 3.2([7]) *Let G_1 and G_2 be two graphs. The union $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The join is denoted $V(G_1) + V(G_2)$ and consists of $V(G_1) \cup V(G_2)$ and all edges joining $V(G_1)$ with $V(G_2)$.*

Let us first consider the join of the graph K_1 with cycle C_n.

Theorem 3.3 *Let $K_1 + C_n$ be a graph with $n+1$ vertices then*

$$i(L(K_1 + C_n)) = 2.$$

Proof The graph $K_1 + C_n$ consists of edges of cycle graph C_n and edges of star graph $K_{1,n}$. Let $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ and $E(K_{1,n}) = \{h_1, h_2, \ldots, h_n\}$. Assume that $V(K_1 + C_n) = V(G_1) \cup V(G_2)$ such that $V(G_1) = \{e_1, e_2, \ldots, e_n\}$ and $V(G_2) = \{h_1, h_2, \ldots, h_n\}$. Let $X \subseteq V(L(K_1 + C_n))$ with $|X| \leq n$ and $|X \cap V(G_1)| = a, |X \cap V(G_2)| = b$. Therefore there are have five cases according to a and b.

Case 1 If $a = 0$, $b \leq n$ and $|X| = b$ then $|\partial(X)| = b.(n-b) + 2b$. Hence

$$\frac{|\partial(X)|}{|X|} = \frac{b(n-b)+2b}{b}.$$

The function $n - b + 2$ takes its minimum value at $b = n$ and $i(L(K_1 + C_n)) = 2$.

Case 2 If $a = b$ and $|X| = a + b$ then $|\partial(X)| = 2 + b.(n-b) + 2$. Thus

$$\frac{|\partial(X)|}{|X|} = \frac{2 + b(n-b) + 2}{a+b} = \frac{2 + b(n-b) + 2}{2b}.$$

The function $\frac{2 + b(n-b) + 2}{2b}$ takes its minimum value at $a = b = \lfloor \frac{n}{2} \rfloor$ and we have

$$i(L(K_1 + C_n)) = \frac{2 + \lfloor \frac{n}{2} \rfloor (n - \lfloor \frac{n}{2} \rfloor) + 2}{2 \lfloor \frac{n}{2} \rfloor}.$$

Case 3 If $a \neq b$, $0 < a < n$, $0 \leq b < n$, $a < b$ and $|X| = a + b$, then $|\partial(X)| = 2 + b(n-b) + (b-a) \times 2$. Therefore we get

$$\begin{aligned} \frac{|\partial(X)|}{|X|} &= \frac{2 + b(n-b) + 2(b-a)}{a+b} \geq \frac{2 + b(n-b) + 2}{2b} \\ &\geq \frac{4 + ba}{2b} \geq 2. \end{aligned}$$

Case 4 If $a \neq b$, $0 < a < n$, $0 \leq b < n$, $a > b$ and $|X| = a + b$, then $|\partial(X)| = 2 + b(n-b) + 2(a-b)$. Thus

$$\begin{aligned} \frac{|\partial(X)|}{|X|} &= \frac{2 + b(n-b) + 2(a-b)}{a+b} \geq \frac{2 + b(n-b) + 2}{2b} \\ &\geq \frac{4 + ba}{2b} \geq 2 \end{aligned}$$

Case 5 If $a = n$, $b = 0$ and $|X| = n$ then $|\partial(X)| = 2n$. Hence $\frac{|\partial(X)|}{|X|} = 2$.

Combining Cases $1-4$, the proof is completed. □

Theorem 3.4 *Let K_1+P_n be a graph with $n+1$ vertices then*

$$i(L(K_1+P_n))=2.$$

Proof The proof is similar to that of Theorem 3.3. □

Finally we consider the cartesian product of two graphs.

Definition 3.3([7]) *The Cartesian product $G_1\times G_2$ of graphs G_1 and G_2 has $V(G_1)\times V(G_2)$ as its vertex set and (u_1,u_2) is adjacent to (v_1,v_2) if either $u_1=v_1$ and u_2 is adjacent to v_2 or $u_2=v_2$ and u_1 is adjacent to v_1.*

Theorem 3.5 following is given by Mohar in [8].

Theorem 3.5([8]) *If G is a graph having an even number number of vertices for every $n\geq 1$,*

$$i(K_{2n}\times G)=min\{i(G),n\}.$$

By applying Theorem 3.5, we can easily get the following observation.

• Let $K_2\times P_n$ be the cartesian product of K_2 and P_n be a graph with $2n$ vertices. Then $i(K_2\times P_n)=i(P_n)$.

• Let $K_2\times C_n$ be the cartesian product of K_2 and C_n be a graph with $2n$ vertices. Then $i(K_2\times C_n)=i(C_n)$.

The following theorems give the isoperimetric number of graphs $L(K_2\times P_n)$ and $L(K_2\times C_n)$.

Theorem 3.6 *Let $K_2\times P_n$ be the cartesian product of K_2 and P_n be a graph with $2n$ vertices. Then*

$$i(L(K_2\times P_n))=\begin{cases}\dfrac{8}{3n-2} & if\ n\ is\ even,\\ \dfrac{8}{3n-3} & if\ n\ is\ odd.\end{cases}$$

Proof Let P_1 and P_2 be two path graphs contained in $K_2\times P_n$. Assume that P_1 has a vertex labelling through v_1-v_n such that $V(P_1)=\{v_i|v_i$ is adjacent to v_{i+1} for $1\leq i\leq n\}$. Similarly P_2 has a labelling through u_1-u_n such that $V(P_2)=\{u_j|u_j$ is adjacent to v_{j+1} for $1\leq i\leq n\}$.

$L(K_2\times P_n)$ has $3n-2$ vertices and let $V(L(K_2\times P_n))=V(G_1)\cup V(G_2)\cup V(G_3)$. Suppose that the edges along the path v_1-v_n which form the vertices of G_1 have a labelling such that $V(G_1)=\{e_i|e_i$ is adjacent to e_{i+1} for $1\leq i\leq n-1\}$. Similarly assume that the edges along the path u_1-u_n which form the vertices of G_1 have a labelling such that $V(G_2)=\{m_i|m_i$ is adjacent to m_{i+1} for $1\leq i\leq n-1\}$. In addition suppose the vertices of G_3 have a labelling such that $V(G_3)=\{k_i|i=j$ for $1\leq i\leq n$ and $1\leq j\leq n$ where v_i and u_j are the vertices of P_1 and P_2 respectively $\}$.

There are three cases according to the cardinality of X where $X \subset V(L(K_2 \times P_n))$ and $|X| \leq \lfloor \frac{3n-2}{2} \rfloor$.

Case 1 If $|X| = 1$ then $\delta = 2$ and $i(L(K_2 \times P_n)) = 2$.

Case 2 If $|X| = 2$ then we have $|\partial(X)| \geq 3$. Then

$$\frac{|\partial(X)|}{|X|} \geq \frac{3}{2}.$$

Case 3 Let $|X| > 2$. The discussion is divided into subcases following.

Subcase 1 Let n be even and $|X| = r$. If $2 < r \leq \frac{3n-2}{2}$, then we have $|\partial(X)| \geq 4$. Hence we have $\frac{|\partial(X)|}{|X|} \geq \frac{4}{r}$. The function $\frac{4}{r}$ takes its minimum value at $r = \frac{3n-2}{2}$ and we get

$$\frac{|\partial(X)|}{|X|} \geq \frac{4}{\frac{3n-2}{2}}.$$

It can be easily seen that there exists a set X such that $X = \{e_1, e_2, \ldots, e_{\frac{n}{2}}, k_1, k_2, \ldots, k_{\frac{n}{2}}, m_1, m_2, \ldots, m_{\frac{n}{2}-1}\}$ and we have $|\partial(X)| = 4$. Therefore, we get

$$i(L(K_2 \times P_n)) = \frac{4}{\frac{3n-2}{2}}.$$

Whence, $i(L(K_2 \times P_n)) = \frac{8}{3n-2}$ for n even.

Subcase 2 Let n be odd and $|X| = r$. If $2 < r \leq \frac{3n-3}{2}$, then we have $|\partial(X)| \geq 4$. Hence we have $\frac{|\partial(X)|}{|X|} \geq \frac{4}{r}$. The function $\frac{4}{r}$ takes its minimum value at $r = \frac{3n-3}{2}$ and we get that

$$\frac{|\partial(X)|}{|X|} \geq \frac{4}{\frac{3n-3}{2}}.$$

It can be easily seen that there exists a set X such that $X = \{e_1, e_2, \ldots, e_{\frac{n-1}{2}}, k_1, k_2, \ldots, k_{\frac{n-1}{2}}, m_1, m_2, \ldots, m_{\frac{n-1}{2}}\}$ and we have $|\partial(X)| = 4$. Therefore we get $i(L(K_2 \times P_n)) = \frac{4}{\frac{3n-3}{2}}$. Therefore, $i(L(K_2 \times P_n)) = \frac{8}{3n-3}$ for n odd. □

Theorem 3.7 *Let $K_2 \times C_n$ be the cartesian product of K_2 and P_n be a graph with $2n$ vertices. Then*

$$i(L(K_2 \times C_n)) = \frac{8}{\lceil \frac{3n}{2} \rceil}.$$

Proof The proof is similar to that of Theorem 3.6. □

§4. Conclusion

In this paper isoperimetric number of line graphs are studied. Some results for the isoperimetric number of graphs obtained by graph operations such as complement, join operation and cartesian product are obtained. To make further progress in this direction, one could try to characterize the graphs with given isoperimetric number.

References

[1] Aytac V., Vulnerabiltiy in graphs: the neighbour-integrity of line graphs, *International Journal of Computer Mathematics*, Vol. 82, No.1, January 2005, 35-40.

[2] Azizoglu M.C., Egecioglu O., The edge-isoperimetric number of generalized cylinders, *Technical Report TRCS99-13*, Department of Computer Science, University of California, Santa Barbara, April 1999.

[3] Azizoglu M.C., Egecioglu O., The isoperimetric number and the bisection width of generalized cylinders, *Electronic Notes in Discrete Mathematics*, Vol. 11, July 2002.

[4] Bollabás B. and Leader I., An isoperimetric inequlaties in the grid, *Combinatorica*, 11 (4), 1991, 299-314.

[5] Chartrand G. and Lesniak I, *Graphs and Digraphs*, Monterey, CA: Wadsworth and Brooks, 1986,

[6] Gross J., Yellen J., *Graph Theory and Its Application*, CRC Press, 1999.

[7] Harary F., *Graph Theory*, Reading, MA: Addison-Wesley, 1972.

[8] Mohar B., Isoperimetric number of graphs, *J. Comb. Theory*(B), 47 (1989), 274-291.

[9] Sivasubramanian S., Approximating the isoperimetric number of strongly regular graphs, *Electronic Journal of Linear Algebra*, Volume 13, pp. 111-121, April 2005