International J.Math. Combin. Vol.4(2011), 70-75

New Version of Spacelike Horizontal Biharmonic Curves with Timelike Binormal According to Flat Metric in Lorentzian Heisenbegr Group Heis^3

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Abstract: In this paper, we study spacelike biharmonic curves with timelike binormal according to flat metric in the Lorentzian Heisenberg group Heis^3. We determine the parametric representation of the spacelike horizontal biharmonic curves with timelike binormal according to flat metric.

Key Words: Biharmonic curve, Heisenberg group, Flat metric.

AMS(2010): 31B30, 58E20

§1. Introduction

Let (N,h) and (M,g) be Riemannian manifolds. A smooth map $\phi : N \longrightarrow M$ is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left|\mathcal{T}(\phi)\right|^2 dv_h,$$

where the section $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr}R\left(\mathcal{T}(\phi), d\phi\right) d\phi, \tag{1.1}$$

and called the *bitension field* of ϕ. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study spacelike biharmonic curves with timelike binormal according to flat metric in the Lorentzian Heisenberg group Heis^3. We determine the parametric representation of the spacelike horizontal biharmonic curves with timelike binormal according to flat metric.

§2. The Lorentzian Heisenberg Group Heis^3

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x,y,z) * (\overline{x},\overline{y},\overline{z}) = (x+\overline{x}, y+\overline{y}, z+\overline{z} - \overline{x}y + x\overline{y}).$$

[1]Received March 30, 2011. Accepted November 26, 2011.

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = dx^2 + (xdy + dz)^2 - ((1-x)\,dy - dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \frac{\partial}{\partial y} + (1-x)\,\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right\}. \tag{2.1}$$

The characteristic properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = 0,\quad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,\quad [\mathbf{e}_2, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1,\quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

Proposition 2.1 . *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \end{pmatrix}, \tag{2.3}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

So we obtain that

$$R(\mathbf{e}_1, \mathbf{e}_3) = R(\mathbf{e}_1, \mathbf{e}_2) = R(\mathbf{e}_2, \mathbf{e}_3) = 0. \tag{2.4}$$

Then, the Lorentz metric g is flat.

§3. Spacelike Horizontal Biharmonic Curves with Timelike Binormal According to Flat Metric in the Lorentzian Heisenberg Group Heis^3

An arbitrary curve $\gamma : I \longrightarrow Heis^3$ is spacelike, timelike or null, if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Let $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike curve with timelike binormal and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa_1\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa_1\mathbf{t} + \kappa_2\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= \kappa_2\mathbf{n}, \end{aligned} \tag{3.1}$$

where κ_1, κ_2 are curvature function and torsion function, respectively and

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) &= 1,\ g(\mathbf{n},\mathbf{n}) = 1,\ g(\mathbf{b},\mathbf{b}) = -1, \\ g(\mathbf{t},\mathbf{n}) &= g(\mathbf{t},\mathbf{b}) = g(\mathbf{n},\mathbf{b}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3, \\ \mathbf{n} &= n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3, \\ \mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3. \end{aligned}$$

Theorem 3.1 *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric, then*

$$\begin{aligned} &\kappa_1 = \text{constant} \neq 0, \\ &\kappa_1^2 - \kappa_2^2 = 0, \\ &\kappa_2 = \text{constant}. \end{aligned} \tag{3.2}$$

Lemma 3.2 *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal, then γ is a helix.*

Theorem 3.3 *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{\cosh^2\varphi}{\kappa_1}\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + C_1, \\ y(s) &= -\frac{\cosh^2\varphi}{\kappa_1}\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + s\sinh\varphi + C_2, \\ z(s) &= -\frac{\cosh^3\varphi}{\kappa_1}\left(\frac{s}{2} - \frac{\cosh\varphi}{\kappa_1}\sin 2\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right]\right) \\ &\quad -\frac{1}{\kappa_1}\left(\cosh^2\varphi - \frac{\sinh\varphi\cosh^3\varphi}{\kappa_1}\right)\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + C_3, \end{aligned} \tag{3.3}$$

where C_1, C_2, C_3 are constants of integration.

Proof Assume that γ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric in the Lorentzian Heisenberg group Heis^3. Using Lemma 3.2 without loss of generality, we take the axis of γ is parallel to the spacelike vector $\mathbf{e}_3$. Then,

$$g(\mathbf{t}, \mathbf{e}_3) = t_3 = \sinh\varphi, \tag{3.4}$$

where φ is constant angle.

Direct computations show that

$$\mathbf{t} = \cosh\varphi\cos\Bbbk\mathbf{e}_1 + \cosh\varphi\sin\Bbbk\mathbf{e}_2 + \sinh\varphi\mathbf{e}_3. \tag{3.5}$$

Using above equation and Frenet equations, we obtain

$$\mathbb{k} = \frac{\kappa_1 s}{\cosh\varphi} + \aleph, \tag{3.6}$$

where $\aleph$ is a constant of integration.

From these we get the following formula

$$\mathbf{t} = \cosh\varphi\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right]\mathbf{e}_1 + \cosh\varphi\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right]\mathbf{e}_2 + \sinh\varphi\mathbf{e}_3. \tag{3.7}$$

Therefore, Equation (3.9) becomes

$$\begin{aligned}\mathbf{t} = (&\cosh\varphi\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right], \cosh\varphi\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + \sinh\varphi, \\ &(1-x)\cosh\varphi\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] - x\sinh\varphi).\end{aligned} \tag{3.8}$$

Now using Equation (3.10) we obtain

$$\begin{aligned}\frac{dx}{ds} &= \cosh\varphi\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right], \\ \frac{dy}{ds} &= \cosh\varphi\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + \sinh\varphi, \\ \frac{dz}{ds} &= -\frac{\cosh^3\varphi}{\kappa_1}\sin^2\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] \\ &\quad +(\cosh\varphi - \frac{\sinh\varphi\cosh^2\varphi}{\kappa_1})\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right].\end{aligned} \tag{3.9}$$

With direct computations on above system we have Equation (3.3). The proof is completed.□

Using Mathematica in above Theorem, we have following figure.

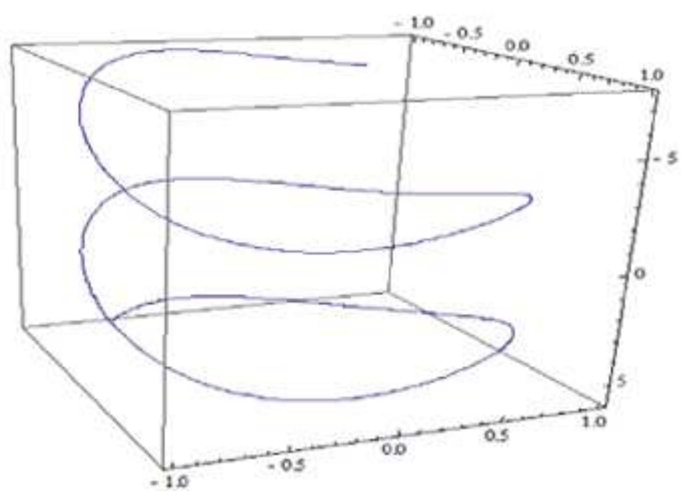

Fig.1

Theorem 3.4 *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike horizontal biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{1}{\kappa_1}\sin[\kappa_1 s+\aleph]+C_1, \\ y(s) &= -\frac{1}{\kappa_1}\cos[\kappa_1 s+\aleph]+C_2, \\ z(s) &= -\frac{1}{\kappa_1}(\frac{s}{2}-\frac{1}{\kappa_1}\sin 2[\kappa_1 s+\aleph])-\frac{1}{\kappa_1}\cos[\kappa_1 s+\aleph]+C_3, \end{aligned}$$

where C_1, C_2, C_3 are constants of integration.

Corollary 3.5 *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then*

$$\kappa_1 = \mp\kappa_2. \tag{3.10}$$

Theorem 3.6 *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike horizontal biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of γ in terms of torsion are*

$$\begin{aligned} x(s) &= \mp\frac{1}{\kappa_2}\sin[\mp\kappa_2 s+\aleph]+C_1, \\ y(s) &= \mp\frac{1}{\kappa_2}\cos[\mp\kappa_2 s+\aleph]+C_2, \\ z(s) &= \mp\frac{1}{\kappa_2}(\frac{s}{2}\mp\frac{1}{\kappa_2}\sin 2[\mp\kappa_2 s+\aleph])\mp\frac{1}{\kappa_2}\cos[\mp\kappa_2 s+\aleph]+C_3, \end{aligned} \tag{3.11}$$

where C_1, C_2, C_3 are constants of integration.

Proof Using Equation (3.10) in Equation (3.3), we obtain Equation (3.11). Thus, the proof is completed. □

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