

Neutrosophic Rings I

Agboola A.A.A., Akinola A.D. and Oyebola O.Y.

(Department of Mathematics, College of Natural Sciences, University of Agriculture, Abeokuta, Nigeria)

E-mail: aaaola2003@yahoo.com, abolaopeyemi@yahoo.co.uk, theoyesquare@yahoo.ca

Abstract: In this paper, we present some elementary properties of neutrosophic rings. The structure of neutrosophic polynomial rings is also presented. We provide answers to the questions raised by Vasantha Kandasamy and Florentin Smarandache in [1] concerning principal ideals, prime ideals, factorization and Unique Factorization Domain in neutrosophic polynomial rings.

Key Words: Neutrosophy, neutrosophic, neutrosophic logic, fuzzy logic, neutrosophic ring, neutrosophic polynomial ring, neutrosophic ideal, pseudo neutrosophic ideal, neutrosophic R-module.

AMS(2010): 03B60, 12E05, 97H40

§1. Introduction

Neutrosophy is a branch of philosophy introduced by Florentin Smarandache in 1980. It is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set and neutrosophic statistics. While neutrosophic set generalizes the fuzzy set, neutrosophic probability generalizes the classical and imprecise probability, neutrosophic statistics generalizes classical and imprecise statistics, neutrosophic logic however generalizes fuzzy logic, intuitionistic logic, Boolean logic, multi-valued logic, paraconsistent logic and dialetheism. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. The use of neutrosophic theory becomes inevitable when a situation involving indeterminacy is to be modeled since fuzzy set theory is limited to modeling a situation involving uncertainty.

The introduction of neutrosophic theory has led to the establishment of the concept of neutrosophic algebraic structures. Vasantha Kandasamy and Florentin Smarandache for the first time introduced the concept of neutrosophic algebraic structures in [2] which has caused a paradigm shift in the study of algebraic structures. Some of the neutrosophic algebraic structures introduced and studied in [2] include neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and so on. The study of neutrosophic rings was

¹Received June 9, 2011. Accepted November 1, 2011.

introduced for the first time by Vasantha Kandasamy and Florentin Smarandache in [1]. Some of the neutrosophic rings studied in [1] include neutrosophic polynomial rings, neutrosophic matrix rings, neutrosophic direct product rings, neutrosophic integral domains, neutrosophic unique factorization domains, neutrosophic division rings, neutrosophic integral quaternions, neutrosophic rings of real quaternions, neutrosophic group rings and neutrosophic semigroup rings.

In Section 2 of this paper, we present elementary properties of neutrosophic rings. Section 3 is devoted to the study of structure of neutrosophic polynomial rings and we present algebraic operations on neutrosophic polynomials. In section 4, we present factorization in neutrosophic polynomial rings. We show that Division Algorithm is generally not true for neutrosophic polynomial rings. We show that a neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ cannot be an Integral Domain even if R is an Integral Domain and also we show that $\langle R \cup I \rangle [x]$ cannot be a Unique Factorization Domain even if R is a Unique Factorization Domain. In section 5 of this paper, we present neutrosophic ideals in neutrosophic polynomial rings and we show that every non-zero neutrosophic principal ideal is not a neutrosophic prime ideal.

§2. Elementary Properties of Neutrosophic Rings

In this section we state for emphasis some basic definitions and results but for further details about neutrosophic rings, the reader should see [1].

Definition 2.1([1]) *Let $(R, +, \cdot)$ be any ring. The set*

$$\langle R \cup I \rangle = \{a + bI : a, b \in R\}$$

is called a neutrosophic ring generated by R and I under the operations of R .

Example 2.2 $\langle \mathbb{Z} \cup I \rangle$, $\langle \mathbb{Q} \cup I \rangle$, $\langle \mathbb{R} \cup I \rangle$ and $\langle \mathbb{C} \cup I \rangle$ are neutrosophic rings of integer, rational, real and complex numbers respectively.

Theorem 2.3 *Every neutrosophic ring is a ring and every neutrosophic ring contains a proper subset which is just a ring.*

Definition 2.4 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is said to be commutative if $\forall x, y \in \langle R \cup I \rangle$, $xy = yx$.*

If in addition there exists $1 \in \langle R \cup I \rangle$ such that $1.r = r.1 = r$ for all $r \in \langle R \cup I \rangle$ then we call $\langle R \cup I \rangle$ a commutative neutrosophic ring with unity.

Definition 2.5 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring of $\langle R \cup I \rangle$ if $P = \langle S \cup nI \rangle$ where S is a subring of R and n an integer. P is said to be generated by S and nI under the operations of R .*

Definition 2.6 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$ which is just a ring. Then P is called a subring.*

Definition 2.7 Let T be a non-empty set together with two binary operations $+$ and $.$ T is said to be a pseudo neutrosophic ring if the following conditions hold:

- (i) T contains elements of the form $(a+bI)$, where a and b are real numbers and $b \neq 0$ for at least one value;
- (ii) $(T, +)$ is an Abelian group;
- (iii) $(T, .)$ is a semigroup;
- (iv) $\forall x, y, z \in T, x(y+z) = xy + xz$ and $(y+z)x = yx + zx$.

Definition 2.8 Let $\langle R \cup I \rangle$ be any neutrosophic ring. A non-empty subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic ideal of $\langle R \cup I \rangle$ if the following conditions hold:

- (i) P is a neutrosophic subring of $\langle R \cup I \rangle$;
- (ii) for every $p \in P$ and $r \in \langle R \cup I \rangle$, $rp \in P$ and $pr \in P$.

If only $rp \in P$, we call P a left neutrosophic ideal and if only $pr \in P$, we call P a right neutrosophic ideal. When $\langle R \cup I \rangle$ is commutative, there is no distinction between rp and pr and therefore P is called a left and right neutrosophic ideal or simply a neutrosophic ideal.

Definition 2.9 Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a pseudo neutrosophic subring of $\langle R \cup I \rangle$. P is said to be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$ if $\forall p \in P$ and $r \in \langle R \cup I \rangle$, $rp, pr \in P$.

Theorem 2.10([1]) Let $\langle \mathcal{Z} \cup I \rangle$ be a neutrosophic ring. Then $\langle \mathcal{Z} \cup I \rangle$ has a pseudo ideal P such that

$$\langle \mathcal{Z} \cup I \rangle \cong \mathcal{Z}_n.$$

Definition 2.11 Let $\langle R \cup I \rangle$ be a neutrosophic ring.

- (i) $\langle R \cup I \rangle$ is said to be of characteristic zero if $\forall x \in R, nx = 0$ implies that $n = 0$ for an integer n ;
- (ii) $\langle R \cup I \rangle$ is said to be of characteristic n if $\forall x \in R, nx = 0$ for an integer n .

Definition 2.12 An element x in a neutrosophic ring $\langle R \cup I \rangle$ is called a left zero divisor if there exists a nonzero element $y \in \langle R \cup I \rangle$ such that $xy = 0$.

A right zero divisor can be defined similarly. If an element $x \in \langle R \cup I \rangle$ is both a left and a right zero divisor, it is then called a zero divisor.

Definition 2.13 Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is called a neutrosophic integral domain if $\langle R \cup I \rangle$ is commutative with no zero divisors.

Definition 2.14 Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is called a neutrosophic division ring if $\langle R \cup I \rangle$ is non-commutative and has no zero divisors.

Definition 2.15 An element x in a neutrosophic ring $\langle R \cup I \rangle$ is called an idempotent element if $x^2 = x$.

Example 2.16 In the neutrosophic ring $\langle \mathcal{Z}_2 \cup I \rangle$, 0 and 1 are idempotent elements.

Definition 2.17 An element $x = a + bI$ in a neutrosophic ring $\langle R \cup I \rangle$ is called a neutrosophic idempotent element if $b \neq 0$ and $x^2 = x$.

Example 2.18 In the neutrosophic ring $\langle \mathbb{Z}_3 \cup I \rangle$, I and $1+2I$ are neutrosophic idempotent elements.

Definition 2.19 Let $\langle R \cup I \rangle$ be a neutrosophic ring. An element $x = a + bI$ with $a \neq \pm b$ is said to be a neutrosophic zero divisor if there exists $y = c + dI$ in $\langle R \cup I \rangle$ with $c \neq \pm d$ such that $xy = yx = 0$.

Definition 2.20 Let $x = a + bI$ with $a, b \neq 0$ be a neutrosophic element in the neutrosophic ring $\langle R \cup I \rangle$. If there exists an element $y \in R$ such that $xy = yx = 0$, then y is called a semi neutrosophic zero divisor.

Definition 2.21 An element $x = a + bI$ with $b \neq 0$ in a neutrosophic ring $\langle R \cup I \rangle$ is said to be a neutrosophic nilpotent element if there exists a positive integer n such that $x^n = 0$.

Example 2.22 In the neutrosophic ring $\langle \mathbb{Z}_4 \cup I \rangle$ of integers modulo 4, $2+2I$ is a neutrosophic nilpotent element.

Example 2.23 Let $\langle M_{2 \times 2} \cup I \rangle$ be a neutrosophic ring of all 2×2 matrices. An element $A = \begin{bmatrix} 0 & 2I \\ 0 & 0 \end{bmatrix}$ is neutrosophic nilpotent since $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Definition 2.24 Let r be a fixed element of the neutrosophic ring $\langle R \cup I \rangle$. We call the set

$$N(r) = \{x \in \langle R \cup I \rangle : xr = rx\}$$

the normalizer of r in $\langle R \cup I \rangle$.

Example 2.25 Let M be a neutrosophic ring defined by

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \langle \mathbb{Z}_2 \cup I \rangle \right\}.$$

It is clear that M has 16 elements.

(i) The normalizer of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in M is obtained as

$$N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix} \right\}.$$

(ii) The normalizer of $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ in M is obtained as

$$N\left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}\right) =$$

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & 1+I \\ 0 & 0 \end{bmatrix} \right\}.$$

It is clear that $N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ and $N\left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}\right)$ are pseudo neutrosophic subrings of M and in fact they are pseudo neutrosophic ideals of M . These emerging facts are put together in the next proposition.

Proposition 2.26 *Let $N(r)$ be a normalizer of an element in a neutrosophic ring $\langle R \cup I \rangle$. Then*

- (i) $N(r)$ is a pseudo neutrosophic subring of $\langle R \cup I \rangle$;
- (ii) $N(r)$ is a pseudo neutrosophic ideal of $\langle R \cup I \rangle$.

Definition 2.27 *Let P be a proper subset of the neutrosophic ring $\langle R \cup I \rangle$. The set*

$$Ann_l(P) = \{x \in \langle R \cup I \rangle : xp = 0 \ \forall p \in P\}$$

is called a left annihilator of P and the set

$$Ann_r(P) = \{y \in \langle R \cup I \rangle : py = 0 \ \forall p \in P\}$$

is called a right annihilator of P . If $\langle R \cup I \rangle$ is commutative, there is no distinction between left and right annihilators of P and we write $Ann(P)$.

Example 2.28 Let M be the neutrosophic ring of Example 2.25. If we take

$$P = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & 1+I \\ 0 & 0 \end{bmatrix} \right\},$$

then, the left annihilator of P is obtained as

$$Ann_l(P) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix} \right\}$$

which is a left pseudo neutrosophic ideal of M .

Proposition 2.29 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$. Then the left(right) annihilator of P is a left(right) pseudo neutrosophic ideal of $\langle R \cup I \rangle$.*

Example 2.30 Consider $\langle \mathbb{Z}_2 \cup I \rangle = \{0, 1, I, 1+I\}$ the neutrosophic ring of integers modulo 2. If $P = \{0, 1+I\}$, then $Ann(P) = \{0, I\}$.

Example 2.31 Consider $\langle \mathbb{Z}_3 \cup I \rangle = \{0, 1, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$ the neutrosophic ring of integers modulo 3. If $P = \{0, I, 2I\}$, then $Ann(P) = \{0, 1+2I, 2+I\}$ which is a pseudo neutrosophic subring and indeed a pseudo neutrosophic ideal.

Proposition 2.32 *Let $\langle R \cup I \rangle$ be a commutative neutrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$. Then $\text{Ann}(P)$ is a pseudo neutrosophic ideal of $\langle R \cup I \rangle$.*

Definition 2.33 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a neutrosophic ideal of $\langle R \cup I \rangle$. The set*

$$\langle R \cup I \rangle / P = \{r + P : r \in \langle R \cup I \rangle\}$$

is called the neutrosophic quotient ring provided that $\langle R \cup I \rangle / P$ is a neutrosophic ring.

To show that $\langle R \cup I \rangle / P$ is a neutrosophic ring, let $x = r_1 + P$ and $y = r_2 + P$ be any two elements of $\langle R \cup I \rangle / P$ and let $+$ and $\cdot$ be two binary operations defined on $\langle R \cup I \rangle / P$ by:

$$\begin{aligned} x + y &= (r_1 + r_2) + P, \\ xy &= (r_1 r_2) + P, \quad r_1, r_2 \in \langle R \cup I \rangle. \end{aligned}$$

It can easily be shown that

- (i) the two operations are well defined;
- (ii) $(\langle R \cup I \rangle / P, +)$ is an abelian group;
- (iii) $(\langle R \cup I \rangle / P, \cdot)$ is a semigroup, and
- (iv) if $z = r_3 + P$ is another element of $\langle R \cup I \rangle / P$ with $r_3 \in \langle R \cup I \rangle$, then we have $z(x + y) = zx + zy$ and $(x + y)z = xz + yz$. Accordingly, $\langle R \cup I \rangle / P$ is a neutrosophic ring with P as an additive identity element.

Definition 2.34 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a neutrosophic ideal of $\langle R \cup I \rangle$. $\langle R \cup I \rangle / P$ is called a false neutrosophic quotient ring if $\langle R \cup I \rangle / P$ is just a ring and not a neutrosophic ring.*

Definition 2.35 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$. $\langle R \cup I \rangle / P$ is called a pseudo neutrosophic quotient ring if $\langle R \cup I \rangle / P$ is a neutrosophic ring. If $\langle R \cup I \rangle / P$ is just a ring, then we call $\langle R \cup I \rangle / P$ a false pseudo neutrosophic quotient ring.*

Definition 2.36 *Let $\langle R \cup I \rangle$ and $\langle S \cup I \rangle$ be any two neutrosophic rings. The mapping $\phi : \langle R \cup I \rangle \rightarrow \langle S \cup I \rangle$ is called a neutrosophic ring homomorphism if the following conditions hold:*

- (i) ϕ is a ring homomorphism;
- (ii) $\phi(I) = I$.

The set $\{x \in \langle R \cup I \rangle : \phi(x) = 0\}$ is called the kernel of ϕ and is denoted by $\text{Ker}\phi$.

Theorem 2.37 *Let $\phi : \langle R \cup I \rangle \rightarrow \langle S \cup I \rangle$ be a neutrosophic ring homomorphism and let $K = \text{Ker}\phi$ be the kernel of ϕ . Then:*

- (i) K is always a subring of $\langle R \cup I \rangle$;
- (ii) K cannot be a neutrosophic subring of $\langle R \cup I \rangle$;
- (iii) K cannot be an ideal of $\langle R \cup I \rangle$.

Proof (i) It is Clear. (ii) Since $\phi(I) = I$, it follows that $I \notin K$ and the result follows. (iii) Follows directly from (ii). $\square$

Example 2.38 Let $\langle R \cup I \rangle$ be a neutrosophic ring and let $\phi : \langle R \cup I \rangle \rightarrow \langle R \cup I \rangle$ be a mapping defined by $\phi(r) = r \ \forall r \in \langle R \cup I \rangle$. Then ϕ is a neutrosophic ring homomorphism.

Example 2.39 Let P be a neutrosophic ideal of the neutrosophic ring $\langle R \cup I \rangle$ and let $\phi : \langle R \cup I \rangle \rightarrow \langle R \cup I \rangle / P$ be a mapping defined by $\phi(r) = r + P, \ \forall r \in \langle R \cup I \rangle$. Then $\forall r, s \in \langle R \cup I \rangle$, we have

$$\phi(r + s) = \phi(r) + \phi(s), \quad \phi(rs) = \phi(r)\phi(s),$$

which shows that ϕ is a ring homomorphism. But then,

$$\phi(I) = I + P \neq I.$$

Thus, ϕ is not a neutrosophic ring homomorphism. This is another marked difference between the classical ring concept and the concept of neutrosophic ring.

Proposition 2.40 Let $(\langle R \cup I \rangle, +)$ be a neutrosophic abelian group and let $Hom(\langle R \cup I \rangle, \langle R \cup I \rangle)$ be the set of neutrosophic endomorphisms of $(\langle R \cup I \rangle, +)$ into itself. Let $+$ and $\cdot$ be addition and multiplication in $Hom(\langle R \cup I \rangle, \langle R \cup I \rangle)$ defined by

$$\begin{aligned} (\phi + \psi)(x) &= \phi(x) + \psi(x), \\ (\phi \cdot \psi)(x) &= \phi(\psi(x)), \forall \phi, \psi \in Hom(\langle R \cup I \rangle, \langle R \cup I \rangle), x \in \langle R \cup I \rangle. \end{aligned}$$

Then $(Hom(\langle R \cup I \rangle, \langle R \cup I \rangle), +, \cdot)$ is a neutrosophic ring.

Proof The proof is the same as in the classical ring. □

Definition 2.41 Let R be an arbitrary ring with unity. A neutrosophic left R -module is a neutrosophic abelian group $(\langle M \cup I \rangle, +)$ together with a scalar multiplication map $\cdot : R \times \langle M \cup I \rangle \rightarrow \langle M \cup I \rangle$ that satisfies the following conditions:

- (i) $r(m + n) = rm + rn$;
- (ii) $(r + s)m = rm + sm$;
- (iii) $(rs)m = r(sm)$;
- (iv) $1.m = m$, where $r, s \in R$ and $m, n \in \langle M \cup I \rangle$.

Definition 2.42 Let R be an arbitrary ring with unity. A neutrosophic right R -module is a neutrosophic abelian group $(\langle M \cup I \rangle, +)$ together with a scalar multiplication map $\cdot : \langle M \cup I \rangle \times R \rightarrow \langle M \cup I \rangle$ that satisfies the following conditions:

- (i) $(m + n)r = mr + nr$;
- (ii) $m(r + s) = mr + ms$;
- (iii) $m(rs) = (mr)s$;
- (iv) $m.1 = m$, where $r, s \in R$ and $m, n \in \langle M \cup I \rangle$.

If R is a commutative ring, then a neutrosophic left R -module $\langle M \cup I \rangle$ becomes a neutrosophic right R -module and we simply call $\langle M \cup I \rangle$ a neutrosophic R -module.

Example 2.43 Let $(\langle M \cup I \rangle, +)$ be a neutrosophic abelian group and let $\mathcal{Z}$ be the ring of integers. If we define the mapping $f : \mathcal{Z} \times \langle M \cup I \rangle \rightarrow \langle M \cup I \rangle$ by $f(n, m) = nm, \forall n \in \mathcal{Z}, m \in \langle M \cup I \rangle$, then $\langle M \cup I \rangle$ becomes a neutrosophic $\mathcal{Z}$ -module.

Example 2.44 Let $\langle R \cup I \rangle[x]$ be a neutrosophic ring of polynomials where R is a commutative ring with unity. Obviously, $(\langle R \cup I \rangle[x], +)$ is a neutrosophic abelian group and the scalar multiplication map $\cdot : R \times \langle R \cup I \rangle[x] \rightarrow \langle R \cup I \rangle[x]$ satisfies all the axioms of the neutrosophic R -module. Hence, $\langle R \cup I \rangle[x]$ is a neutrosophic R -module.

Proposition 2.45 Let $(\langle R \cup I \rangle, +)$ be a neutrosophic abelian group and let $\text{Hom}(\langle R \cup I \rangle, \langle R \cup I \rangle)$ be the neutrosophic ring obtained in Proposition (2.40). Let $\cdot : \text{Hom}(\langle R \cup I \rangle, \langle R \cup I \rangle) \times \langle R \cup I \rangle \rightarrow \langle R \cup I \rangle$ be a scalar multiplication defined by $\cdot(f, r) = fr, \forall f \in \text{Hom}(\langle R \cup I \rangle, \langle R \cup I \rangle), r \in \langle R \cup I \rangle$. Then $\langle R \cup I \rangle$ is a neutrosophic left $\text{Hom}(\langle R \cup I \rangle, \langle R \cup I \rangle)$ -module.

Proof Suppose that $\text{Hom}(\langle R \cup I \rangle, \langle R \cup I \rangle)$ is a neutrosophic ring. Then by Theorem (2.3), it is also a ring. It is clear that $\cdot(f, r) = fr$ is the image of r under f and it is an element of $\langle R \cup I \rangle$. It can easily be shown that the scalar multiplication " $\cdot$ " satisfies the axioms of a neutrosophic left R -module. Hence, $\langle R \cup I \rangle$ is a neutrosophic left $\text{Hom}(\langle R \cup I \rangle, \langle R \cup I \rangle)$ -module. $\square$

Definition 2.46 Let $\langle M \cup I \rangle$ be a neutrosophic left R -module. The set $\{r \in R : rm = 0 \ \forall m \in \langle M \cup I \rangle\}$ is called the annihilator of $\langle M \cup I \rangle$ and is denoted by $\text{Ann}(\langle M \cup I \rangle)$. $\langle M \cup I \rangle$ is said to be faithful if $\text{Ann}(\langle M \cup I \rangle) = (0)$. It can easily be shown that $\text{Ann}(\langle M \cup I \rangle)$ is a pseudo neutrosophic ideal of $\langle M \cup I \rangle$.

§3. Neutrosophic Polynomial Rings

In this section and Sections 4 and 5, unless otherwise stated, all neutrosophic rings will be assumed to be commutative neutrosophic rings with unity and x will be an indeterminate in $\langle R \cup I \rangle[x]$.

Definition 3.1 (i) By the neutrosophic polynomial ring in x denoted by $\langle R \cup I \rangle[x]$ we mean the set of all symbols $\sum_{i=1}^n a_i x^i$ where n can be any nonnegative integer and where the coefficients $a_i, i = n, n-1, \dots, 2, 1, 0$ are all in $\langle R \cup I \rangle$.

(ii) If $f(x) = \sum_{i=1}^n a_i x^i$ is a neutrosophic polynomial in $\langle R \cup I \rangle[x]$ such that $a_i = 0, \forall i = n, n-1, \dots, 2, 1, 0$, then we call $f(x)$ a zero neutrosophic polynomial in $\langle R \cup I \rangle[x]$.

(iii) If $f(x) = \sum_{i=1}^n a_i x^i$ is a nonzero neutrosophic polynomial in $\langle R \cup I \rangle[x]$ with $a_n \neq 0$, then we call n the degree of $f(x)$ denoted by $\deg f(x)$ and we write $\deg f(x) = n$.

(iv) Two neutrosophic polynomials $f(x) = \sum_{i=1}^n a_i x^i$ and $g(x) = \sum_{j=1}^m b_j x^j$ in $\langle R \cup I \rangle[x]$ are said to be equal written $f(x) = g(x)$ if and only if for every integer $i \geq 0$, $a_i = b_i$ and $n = m$.

(v) A neutrosophic polynomial $f(x) = \sum_{i=1}^n a_i x^i$ in $\langle R \cup I \rangle[x]$ is called a strong neutrosophic polynomial if for every $i \geq 0$, each a_i is of the form $(a + bI)$ where $a, b \in R$ and $b \neq 0$.

$f(x) = \sum_{i=1}^n a_i x^i$ is called a mixed neutrosophic polynomial if some $a_i \in R$ and some a_i are of the form $(a + bI)$ with $b \neq 0$. If every $a_i \in R$ then $f(x) = \sum_{i=1}^n a_i x^i$ is called a polynomial.

Example 3.2 $\langle \mathcal{Z} \cup I \rangle [x]$, $\langle \mathcal{Q} \cup I \rangle [x]$, $\langle \mathcal{R} \cup I \rangle [x]$, $\langle \mathcal{C} \cup I \rangle [x]$ are neutrosophic polynomial rings of integers, rationals, real and complex numbers respectively each of zero characteristic.

Example 3.3 Let $\langle \mathcal{Z}_n \cup I \rangle$ be the neutrosophic ring of integers modulo n . Then $\langle \mathcal{Z}_n \cup I \rangle [x]$ is the neutrosophic polynomial ring of integers modulo n . The characteristic of $\langle \mathcal{Z}_n \cup I \rangle [x]$ is n . If $n = 3$ and $\langle \mathcal{Z}_3 \cup I \rangle [x] = \{ax^2 + bx + c : a, b, c \in \langle \mathcal{Z}_3 \cup I \rangle\}$, then $\langle \mathcal{Z}_3 \cup I \rangle [x]$ is a neutrosophic polynomial ring of integers modulo 3.

Example 3.4 Let $f(x), g(x) \in \langle \mathcal{Z} \cup I \rangle [x]$ such that $f(x) = 2Ix^2 + (2 + I)x + (1 - 2I)$ and $g(x) = x^3 - (1 - 3I)x^2 + 3Ix + (1 + I)$. Then $f(x)$ and $g(x)$ are strong and mixed neutrosophic polynomials of degrees 2 and 3 respectively.

Definition 3.5 Let α be a fixed element of the neutrosophic ring $\langle R \cup I \rangle$. The mapping $\phi_\alpha : \langle R \cup I \rangle [x] \rightarrow \langle R \cup I \rangle$ defined by

$$\phi_\alpha (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0$$

is called the neutrosophic evaluation map. It can be shown that ϕ_α is a neutrosophic ring homomorphism. If $R = \mathcal{Z}$ and $f(x) \in \langle \mathcal{Z} \cup I \rangle [x]$ such that $f(x) = 2Ix^2 + x - 3I$, then $\phi_{1+I}(f(x)) = 1 + 6I$ and $\phi_I(f(x)) = 0$. The last result shows that $f(x)$ is in the kernel of ϕ_I .

Theorem 3.6([1]) Every neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ contains a polynomial ring $R[x]$.

Theorem 3.7 The neutrosophic ring $\langle R \cup I \rangle$ is not an integral domain (ID) even if R is an ID.

Proof Suppose that $\langle R \cup I \rangle$ is an ID. Obviously, $R \subset \langle R \cup I \rangle$. Let $x = (\alpha - \alpha I)$ and $y = \beta I$ be two elements of $\langle R \cup I \rangle$ where α and β are non-zero positive integers. Clearly, $x \neq 0$ and $y \neq 0$ and since $I^2 = I$, we have $xy = 0$ which shows that x and y are neutrosophic zero divisors in $\langle R \cup I \rangle$ and consequently, $\langle R \cup I \rangle$ is not an ID. $\square$

Theorem 3.8 The neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ is not an ID even if R is an ID.

Proof Suppose that R is an ID. Then $R[x]$ is also an ID and $R[x] \subset \langle R \cup I \rangle [x]$. But then by Theorem 3.7, $\langle R \cup I \rangle$ is not an ID and therefore $\langle R \cup I \rangle [x]$ cannot be an ID. $\square$

Example 3.9 Let $\langle \mathcal{Z} \cup I \rangle [x]$ be the neutrosophic polynomial ring of integers and let $f(x)$, $g(x)$, $p(x)$ and $q(x)$ be neutrosophic polynomials in $\langle \mathcal{Z} \cup I \rangle$ given by $f(x) = (2 - 2I)x^2 + 3Ix - I$, $g(x) = Ix + (1 + I)$, $p(x) = (8 - 8I)x^5$ and $q(x) = 7Ix^3$. Then $f(x)g(x) = (2 + I)x^2 + 5Ix - 2I$ and $p(x)q(x) = 0$. Now $\deg f(x) + \deg g(x) = 3$, $\deg(f(x)g(x)) = 2 < 3$, $\deg p(x) + \deg q(x) = 8$ and $\deg(p(x)q(x)) = 0 < 8$. The causes of these phenomena are the existence of neutrosophic zero divisors in $\langle \mathcal{Z} \cup I \rangle$ and $\langle \mathcal{Z} \cup I \rangle [x]$ respectively. We register these observations in the following theorem.

Theorem 3.10 Let $\langle R \cup I \rangle$ be a commutative neutrosophic ring with unity. If $f(x) = \sum_{i=1}^n a_i x^i$ and $g(x) = \sum_{j=1}^m b_j x^j$ are two non-zero neutrosophic polynomials in $\langle R \cup I \rangle[x]$ with R an ID or not such that $a_n = (\alpha - \alpha I)$ and $b_m = \beta I$ where α and β are non-zero positive integers, then

$$\deg(f(x)g(x)) < \deg f(x) + \deg g(x).$$

Proof Suppose that $f(x) = \sum_{i=1}^n a_i x^i$ and $g(x) = \sum_{j=1}^m b_j x^j$ are two non-zero neutrosophic polynomials in $\langle R \cup I \rangle[x]$ with $a_n = (\alpha - \alpha I)$ and $b_m = \beta I$ where α and β are non-zero positive integers. Clearly, $a_n \neq 0$ and $b_m \neq 0$ but then $a_n b_m = 0$ and consequently,

$$\begin{aligned} \deg(f(x)g(x)) &= (n-1) + (m-1) \\ &= (n+m) - 2 < (n+m) \\ &= \deg f(x) + \deg g(x). \end{aligned}$$

□

§4. Factorization in Neutrosophic Polynomial Rings

Definition 4.1 Let $f(x) \in \langle R \cup I \rangle[x]$ be a neutrosophic polynomial. Then

- (i) $f(x)$ is said to be neutrosophic reducible in $\langle R \cup I \rangle[x]$ if there exists two neutrosophic polynomials $p(x), q(x) \in \langle R \cup I \rangle[x]$ such that $f(x) = p(x).q(x)$.
- (ii) $f(x)$ is said to be semi neutrosophic reducible if $f(x) = p(x).q(x)$ but only one of $p(x)$ or $q(x)$ is a neutrosophic polynomial in $\langle R \cup I \rangle[x]$.
- (iii) $f(x)$ is said to be neutrosophic irreducible if $f(x) = p(x).q(x)$ but either $p(x)$ or $q(x)$ equals I or 1 .

Definition 4.2 Let $f(x)$ and $g(x)$ be two neutrosophic polynomials in the neutrosophic polynomial ring $\langle R \cup I \rangle[x]$. Then

- (i) The pair $f(x)$ and $g(x)$ are said to be relatively neutrosophic prime if the $\gcd(f(x), g(x)) = r(x)$ is not possible for a neutrosophic polynomial $r(x) \in \langle R \cup I \rangle[x]$.
- (ii) The pair $f(x)$ and $g(x)$ are said to be strongly relatively neutrosophic prime if their $\gcd(f(x), g(x)) = 1$ or I .

Definition 4.3 A neutrosophic polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \langle \mathcal{Z} \cup I \rangle[x]$ is said to be neutrosophic primitive if the $\gcd(a_n, a_{n-1}, \dots, a_1, a_0) = 1$ or I .

Definition 4.3 Let $f(x) = \sum_{i=1}^n a_i x^i$ be a neutrosophic polynomial in $\langle R \cup I \rangle[x]$. $f(x)$ is said to be a neutrosophic monic polynomial if $a_n = 1$.

Example 4.5 Let us consider the neutrosophic polynomial ring $\langle \mathcal{R} \cup I \rangle[x]$ of all real numbers and let $f(x)$ and $d(x)$ be two neutrosophic polynomials in $\langle \mathcal{R} \cup I \rangle[x]$.

- (i) If $f(x) = 2Ix^2 - (1 + 7I)x + 6I$ and $d(x) = x - 3I$, then by dividing $f(x)$ by $d(x)$ we obtain the quotient $q(x) = 2Ix - (1 + I)$ and the remainder $r(x) = 0$ and hence $f(x) \equiv (2Ix - (1 + I))(x - 3I) + 0$.

(ii) If $f(x) = 2Ix^3 + (1 + I)$ and $d(x) = Ix + (2 - I)$, then $q(x) = 2Ix^2 - 2Ix + 2I$, $r(x) = 1 - I$ and $f(x) \equiv (2Ix^2 - 2Ix + 2I)(Ix + (2 - I)) + (1 - I)$.

(iii) If $f(x) = (2 + I)x^2 + 2Ix + (1 + I)$ and $d(x) = (2 + I)x + (2 - I)$, then $q(x) = x - (1 - \frac{4}{3}I)$, $r(x) = 3 - \frac{4}{3}I$ and $f(x) \equiv (x - (1 - \frac{4}{3}I))((2 + I)x + (2 - I)) + (3 - \frac{4}{3}I)$.

(iv) If $f(x) = Ix^2 + x - (1 + 5I)$ and $d(x) = x - (1 + I)$, then $q(x) = Ix + (1 + 2I)$, $r(x) = 0$ and $f(x) \equiv (Ix + (1 + 2I))(x - (1 + I)) + 0$.

(v) If $f(x) = x^2 - Ix + (1 + I)$ and $d(x) = x - (1 - I)$, then $q(x) = x + (1 - 2I)$, $r(x) = 2$ and $f(x) \equiv (x + (1 - 2I))(x - (1 - I)) + 2$.

The examples above show that for each pair of the neutrosophic polynomials $f(x)$ and $d(x)$ considered there exist unique neutrosophic polynomials $q(x), r(x) \in \langle \mathcal{R} \cup I \rangle[x]$ such that $f(x) = q(x)d(x) + r(x)$ where $\deg r(x) < \deg d(x)$. However, this is generally not true. To see this let us consider the following pairs of neutrosophic polynomials in $\langle \mathcal{R} \cup I \rangle[x]$:

- (i) $f(x) = 4Ix^2 + (1 + I)x - 2I$, $d(x) = 2Ix + (1 + I)$;
- (ii) $f(x) = 2Ix^2 + (1 + I)x + (1 - I)$, $d(x) = 2Ix + (3 - 2I)$;
- (iii) $f(x) = (-8I)x^2 + (7 + 5I)x + (2 - I)$, $d(x) = Ix + (1 + I)$;
- (iv) $f(x) = Ix^2 - 2Ix + (1 + I)$, $d(x) = Ix - (1 - I)$.

In each of these examples, it is not possible to find $q(x), r(x) \in \langle \mathcal{R} \cup I \rangle[x]$ such that $f(x) = q(x)d(x) + r(x)$ with $\deg r(x) < \deg d(x)$. Hence Division Algorithm is generally not possible for neutrosophic polynomial rings. However for neutrosophic polynomial rings in which all neutrosophic polynomials are neutrosophic monic, the Division Algorithm holds generally. The question of whether Division Algorithm is true for neutrosophic polynomial rings raised by Vasantha Kandasamy and Florentin Smarandache in [1] is thus answered.

Theorem 4.6 *If $f(x)$ and $d(x)$ are neutrosophic polynomials in the neutrosophic polynomial ring $\langle R \cup I \rangle[x]$ with $f(x)$ and $d(x)$ neutrosophic monic, there exist unique neutrosophic polynomials $q(x), r(x) \in \langle R \cup I \rangle[x]$ such that $f(x) = q(x)d(x) + r(x)$ with $\deg r(x) < \deg d(x)$.*

Proof The proof is the same as the classical case. □

Theorem 4.7 *Let $f(x)$ be a neutrosophic monic polynomial in $\langle R \cup I \rangle[x]$ and for $u \in \langle R \cup I \rangle$, let $d(x) = x - u$. Then $f(u)$ is the remainder when $f(x)$ is divided by $d(x)$. Furthermore, if $f(u) = 0$ then $d(x)$ is a neutrosophic factor of $f(x)$.*

Proof Since $f(x)$ and $d(x)$ are neutrosophic monic in $\langle R \cup I \rangle[x]$, there exists $q(x)$ and $r(x)$ in $\langle R \cup I \rangle[x]$ such that $f(x) = q(x)(x - u) + r(x)$, with $r(x) = 0$ or $\deg r(x) < \deg d(x) = 1$. Hence $r(x) = r \in \langle R \cup I \rangle$. Now, $\phi_u(f(x)) = 0 + r(u) = r(u) = r \in \langle R \cup I \rangle$. If $f(u) = 0$, it follows that $r(x) = 0$ and consequently, $d(x)$ is a neutrosophic factor of $f(x)$. □

Observation 4.8 Since the indeterminacy factor I has no inverse, it follows that the neutrosophic rings $\langle \mathcal{Q} \cup I \rangle, \langle \mathcal{R} \cup I \rangle, \langle \mathcal{C} \cup I \rangle$ cannot be neutrosophic fields and consequently neutrosophic equations of the form $(a + bI)x = (c + dI)$ are not solvable in $\langle \mathcal{Q} \cup I \rangle, \langle \mathcal{R} \cup I \rangle, \langle \mathcal{C} \cup I \rangle$ except $b \equiv 0$.

Definition 4.9 *Let $f(x)$ be a neutrosophic polynomial in $\langle R \cup I \rangle[x]$ with $\deg f(x) \geq 1$. An*

element $u \in \langle R \cup I \rangle$ is said to be a neutrosophic zero of $f(x)$ if $f(u) = 0$.

Example 4.10 (i) Let $f(x) = 6x^2 + Ix - 2I \in \langle \mathcal{Q} \cup I \rangle[x]$. Then $f(x)$ is neutrosophic reducible and $(2x-I)$ and $(3x+2I)$ are the neutrosophic factors of $f(x)$. Since $f(\frac{1}{2}I) = 0$ and $f(-\frac{2}{3}I) = 0$, then $\frac{1}{2}I, -\frac{2}{3}I \in \langle \mathcal{Q} \cup I \rangle$ are the neutrosophic zeroes of $f(x)$. Since $f(x)$ is of degree 2 and it has two zeroes, then the Fundamental Theorem of Algebra is obeyed.

(ii) Let $f(x) = 4Ix^2 + (1+I)x - 2I \in \langle \mathcal{Q} \cup I \rangle[x]$. $f(x)$ is neutrosophic reducible and $p(x) = 2Ix + (1+I)$ and $q(x) = (1+I)x - I$ are the neutrosophic factors of $f(x)$. But then, $f(x)$ has no neutrosophic zeroes in $\langle \mathcal{Q} \cup I \rangle$ and even in $\langle \mathcal{R} \cup I \rangle$ and $\langle \mathcal{C} \cup I \rangle$ since I^{-1} , the inverse of I does not exist.

(iii) $Ix^2 - 2$ is neutrosophic irreducible in $\langle \mathcal{Q} \cup I \rangle[x]$ but it is neutrosophic reducible in $\langle \mathcal{R} \cup I \rangle[x]$ since $Ix^2 - 2 = (Ix - \sqrt{2})(Ix + \sqrt{2})$. However since $\langle \mathcal{R} \cup I \rangle$ is not a field, $Ix^2 - 2$ has no neutrosophic zeroes in $\langle \mathcal{R} \cup I \rangle$.

Theorem 4.11 Let $f(x)$ be a neutrosophic polynomial of degree > 1 in the neutrosophic polynomial ring $\langle R \cup I \rangle[x]$. If $f(x)$ has neutrosophic zeroes in $\langle R \cup I \rangle$, then $f(x)$ is neutrosophic reducible in $\langle R \cup I \rangle[x]$ and not the converse.

Theorem 4.12 Let $f(x)$ be a neutrosophic polynomial in $\langle R \cup I \rangle[x]$. The factorization of $f(x)$ if possible over $\langle R \cup I \rangle[x]$ is not unique.

Proof Let us consider the neutrosophic polynomial $f(x) = 2Ix^2 + (1+I)x + 2I$ in the neutrosophic ring of polynomials $\langle \mathcal{Z}_3 \cup I \rangle[x]$. $f(I) = 0$ and by Theorem 4.11, $f(x)$ is neutrosophic reducible in $\langle \mathcal{Z}_3 \cup I \rangle[x]$ and hence $f(x)$ can be expressed as $f(x) = (2Ix+1)(x-I) = (2Ix+1)(x+2I)$. However, $f(x)$ can also be expressed as $f(x) = [(1+I)x+I][Ix+(1+I)]$. This shows that the factorization of $f(x)$ is not unique in $\langle \mathcal{Z}_3 \cup I \rangle[x]$. We note that the first factorization shows that $f(x)$ has $I \in \langle \mathcal{Z}_3 \cup I \rangle$ as a neutrosophic zero but the second factorization shows that $f(x)$ has no neutrosophic zeroes in $\langle \mathcal{Z}_3 \cup I \rangle$. This is different from what obtains in the classical rings of polynomials. $\square$

Observation 4.13 Let us consider the neutrosophic polynomial ring $\langle R \cup I \rangle[x]$. It has been shown in Theorem 3.8 that $\langle R \cup I \rangle[x]$ cannot be a neutrosophic ID even if R is an ID. Also by Theorem 4.12, factorization in $\langle R \cup I \rangle[x]$ is generally not unique. Consequently, $\langle R \cup I \rangle[x]$ cannot be a neutrosophic Unique Factorization Domain (UFD) even if R is a UFD. Thus Gauss's Lemma, which asserts that $R[x]$ is a UFD if and only if R is a UFD does not hold in the setting of neutrosophic polynomial rings. Also since $I \in \langle R \cup I \rangle$ and I^{-1} , the inverse of I does not exist, then $\langle R \cup I \rangle$ cannot be a field even if R is a field and consequently $\langle R \cup I \rangle[x]$ cannot be a neutrosophic UFD. Again, the question of whether $\langle R \cup I \rangle[x]$ is a neutrosophic UFD given that R is a UFD raised by Vasantha Kandasamy and Florentin Smarandache in [1] is answered.

§5. Neutrosophic Ideals in Neutrosophic Polynomial Rings

Definition 5.1 Let $\langle R \cup I \rangle[x]$ be a neutrosophic ring of polynomials. An ideal J of $\langle R \cup I \rangle[x]$

is called a neutrosophic principal ideal if it can be generated by an irreducible neutrosophic polynomial $f(x)$ in $\langle R \cup I \rangle[x]$.

Definition 5.2 A neutrosophic ideal P of a neutrosophic ring of polynomials $\langle R \cup I \rangle[x]$ is called a neutrosophic prime ideal if $f(x)g(x) \in P$, then $f(x) \in P$ or $g(x) \in P$ where $f(x)$ and $g(x)$ are neutrosophic polynomials in $\langle R \cup I \rangle[x]$.

Definition 5.3 A neutrosophic ideal M of a neutrosophic ring of polynomials $\langle R \cup I \rangle[x]$ is called a neutrosophic maximal ideal of $\langle R \cup I \rangle[x]$ if $M \neq \langle R \cup I \rangle[x]$ and no proper neutrosophic ideal N of $\langle R \cup I \rangle[x]$ properly contains M that is if $M \subseteq N \subseteq \langle R \cup I \rangle[x]$ then $M = N$ or $N = \langle R \cup I \rangle[x]$.

Example 5.4 Let $\langle \mathcal{Z}_2 \cup I \rangle[x] = \{ax^2 + bx + c : a, b, c \in \langle \mathcal{Z}_2 \cup I \rangle\}$ and consider $f(x) = Ix^2 + Ix + (1 + I) \in \langle \mathcal{Z}_2 \cup I \rangle[x]$. The neutrosophic ideal $J = \langle f(x) \rangle$ generated by $f(x)$ is neither a neutrosophic principal ideal nor a neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle[x]$. This is so because $f(x)$ is neutrosophic reducible in $\langle \mathcal{Z}_2 \cup I \rangle[x]$ eventhough it does not have zeroes in $\langle \mathcal{Z}_2 \cup I \rangle$. Also, $(Ix + (1 + I))(Ix + 1) \in J$ but $(Ix + (1 + I)) \notin J$ and $(Ix + 1) \notin J$. Hence J is not a neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle[x]$. However, $\langle 0 \rangle$ is the only neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle[x]$ which is not a neutrosophic maximal ideal.

Theorem 5.5 Let $\langle R \cup I \rangle[x]$ be a neutrosophic ring of polynomials. Every neutrosophic principal ideal of $\langle R \cup I \rangle[x]$ is not prime.

Proof Consider the neutrosophic polynomial ring $\langle \mathcal{Z}_3 \cup I \rangle[x] = \{x^3 + ax + b : a, b \in \langle \mathcal{Z}_3 \cup I \rangle\}$ and Let $f(x) = x^3 + Ix + (1 + I)$. It can be shown that $f(x)$ is neutrosophic irreducible in $\langle \mathcal{Z}_3 \cup I \rangle[x]$ and therefore $\langle f(x) \rangle$, the neutrosophic ideal generated by $f(x)$ is principal and not a prime ideal. We have also answered the question of Vasantha Kandasamy and Florentin Smarandache in [1] of wether every neutrosophic principal ideal of $\langle R \cup I \rangle[x]$ is also a neutrosophic prime ideal. $\square$

References

- [1] Vasantha Kandasamy W.B. and Florentin Smarandache, *Neutrosophic Rings*, Hexis, Phoenix, Arizona, 2006.
- [2] Vasantha Kandasamy W.B. and Florentin Smarandache, *Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures*, Hexis, Phoenix, Arizona, 2006.
- [3] Vasantha Kandasamy W.B. and Florentin Smarandache, *Basic Neutrosophic Algebraic Structures and their Applications to Fuzzy and Neutrosophic Models*, Hexis, Church Rock, 2004.
- [4] Vasantha Kandasamy W.B., Gaussian polynomial rings, *Octagon*, Vol.5(1997), 58-59.
- [5] Vasantha Kandasamy W.B., Inner associative rings, *J. of Math. Res. and Expo.*, Vol.18(1998), 217-218.
- [6] Vasantha Kandasamy W.B., *CN rings*, *Octagon*, Vol.9(2001), 343-344.
- [7] Vasantha Kandasamy W.B., On locally semi unitary rings, *Octagon*, Vol.9(2001), 260-262.

- [8] Vougiouklis Thomas, On rings with zero divisors strong V-groups, *Comment Math. Univ. Carolin J.*, Vol.31(1990), 431-433.
- [9] Wilson John S., A note on additive subgroups of finite rings, *J. Algebra*, Vol.234(2000), 362-366.
- [10] Florentin Smarandache, *A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability*(3rd edition), American Research Press, Rehoboth, 2003.
- [11] Agboola A.A.A. and Akinola L.S., On the Bicoset of a Bivector Space, *Int. J. of Math. Comb.* Vol.4(2009), 1-8.
- [12] Fraleigh, J.B., *A First Course in Abstract Algebra*(5th Edition), Addison-Wesley, 1994.
- [13] Lang S., *Algebra*, Addison Wesley, 1984.
- [14] Atiyah M.F. and MacDonald I.G., *Introduction to Commutative Algebra*, Addison-Wesley, 1969.
- [15] Herstein I.N., *Topics in Algebra*, John Wiley and Sons, 1964.
- [16] Herstein I.N., *Topics in Ring Theory*, University of Chicago Press, 1969.