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Fibonacci and Super Fibonacci Graceful Labelings of Some Cycle Related Graphs

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Abstract: We investigate Fibonacci and super Fibonacci graceful labelings for some cycle related graphs. We prove that the path union of k-copies of C_m where $m \equiv 0\ (mod\ 3)$ is a Fibonacci graceful graph. We also discuss the embedding of cycle in the context of these labelings. This work is a nice combination of graph theory and elementary number theory.

Key Words: Graph labeling, Fibonacci graceful, super Fibonacci graceful graph.

AMS(2010): 05C78

§1. Introduction and Definitions

We begin with simple, finite, undirected and non-trivial graph $G = (V, E)$, with vertex set V and edge set E. In the present work C_n denote the cycle with n vertices and P_n denote the path of n vertices. In the wheel $W_n = C_n + K_1$ the vertex corresponding to K_1 is called the apex vertex and the vertices corresponding to C_n are called the rim vertices where $n \geqslant 3$. Throughout this paper $|V|$ and $|E|$ are used for cardinality of vertex set and edge set respectively. We assume $F_1 = 1, F_2 = 2$ and for each positive integer n, $F_{n+2} = F_{n+1} + F_n$. For each positive integer n, F_n is called the nth Fibonacci number. For various graph theoretic notations and terminology we follow Gross and Yellen [3] while for number theory we follow Burton [1]. We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1 *If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.*

Vast amount of literature is available in printed as well as in electronic form on different types of graph labeling. More than 1200 research papers have been published so far in last four decades. Most interesting graph labeling problems have following three important ingredients.

- a set of numbers from which vertex labels are chosen;

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- a rule that assigns a value to each edge;
- a condition that these values must satisfy.

Most of the graph labeling techniques trace their origin to graceful labeling introduced by Rosa [5].

Definition 1.2 *Let $G = (V, E)$ be a graph with q edges. A graceful labeling of G is an injective function $f : V \to \{0, 1, 2, \cdots, q\}$ such that the induced edge labeling $f(uv) = |f(u) - f(v)|$ is a bijection from E onto the set $\{1, 2, \cdots, q\}$. If a graph G admits a graceful labeling then G is called graceful graph.*

The problem of characterizing all graceful graphs and the graceful tree conjecture provided the reason for different ways of labeling of graphs. Some variations of graceful labeling are also introduced recently such as edge graceful labeling, Fibonacci graceful labeling, odd graceful labeling. For a detailed survey on graph labeling we refer to Gallian [2]. The present work is aimed to discuss Fibonacci graceful labeling.

Definition 1.3 *A Fibonacci graceful labeling of G is an injective function $f : V \to \{0, 1, 2, \cdots, F_q\}$ such that the induced edge labeling $f(uv) = |f(u)-f(v)|$ is a bijection onto the set $\{F_1, F_2, \cdots, F_q\}$. If a graph G admits a Fibonacci graceful labeling then G is called a Fibonacci graceful graph.*

The notion of a Fibonacci graceful labeling was originated by Kathiresan and Amutha [4]. They have proved that K_n is Fibonacci graceful if and only if $n \leqslant 3$ and path P_n is Fibonacci graceful.

Illustration 1.4 The Fibonacci graceful labeling of $K_{1,6}$ and C_6 are shown in Fig.1.

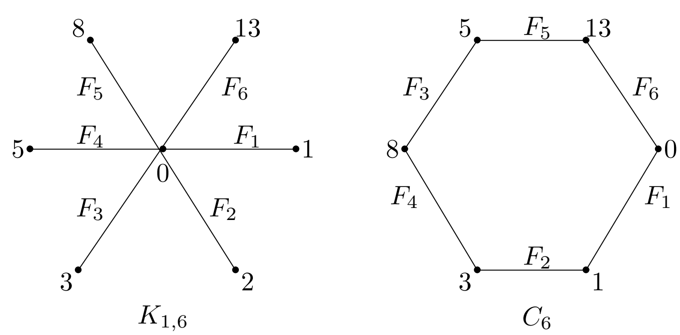

Fig.1

Definition 1.5 *Let $G = (V, E)$ be a graph with q edges. A super Fibonacci graceful labeling of G is an injective function $f : V \to \{0, F_1, F_2, \cdots, F_q\}$ such that the induced edge labeling $f(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \cdots, F_q\}$. If a graph G admits a super Fibonacci graceful labeling then G is called a super Fibonacci graceful graph.*

With reference to Definitions 1.3 and 1.5 we observe that in any (super) Fibonacci graceful graph there are two vertices having labels 0 and F_q and these vertices are adjacent.

Definition 1.6 *The graph obtained by identifying a vertex of a cycle C_n with a vertex of a cycle C_m is the graph with $|V| = m+n-1$, $|E| = m+n$ and is denoted by $< C_n : C_m >$.*

Definition 1.7 *The graph $G =< C_n : P_k : C_m >$ is the graph obtained by identifying one end vertex of P_k with a vertex of C_n and the other end vertex of P_k with a vertex of C_m.*

Definition 1.8([6]) *Let $G_1, G_2, \cdots, G_k$, $k \geqslant 2$ be k copies of a fixed graph G. Then the graph obtained by joining a vertex of G_i to the corresponding vertex of G_{i+1} by an edge for $i = 1, 2, \cdots, k-1$ is called a path union of $G_1, G_2, \cdots, G_k$.*

Motivated through this definition we define the following.

Definition 1.9 *Let $G_1, G_2, \cdots, G_k$, $k \geqslant 2$ be k graphs of a graph family. Adding an edge between G_i to G_{i+1} for $i = 1, 2, \cdots, k-1$ is called an arbitrary path union of $G_1, G_2, \cdots, G_k$.*

In the next section we investigate some new results on Fibonacci graceful graphs.

§2. Some results on Fibonacci Graceful Graphs

Theorem 2.1 *The graph obtained by joining a vertex of C_{3m} and a vertex of C_{3n} by an edge admits a Fibonacci graceful labeling.*

Proof Let the graph $G =< C_{3m} : P_2 : C_{3n} >$ is obtained by joining a vertex of a cycle C_n with a vertex of a cycle C_m by an edge.

Let the vertices of C_{3m} and C_{3n} in order be $v_0, v_1, v_2, \cdots, v_{3m-1}$ and $u_0, u_1, u_2, \cdots, u_{3n-1}$ respectively. Let u_o and v_0 be joined by an edge e. Then the vertex set of the graph is $V = \{v_0, v_1, v_2, \cdots, v_{3m-1}, u_0, u_1, u_2, \cdots, u_{3n-1}\}$ and the number of edges of G is $|E| = q = 3(m+n)+1$. Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(v_0) = 0$; for $i = 1, 2, 3, \cdots, 3m-1$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3); \end{cases}$$

$f(u_0) = F_q$ and for $i = 1, 2, 3, \cdots, 3n-1$

$$f(u_i) = \begin{cases} F_q + F_{3m+i} & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_q + F_{3m+i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_q + F_{3m+i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling for G. That is, G is a Fibonacci graceful graph. □

Illustration 2.2 The Fibonacci graceful labeling of the graph joining a vertex of C_9 and a vertex of C_6 by an edge is as shown in Fig.2.

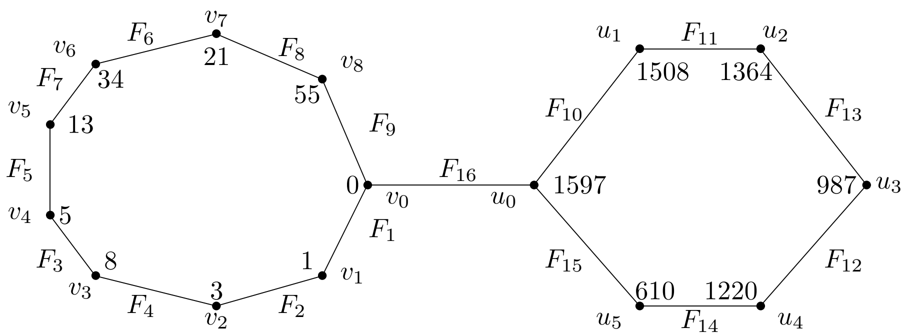

Fig.2

Theorem 2.3 *The graph obtained by joining a vertex of C_{3m} and a vertex of C_{3n} by a path P_3 admits Fibonacci graceful labeling.*

Proof Let the graph $G =< C_{3m} : P_3 : C_{3n} >$ is obtained by joining a vertex of a cycle C_{3m} with a vertex of a cycle C_{3n} by a path P_3.

Let the vertices of C_{3m} and C_{3n} be $v_0, v_1, v_2, \cdots, v_{3m-1}$ and $u_0, u_1, u_2, \cdots, u_{3n-1}$ respectively. Let u_o and v_0 be joined by a path $P_3 = u_0, w_1, v_0$. Here $V = \{v_0, v_1, v_2, \cdots, v_{3m-1}, w_1, u_0, u_1, u_2, \cdots, u_{3n-1}\}$ and the number of edges of G is $|E| = q = 3(m+n)+2$. Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(v_0) = 0$; for $i = 1, 2, \cdots, 3m-1$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3); \end{cases}$$

$f(w_1) = F_q$; $f(u_0) = F_{q-2}$ and for $i = 1, 2, \cdots, 3n-1$,

$$f(u_i) = \begin{cases} F_{q-2} + F_{3m+i} & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{q-2} + F_{3m+i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{q-2} + F_{3m+i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling of the graph G. That is, G is a Fibonacci graceful graph. □

Illustration 2.4 The Fibonacci graceful labeling of the graph joining a vertex of C_9 and a vertex of C_6 by a path P_3 is as shown in Fig.3.

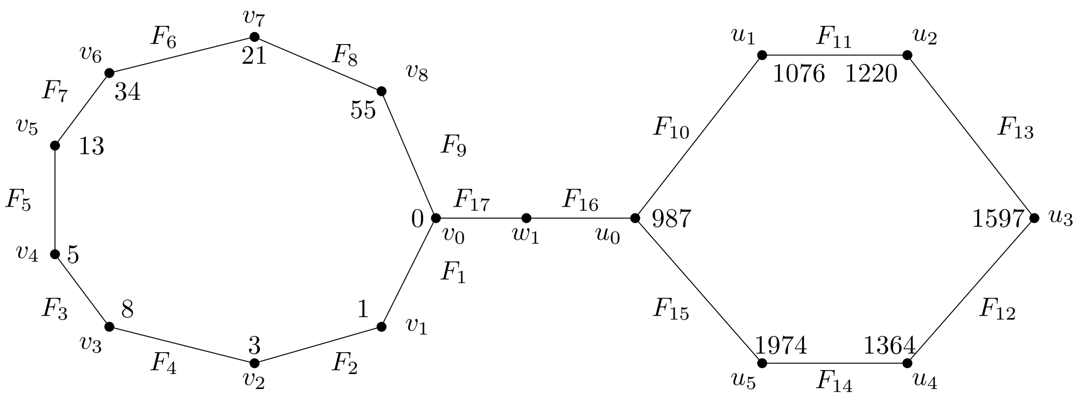

Fig.3

Theorem 2.5 *The graph obtained by joining a vertex of C_{3m} and a vertex of C_{3n} by a path P_k admits Fibonacci graceful labeling.*

Proof Let the graph $G =< C_{3m} : P_k :\ C_{3n} >$ is obtained by joining one vertex of a cycle C_n with one vertex of a cycle C_m by a path of length k.

Let the vertices of C_{3m} and C_{3n} be $v_0, v_1, v_2, \cdots, v_{3m-1}$ and $u_0, u_1, u_2, \cdots, u_{3n-1}$ respectively. Let v_0 and u_0 be joined by a path $P_k = w_0, w_1, w_2, \cdots, w_{k-1}$ on k vertices with $v_0 = w_0$ and $u_0 = w_{k-1}$. The vertex set of G is $V = \{v_0, v_1, \cdots, v_{3m-1}, u_0, u_1, \cdots, u_{3n-1}, w_1, w_2, \cdots, w_{k-2}\}$ and the number of edges of G is $|E| = q = 3(m + n) + k - 1$.

Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(v_0) = 0$; for $i = 1, 2, 3, \cdots, 3m - 1$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3); \end{cases}$$

for $i = 1, 2, 3, \cdots, k - 1$, $f(w_i) = \sum_{j=1}^{i} (-1)^{j-1} F_{q-(j-1)}$ and for $i = 1, 2, \cdots, 3n - 1$,

$$f(u_i) = \begin{cases} f(w_k) + F_{3m+i} & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ f(w_k) + F_{3m+i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ f(w_k) + F_{3m+i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling of the graph G. That is, G is a Fibonacci graceful graph. □

Illustration 2.6 A Fibonacci graceful labeling of the graph obtained by joining a vertex of C_9 and a vertex of C_6 by a path P_6 is shown in the following Fig.4.

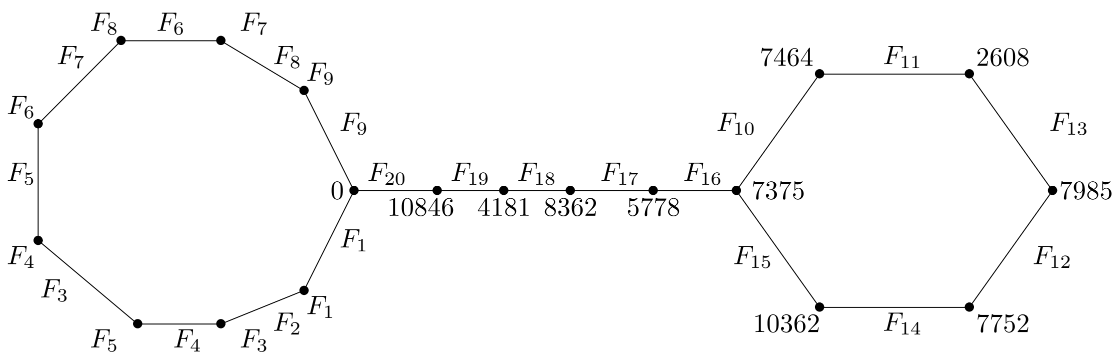

Fig.4

Theorem 2.7 *An arbitrary path union of $k-$copies of cycles C_{3m} is a Fibonacci graceful graph.*

Proof Let the graph G be obtained by attaching cycles $C^i_{3n_i}$ of length $3n_i$ at each of the vertices v_i of a path $P = v_0v_1v_2 \cdots v_{k-1}$ on k vertices. So the number of edges $|E| = q = 3(n_0+n_1+\cdots+n_{k-1})+k-1$. Let the vertices of each of the cycles $C^i_{3n_i}$ be $u_{i,0}, u_{i,1}, \cdots , u_{i,3n_i-1}$ for each $i = 0, 1, 2, \cdots , k-1$. Let the vertices $u_{0,0}, u_{1,0}, \cdots , u_{k-1,0}$ forms a path $P = u_{0,0}\ u_{1,0}\ \cdots\ u_{k-1,0}$. Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots , F_q\}$ as follows:

$f(u_{0,0}) = 0$; for $i = 1, 2, \cdots , k-1$, $f(u_{i,0}) = \sum_{j=1}^{i}(-1)^{j-1}F_{q-(j-1)}$; for $i = 1, 2, \cdots , n_0-1$;

$$f(u_{0,i}) = \begin{cases} F_i & \text{if } i \equiv 1 \ (\ mod\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (\ mod\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (\ mod\ 3); \end{cases}$$

for $j = 1, 2, \cdots , k-1$ and $i = 1, 2, \cdots , 3n_j - 1$,

$$f(u_{j,i}) = \begin{cases} f(u_{j,0}) + F_{3mj+i} & \text{if } i \equiv 1 \ (\ mod\ 3); \\ f(u_{j,0}) + F_{3mj+i+1} & \text{if } i \equiv 2 \ (\ mod\ 3); \\ f(u_{j,0}) + F_{3mj+i+2} & \text{if } i \equiv 0 \ (\ mod\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling for graph G. That is, G is a Fibonacci graceful graph. □

Illustration 2.8 In the following Fig.5 the path union of three cycles C_3, C_6 and C_9 with its Fibonacci graceful labeling is shown.

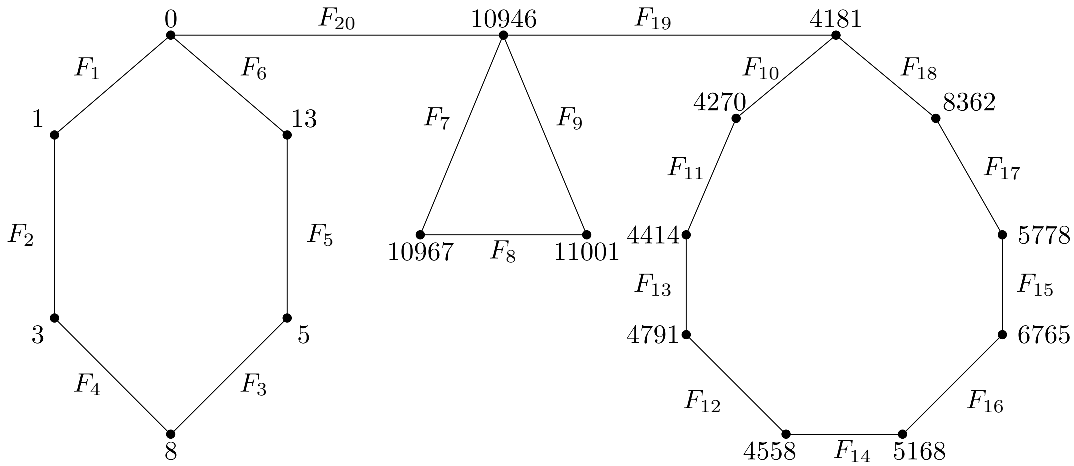

Fig.5

§3. Some Results on Super Fibonacci Graceful Graphs

Theorem 3.1 *One point union of two cycles C_{3m} and C_{3n} is a super Fibonacci graceful graph.*

Proof Let the vertices of C_{3m} and C_{3n} be $v_0, v_1, v_2, \cdots, v_{3m-1}$ and $u_0, u_1, u_2, \cdots, u_{3n-1}$ respectively. One point union of C_{3m} and C_{3n} is obtained by identifying u_0 and v_0. Then the vertex set of the resulting graph G is $V = \{v_0, v_1, v_2, \cdots, v_{3m-1}, u_1, u_2, \cdots, u_{3n-1}\}$ and the number of edges is $|E| = q = 3(m+n)$. Define $f : V \longrightarrow \{0, F_1, F_2, \cdots, F_q\}$ as follows:

$f(v_0) = 0$; for $i = 1, 2, 3, \cdots, 3m-1$

$$f(v_i) = \begin{cases} F_i & \text{if} \quad i \equiv 1 \quad (\,mod\ 3); \\ F_{i+1} & \text{if} \quad i \equiv 2 \quad (\,mod\ 3); \\ F_{i+2} & \text{if} \quad i \equiv 0 \quad (\,mod\ 3); \end{cases}$$

and for $i = 1, 2, 3, \cdots, 3n-1$,

$$f(u_i) = \begin{cases} F_{3m+i} & \text{if} \quad i \equiv 1 \quad (\,mod\ 3); \\ F_{3m+i+1} & \text{if} \quad i \equiv 2 \quad (\,mod\ 3); \\ F_{3m+i+2} & \text{if} \quad i \equiv 0 \quad (\,mod\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a super Fibonacci graceful labeling of the graph G. That is, G is a super Fibonacci graceful graph. □

Illustration 3.2 The super Fibonacci graceful labeling of $< C_9 : C_6 >$ is as shown in Fig.6.

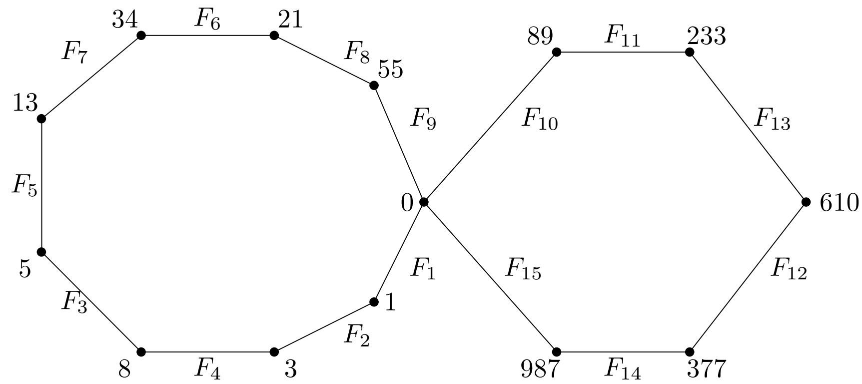

Fig.6

Theorem 3.3 *Every cycle C_n with $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$ is an induced subgraph of a super Fibonacci graceful graph while every cycle C_n with $n \equiv 2 \pmod 3$ can be embedded as a subgraph of a Fibonacci graceful graph.*

Proof Let the cycle C_n has the n vertices $v_0, v_1, \cdots, v_{n-1}$ in order. For the positive integer $n \geqslant 3$ we have the following three possibilities.

Case 1 If $n \equiv 0 \pmod 3$ then the cycle C_n is itself a super Fibonacci graceful.

Case 2 If $n \equiv 1 \pmod 3$ then $n = 3m+1$ for some positive integer m. Consider the graph G obtained from C_{3m+1} by adding an edge v_0v_{3m-1}. Then the number of edges of G is $|E| = q = 3m+2$. Define $f : V(G) \to \{0, F_1, F_2, \cdots, F_q\}$ as $f(v_0) = 0$ and for $i = 1, 2, 3, \cdots, 3m$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \pmod 3; \\ F_{i+1} & \text{if } i \equiv 2 \pmod 3; \\ F_{i+2} & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

So for $i \in \{1, 2, 3, \cdots, 3m\}$

$$f(v_{i-1}v_i) = \begin{cases} |F_{i+1} - F_i| & \text{if } i \equiv 1 \pmod 3; \\ |F_{i-1} - F_{i+1}| & \text{if } i \equiv 2 \pmod 3; \\ |F_i - F_{i+2}| & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

Thus

$$f(v_{i-1}v_i) = \begin{cases} F_{i-1} & \text{if } i \equiv 1 \pmod 3; \\ F_i & \text{if } i \equiv 2 \pmod 3; \\ F_{i+1} & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

Also $f(v_0v_{3m}) = |0 - F_{3m+2}| = F_{3m+2}$ and $f(v_0v_{3m-1}) = |F_{3m} - 0| = F_{3m}$. Here each vertex label is either zero or a Fibonacci number at the most F_q and each edge label is also a Fibonacci number at the most F_q. In view of the above defined labeling pattern f admits a super Fibonacci graceful labeling for graph G. That is, G is a super Fibonacci graceful graph.

Case 3 If $n \equiv 2 \pmod 3$ then $n = 3m + 2$ for some positive integer m. Consider the graph G obtained from C_{3m+2} by adding an edge v_0v_{3m-1} and one more edge $v_{3m}v_{3m+2}$ incident to the vertex v_{3m} and a new vertex v_{3m+2}. Then the number of edges of G is $|E| = q = 3m + 4$. Define $f : V(G) \to \{0, 1, 2, \cdots, F_q\}$ as $f(v_0) = 0$ and for $i = 1, 2, 3, \cdots, 3m$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \pmod 3; \\ F_{i+1} & \text{if } i \equiv 2 \pmod 3; \\ F_{i+2} & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

also $f(v_{3m+1}) = F_{3m+4}$, $f(v_{3m+2}) = 2F_{3m+2}$. So for $i \in \{1, 2, 3, \cdots, 3m\}$ we get that

$$f(v_{i-1}v_i) = \begin{cases} |F_{i+1} - F_i| & \text{if } i \equiv 1 \pmod 3; \\ |F_{i-1} - F_{i+1}| & \text{if } i \equiv 2 \pmod 3; \\ |F_i - F_{i+2}| & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

$$f(v_{i-1}v_i) = \begin{cases} F_{i-1} & \text{if } i \equiv 1 \pmod 3; \\ F_i & \text{if } i \equiv 2 \pmod 3; \\ F_{i+1} & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

Also $f(v_0v_{3m+1}) = |F_{3m+4} - 0| = F_{3m+4} = F_q$, $f(v_{3m}v_{3m-1}) = |F_{3m+2} - F_{3m}| = F_{3m+1}$ and $f(v_{3m}v_{3m+2}) = |F_{3m+2} - 2F_{3m+2}| = F_{3m+2}$.

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling for graph G. That is, G is a Fibonacci graceful graph. □

Remark 3.4 In Case 3, if $n \equiv 2 \pmod 3$ then $f(v_{3m+2}) = 2F_{3m+2}$ which is not a Fibonacci number. Therefore such embedding is not a super Fibonacci graceful. Thus to embed a cycle C_n with $n \equiv 2 \pmod 3$ as a subgraph of a super Fibonacci graceful graph remains an open problem.

Illustration 3.5 A super Fibonacci graceful embedding of the cycle C_7 is shown in Fig.7.

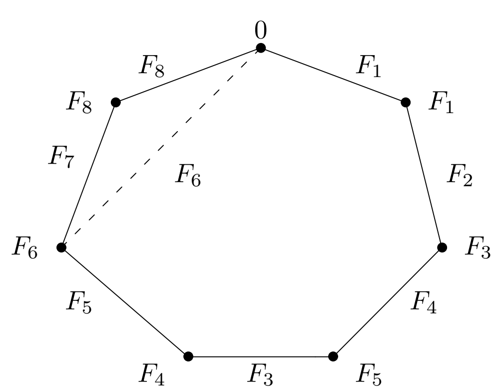

Fig.7

Illustration 3.6 A Fibonacci graceful embedding of the cycle C_8 is shown in Fig.8.

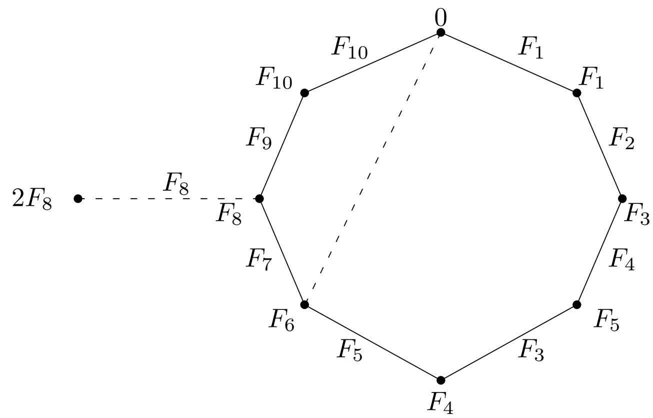

Fig.8

Theorem 3.7 *One point union of k cycles C_n (where $n \equiv 0 \pmod 3$) is a super Fibonacci graceful graph.*

Proof Let the graph G be obtained by taking one point union of k cycles $C^i_{3n_i}$ of order $3n_i$ for each $i = 0, 1, 2, 3 \cdots, k-1$. Let the vertices of each of the cycles $C^i_{3n_i}$ be $u_{i,0}, u_{i,1}, \cdots, u_{i,3n_i-1}$ for each $i = 0, 1, 2, \cdots, k-1$. Let the vertices $u_{0,0}, u_{1,0}, \cdots, u_{k-1,0}$ be identifying to a vertex u_0. So the number of edges $|E| = q = 3(n_0 + n_1 + n_2 + \cdots + n_{k-1})$.

Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(u_0) = 0$; for $i = 1, 2, \cdots, 3n_0 - 1$,

$$f(u_{0,i}) = \begin{cases} F_i & \text{if } i \equiv 1 \pmod 3; \\ F_{i+1} & \text{if } i \equiv 2 \pmod 3; \\ F_{i+2} & \text{if } i \equiv 0 \pmod 3; \end{cases}$$

and for each $j = 1, 2, 3, \cdots, k-1$,

$$f(u_{j,i}) = \begin{cases} F_{(3\sum_{t=0}^{j-1} n_t + i)} & \text{if } i \equiv 1 \pmod 3; \\ F_{(3\sum_{t=0}^{j-1} n_t + i + 1)} & \text{if } i \equiv 2 \pmod 3; \\ F_{(3\sum_{t=0}^{j-1} n_t + i + 2)} & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

In view of the above defined labeling pattern f admits a super Fibonacci graceful labeling of the graph G. That is, G is a super Fibonacci graceful graph. □

Illustration 3.8 A super Fibonacci graceful labeling of the one point union of three cycles C_3, C_6 and C_3 is as shown in Fig.9.

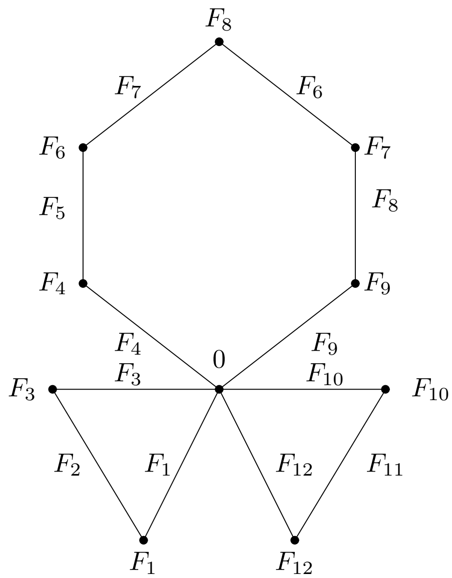

Fig.9

§4. **Concluding Remarks**

Here we investigate four new results corresponding to Fibonacci graceful labeling and three new results corresponding to super Fibonacci graceful labeling of graphs. Analogous results can be derived for other graph families and in the context of different graph labeling problems.

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