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Dual Spacelike Elastic Biharmonic Curves with Timelike Principal Normal According to Dual Bishop Frames

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Abstract: In this paper, we study dual spacelike elastic biharmonic curves with spacelike binormal in dual Lorentzian space $\mathbb{D}_1^3$. We use Noether's Theorem in our main theorem. Finally we obtain Killing vector field acoording to dual spacelike elastic biharmonic curves with spacelike binormal in dual Lorentzian space $\mathbb{D}_1^3$.

Key Words: Dual space curve, dual bishop frame, biharmonic curve.

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§1. Introduction

The Mathematical Theory of Elasticity is occupied with an attempt to reduce to calculation the state of strain, or relative displacement, within a solid body which is subject to the action of an equilibrating system of forces, or is in a state of slight internal relative motion, and with endeavours to obtain results which shall be practically important in applications to architecture, engineering, and all other useful arts in which the material of construction is solid. Its history should embrace that of the progress of our experimental knowledge of the behaviour of strained bodies, so far as it has been embodied in the mathematical theory, of the development of our conceptions in regard to the physical principles necessary to form a foundation for theory, of the growth of that branch of mathematical analysis in which the process of the calculations consists, and of the gradual acquisition of practical rules by the interpretation of analytical results.

Elastic structures con ned to a certain volume or area appear in many situations. For example inner membranes in biological cells separate an inner region from the rest of the cell and consist of an elastic bilayer. The inner structures are con ned by the outer cell membrane. Since the inner membrane contributes to the biological function it is advantageous to include a large membrane area in the cell.

In this paper, we study dual spacelike elastic biharmonic curves with spacelike binormal in dual Lorentzian space $\mathbb{D}_1^3$. We use Noether's Theorem in our main theorem. Finally we obtain Killing vector field acoording to dual spacelike elastic biharmonic curves with spacelike binormal in dual Lorentzian space $\mathbb{D}_1^3$.

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§2. Preliminaries

If φ and φ^* are real numbers and $\varepsilon^2 = 0$ the combination $\hat{\varphi} = \varphi + \varphi^*$ is called a *dual number*. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. Later, E.Study introduced the dual angle subtended by two nonparallel lines $\mathbb{E}^3$, and defined it as $\hat{\varphi} = \varphi + \varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines.

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

A dual point on $\mathbb{D}^3$ corresponds to a line in $\mathbb{E}^3$, two different points of $\mathbb{D}^3$ represents two skew lines in $\mathbb{E}^3$. A differentiable curve on $\mathbb{D}^3$ represents a ruled surface $\mathbb{E}^3$. The set

$$\mathbb{D}^3 = \left\{\hat{\varphi} : \hat{\varphi} = \varphi + \varepsilon\varphi^*,\ \varphi, \varphi^* \in \mathbb{E}^3\right\}$$

is a module over the ring $\mathbb{D}$.

The elements of $\mathbb{D}^3$ are called *dual vectors.* Thus a dual vector $\hat{\varphi}$ can be written

$$\hat{\mathbf{\Omega}} = \mathbf{\Omega} + \varepsilon\mathbf{\Omega}^*,$$

where φ and φ^* are real vectors in $\mathbb{R}^3$.

The *Lorentzian inner product* of dual vectors $\hat{\varphi}$ and $\hat{\psi}$ in $\mathbb{D}^3$ is defined by

$$\left\langle \hat{\mathbf{\Omega}}, \hat{\psi} \right\rangle = \langle \mathbf{\Omega}, \psi \rangle + \varepsilon \left(\langle \mathbf{\Omega}, \psi^* \rangle + \langle \mathbf{\Omega}^*, \psi \rangle \right),$$

with the Lorentzian inner product φ and ψ

$$\langle \mathbf{\Omega}, \psi \rangle = -\Omega_1\psi_1 + \Omega_2\psi_2 + \Omega_3\psi_3,$$

where $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$. Therefore, $\mathbb{D}^3$ with the Lorentzian inner product $\left\langle \hat{\mathbf{\Omega}}, \hat{\psi} \right\rangle$ is called *3-dimensional dual Lorentzian space* and denoted by of $\mathbb{D}^3_1$. For $\hat{\mathbf{\Omega}} \neq 0$, the *norm* $\left\| \hat{\mathbf{\Omega}} \right\|$ of $\hat{\mathbf{\Omega}}$ is defined by

$$\left\| \hat{\mathbf{\Omega}} \right\| = \sqrt{\left\langle \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}} \right\rangle}.$$

A dual vector $\hat{\mathbf{\Omega}} = \varphi + \varepsilon\varphi^*$ is called *dual spacelike vector* if $\left\langle \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}} \right\rangle > 0$ or $\hat{\mathbf{\Omega}} = \mathbf{0}$, *dual timelike vector* if $\left\langle \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}} \right\rangle < 0$ and *dual null (lightlike) vector* if $\left\langle \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}} \right\rangle = 0$ for $\hat{\mathbf{\Omega}} \neq 0$.

Therefore, an arbitrary dual curve, which is a differentiable mapping onto $\mathbb{D}^3_1$, can locally be dual space-like, dual time-like or dual null, if its velocity vector is respectively, dual spacelike, dual timelike or dual null.

§3. Spacelike Dual Biharmonic Curves with Spacelike Principal Normal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\hat{\gamma} = \gamma + \varepsilon\gamma^* : I \subset R \to \mathbb{D}_1^3$ be a C^4 dual spacelike curve with spacelike principal normal by the arc length parameter s. Then the unit tangent vector $\hat{\gamma}' = \hat{\mathbf{t}}$ is defined, and the principal normal is $\hat{\mathbf{n}} = \frac{1}{\hat{\kappa}}\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\|\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}\right\| = \kappa + \varepsilon\kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$**.** Hence, the triple $\left\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\right\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{aligned} \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}} &= \hat{\kappa}\hat{\mathbf{n}}, \\ \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{n}} &= \hat{\kappa}\hat{\mathbf{t}} + \hat{\tau}\hat{\mathbf{b}}, \\ \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{b}} &= \hat{\tau}\hat{\mathbf{n}}, \end{aligned} \tag{3.1}$$

where $\hat{\tau} = \tau + \varepsilon\tau^*$ is the dual torsion of the timelike dual curve $\hat{\gamma}$. Here, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual. In addition,

$$\begin{aligned} g\left(\hat{\mathbf{t}}, \hat{\mathbf{t}}\right) &= 1,\ g\left(\hat{\mathbf{n}}, \hat{\mathbf{n}}\right) = -1,\ g\left(\hat{\mathbf{b}}, \hat{\mathbf{b}}\right) = 1, \\ g\left(\hat{\mathbf{t}}, \hat{\mathbf{n}}\right) &= g\left(\hat{\mathbf{t}}, \hat{\mathbf{b}}\right) = g\left(\hat{\mathbf{n}}, \hat{\mathbf{b}}\right) = 0. \end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\hat{\kappa} \neq 0$ and $\hat{\tau} \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} &\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}} = \hat{k}_1\hat{\mathbf{m}}_1 - \hat{k}_2\hat{\mathbf{m}}_2, \\ &\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{m}}_1 = \hat{k}_1\hat{\mathbf{t}}, \\ &\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{m}}_2 = \hat{k}_2\hat{\mathbf{t}}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &g\left(\hat{\mathbf{t}}, \hat{\mathbf{t}}\right) = 1,\ g\left(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_1\right) = -1,\ g\left(\hat{\mathbf{m}}_2, \hat{\mathbf{m}}_2\right) = 1, \\ &g\left(\hat{\mathbf{t}}, \hat{\mathbf{m}}_1\right) = g\left(\hat{\mathbf{t}}, \hat{\mathbf{m}}_2\right) = g\left(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2\right) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\hat{\mathbf{t}}, \hat{\mathbf{m}}_1, \hat{\mathbf{m}}_1\}$ as Bishop trihedra, $\hat{k}_1$ and $\hat{k}_2$ as Bishop curvatures. Here $\tau(s) = \hat{\theta}'(s)$ and $\hat{\kappa}(s) = \sqrt{\left|\hat{k}_2^2 - \hat{k}_1^2\right|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} \hat{k}_1 &= \hat{\kappa}(s)\sinh\hat{\theta}(s), \\ \hat{k}_2 &= \hat{\kappa}(s)\cosh\hat{\theta}(s). \end{aligned} \tag{3.5}$$

Theorem 3.1 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike binormal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\begin{aligned} &\hat{k}_1^2 - \hat{k}_2^2 = \hat{\Omega}, \\ &\hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1 = 0, \\ &-\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2 = 0, \end{aligned} \tag{3.6}$$

where $\hat{\Omega}$ is dual constant of integration, [5].

Lemma 3.2 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike binormal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\begin{aligned} \hat{k}_1^2 - \hat{k}_2^2 &= \hat{\Omega}, \\ \hat{k}_1'' + \hat{k}_1\hat{\Omega} &= 0, \\ \hat{k}_2'' + \hat{k}_2\hat{\Omega} &= 0, \end{aligned} \tag{3.7}$$

where $\hat{\Omega} = \Omega + \varepsilon\Omega^*$ is constant of integration, [5].

Corollary 3.3 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike binormal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$k_1^2 - k_2^2 = -\Omega, \tag{3.8}$$

$$k_1k_1^* - k_2k_2^* = -\Omega^*. \tag{3.9}$$

§4. Dual Spacelike Elastic Biharmonic Curves with Timelike Normal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Consider regular curve (curves with nonvanishing velocity vector) in dual Lorentzian space $\mathbb{D}_1^3$ defined on a fixed interval $I = [a_1, a_2]$:

$$\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3.$$

We will assume (for technical reasons) that the curvature $\hat{\kappa}$ of $\hat{\gamma}$ is nonvanishing. The elastica minimizes the bending energy

$$\Pi(\hat{\gamma}) = \int_{\hat{\gamma}} \hat{\kappa}(s)^2 \, ds$$

with fixed length and boundary conditions. Accordingly, let α_1 and α_2 be points in $\mathbb{D}_1^3$ and α_1', α_2' nonzero vectors. We will consider the space of smooth curves

$$\Xi = \{\hat{\gamma} : \hat{\gamma}(a_i) = \hat{\alpha}_i, \ \hat{\gamma}'(a_i) = \hat{\alpha}_i'\},$$

and the subspace of unit-speed curves

$$\Xi_u = \{\hat{\gamma} \in \Omega : \|\hat{\gamma}'\| = 1\}.$$

Later on we need to pay more attention to the precise level of differentiability of curves, but we will ignore that for now.

$\Pi^\lambda : \Omega \longrightarrow \mathbb{D}$ is defined by

$$\Pi^\lambda(\hat{\gamma}) = \frac{1}{2}\int_{\hat{\gamma}} \left[\|\hat{\gamma}''\| + \hat{\Lambda}(t)(\|\hat{\gamma}'\| - 1)\right] dt,$$

where $\hat{\Lambda}(t) = \Lambda(t) + \varepsilon\Lambda^*(t)$ is a pointwise dual multiplier, constraining speed.

Theorem 4.1 *(Noether's Theorem) If $\hat{\gamma}$ is a solution curve and W is an infinitesimal symmetry, then*

$$\hat{\gamma}''.W' + \left(\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}'''\right).W$$

is constant. In particular, for a translational symmetry, W is constant; so

$$\left(\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}'''\right).W = \text{constant}.$$

Letting W range over all translations, we get

$$\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}''' = \hat{J}, \tag{4.1}$$

for $\hat{J}$ some constant field and

$$\hat{J} = J + \varepsilon J^*$$

Theorem 4.2 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike elastic biharmonic curves with spacelike binormal according to Bishop frame. Then,*

$$\Lambda(s) = 0 \text{ and } \Lambda^*(s) = 0. \tag{4.2}$$

Proof Now it is helpful to assume dual biharmonic curve $\hat{\gamma}$ is parametrized by arclength s. If we use dual Bishop frame (3.3), yields

$$\begin{aligned}
\hat{\gamma}' &= \hat{\mathbf{t}} \\
\hat{\gamma}'' &= \hat{k}_1\hat{\mathbf{m}}_1 - \hat{k}_2\hat{\mathbf{m}}_2, \\
\hat{\gamma}''' &= \left(\hat{k}_1^2 - \hat{k}_2^2\right)\hat{\mathbf{t}} + \hat{k}_1'\hat{\mathbf{m}}_1 - \hat{k}_2'\hat{\mathbf{m}}_2.
\end{aligned} \tag{4.3}$$

By means of dual function, $\varepsilon^2 = 0$ reduces to

$$\begin{aligned}
\hat{\gamma}' &= \mathbf{t} + \varepsilon\mathbf{t}^*, \\
\hat{\gamma}'' &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2 + \varepsilon(k_1^*\mathbf{m}_1 + k_1\mathbf{m}_1^* - k_2^*\mathbf{m}_2 - k_2\mathbf{m}_2^*), \\
\hat{\gamma}''' &= \left(k_1^2 - k_2^2\right)\mathbf{t} + k_1'\mathbf{m}_1 - k_2'\mathbf{m}_2 + \varepsilon(\left(k_2^2 - k_1^2\right)\mathbf{t}^* \\
&\quad + (2k_2k_2^* - 2k_1k_1^*)\mathbf{t} + k_1^{*\prime}\mathbf{m}_1 + k_1'\mathbf{m}_1^* - k_2^{*\prime}\mathbf{m}_2 - k_2'\mathbf{m}_2^*.
\end{aligned} \tag{4.4}$$

If we calculate the real and dual parts of this equation, we get the following relations

$$\begin{aligned}
\gamma' &= \mathbf{t}, \\
\gamma'' &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2, \\
\gamma''' &= \left(k_2^2 - k_1^2\right)\mathbf{t} + k_1'\mathbf{m}_1 - k_2'\mathbf{m}_2,
\end{aligned}$$

and

$$\begin{aligned}
\gamma^{*\prime} &= \mathbf{t}^*, \\
\gamma^{*\prime\prime} &= k_1^*\mathbf{m}_1 + k_1\mathbf{m}_1^* - k_2^*\mathbf{m}_2 - k_2\mathbf{m}_2^*, \\
\gamma^{*\prime\prime\prime} &= \left(k_2^2 - k_1^2\right)\mathbf{t}^* + (2k_2k_2^* - 2k_1k_1^*)\mathbf{t} \\
&\quad + k_1^{*\prime}\mathbf{m}_1 + k_1'\mathbf{m}_1^* - k_2^{*\prime}\mathbf{m}_2 - k_2'\mathbf{m}_2^*.
\end{aligned}$$

Using (4.1), we get

$$\begin{aligned} J &= \left(k_1^2 - k_2^2 - \Lambda\right)\mathbf{t} + k_1'\mathbf{m}_1 - k_2'\mathbf{m}_2, \\ J^* &= \left(k_1^2 - k_2^2 - \Lambda\right)\mathbf{t}^* + \left(-2k_2k_2^* + 2k_1k_1^* - \Lambda^*\right)\mathbf{t} \\ &\quad + k_1^{*\prime}\mathbf{m}_1 + k_1'\mathbf{m}_1^* - k_2^{*\prime}\mathbf{m}_2 - k_2'\mathbf{m}_2^*. \end{aligned} \tag{4.5}$$

If we take the derivative of $\hat{J}$ with respect to s, we get

$$\begin{aligned} \hat{J}_s &= \left(-\Lambda_s + k_1'k_1 - k_2'k_2\right)\mathbf{t} + \varepsilon[\left(-\Lambda_s + k_1'k_1 - k_2'k_2\right)\mathbf{t}^* \\ &\quad + (-\Lambda_s^* - k_2'k_2^* - k_2^{*\prime}k_2 + k_1'k_1^* + k_1^{*\prime}k_1)\mathbf{t}] \\ &\quad + [k_1'' - k_1\left(k_1^2 - k_2^2 - \Lambda\right)]\mathbf{m}_1 \\ &\quad + \varepsilon[k_1^{*\prime\prime} - \left(k_1^2 - k_2^2 - \Lambda\right)k_1^* + \left(-2k_2k_2^* + 2k_1k_1^* - \Lambda^*\right)k_1]\mathbf{m}_1 \\ &\quad + \varepsilon[k_1'' - k_1\left(k_1^2 - k_2^2 - \Lambda\right)]\mathbf{m}_1^* \\ &\quad - [k_2'' - k_2\left(k_1^2 - k_2^2 - \Lambda\right)]\mathbf{m}_2 \\ &\quad - \varepsilon[k_2^{*\prime\prime} - \left(k_1^2 - k_2^2 - \Lambda\right)k_2^* + \left(-2k_2k_2^* + 2k_1k_1^* - \Lambda^*\right)k_2]\mathbf{m}_2 \\ &\quad - \varepsilon[k_2'' - k_2\left(k_1^2 - k_2^2 - \Lambda\right)]\mathbf{m}_2^*. \end{aligned} \tag{4.6}$$

Then we calculate the real and dual parts of this equation, we get the following relations

$$\begin{aligned} J_s &= \left(-\Lambda_s + k_1'k_1 - k_2'k_2\right)\mathbf{t} + [k_1'' - k_1\left(k_1^2 - k_2^2 - \Lambda\right)]\mathbf{m}_1 \\ &\quad - [k_2'' - k_2\left(k_1^2 - k_2^2 - \Lambda\right)]\mathbf{m}_2, \end{aligned}$$

$$\begin{aligned} J_s^* &= \left(-\Lambda_s + k_1'k_1 - k_2'k_2\right)\mathbf{t}^* + (-\Lambda_s^* - k_2'k_2^* - k_2^{*\prime}k_2 + k_1'k_1^* + k_1^{*\prime}k_1)\mathbf{t} \\ &\quad + [k_1^{*\prime\prime} - \left(k_1^2 - k_2^2 - \Lambda\right)k_1^* + \left(-2k_2k_2^* + 2k_1k_1^* - \Lambda^*\right)k_1]\mathbf{m}_1 \\ &\quad + [k_1'' - k_1\left(k_1^2 - k_2^2 - \Lambda\right)]\mathbf{m}_1^* \\ &\quad - [k_2^{*\prime\prime} - \left(k_1^2 - k_2^2 - \Lambda\right)k_2^* + \left(-2k_2k_2^* + 2k_1k_1^* - \Lambda^*\right)k_2]\mathbf{m}_2 \\ &\quad - [k_2'' - k_2\left(k_1^2 - k_2^2 - \Lambda\right)]\mathbf{m}_2^*. \end{aligned}$$

Thus, by taking into consideration that (3.3) and (3.4), we complete the proof. □

Corollary 4.3 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike elastic biharmonic curves with timelike* *binormal according to Bishop frame. Then,*

$$
\begin{aligned}
J_s &= \left(k_1' k_1 - k_2' k_2\right)\mathbf{t} + [k_1'' - k_1\left(k_1^2 - k_2^2\right)]\mathbf{m}_1 \\
&\quad -[k_2'' - k_2\left(k_1^2 - k_2^2\right)]\mathbf{m}_2
\end{aligned}
$$

$$
\begin{aligned}
J_s^* &= \left(k_1' k_1 - k_2' k_2\right)\mathbf{t}^* + \left(-k_2' k_2^* - k_2^{*\prime} k_2 + k_1' k_1^* + k_1^{*\prime} k_1\right)\mathbf{t} \\
&\quad +[k_1^{*\prime\prime} - \left(k_1^2 - k_2^2\right)k_1^* + \left(-2k_2 k_2^* + 2k_1 k_1^*\right)k_1]\mathbf{m}_1 \\
&\quad +[k_1'' - k_1\left(k_1^2 - k_2^2\right)]\mathbf{m}_1^* \\
&\quad -[k_2^{*\prime\prime} - \left(k_1^2 - k_2^2\right)k_2^* + \left(-2k_2 k_2^* + 2k_1 k_1^*\right)k_2]\mathbf{m}_2 \\
&\quad -[k_2'' - k_2\left(k_1^2 - k_2^2\right)]\mathbf{m}_2^*
\end{aligned} \tag{4.7}
$$

Proof Using (4.2) and (4.6), we have (4.7). This completes the proof. □

Corollary 4.4 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike elastic biharmonic curves with timelike binormal according to Bishop frame. Then $\hat{J}$ is a Killing vector field.*

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