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Divisor Cordial Graphs

R.Varatharajan

(Department of Mathematics, Sri S.R.N.M.College, Sattur - 626 203, Tamil Nadu, India)

S.Navanaeethakrishnan

(Department of Mathematics, V.O.C.College, Tuticorin - 628 008, Tamil Nadu, India)

K.Nagarajan

(Department of Mathematics, Sri S.R.N.M.College, Sattur - 626 203, Tamil Nadu, India)

E-mail: varatharajansrnm@gmail.com, snk.voc@gmail.com, k_nagarajan_srnmc@yahoo.co.in

Abstract: A divisor cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, \cdots, |V|\}$ such that an edge uv is assigned the label 1 if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a divisor cordial labeling, then it is called divisor cordial graph. In this paper, we proved the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial. We also proved that complete graph is not divisor cordial.

Key Words: Cordial labeling, divisor cordial labeling, divisor cordial graph

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§1. Introduction

By a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [5].

First we give the some concepts in number theory [3].

Definition 1.1 *Let a and b be two integers. If a divides b means that there is a positive integer k such that $b = ka$. It is denoted by $a \mid b$.*

If a does not divide b, then we denote $a \nmid b$.

Now we give the definition of divisor function.

Definition 1.2 *The divisor function of integer $d(n)$ is defined by $d(n)$=Σ1. That is, $d(n)$ denotes the number of divisor of an integer n.*

Next we define the divisor summability function.

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Definition 1.3 *Let n be an integer and x be a real number. The divisor summability function is defined as $D(x)= \Sigma d(n)$. That is, $D(x)$ is the sum of the number of divisor of n for $n \leqslant x$.*

The big O notation is defined as follows.

Definition 1.4 *Let $f(x)$ and $g(x)$ be two functions defined on some subset of the real numbers. $f(x) = O(g(x))$ as $x \to \infty$ if and only if there is a positive real number M and a real number x_0 such that $|\ f(x)| \leqslant M \mid g(x)|$ for all $x > x_0$.*

Next, we state Dirichlet's divisor result as follows.

Result 1.5 *$D(x) = xlogx + x(2\gamma - 1) + \Delta(x)$ where γ is the Euler-Mascheroni Constant given by γ=0.577 approximately and $\Delta(x)=O(\sqrt{x})$.*

Graph labeling [4] is a strong communication between number theory [3] and structure of graphs [5]. By combining the divisibility concept in number theory and cordial labeling concept in Graph labeling, we introduce a new concept called divisor cordial labeling. In this paper, we prove the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial and complete graph is not divisor cordial.

A vertex labeling [4] of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a label depending on the vertex label $f(u)$ and $f(v)$. The two best known labeling methods are called graceful and harmonious labelings. Cordial labeling is a variation of both graceful and harmonious labelings [1].

Definition 1.6 *Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

The concept of cordial labeling was introduced by Cahit [1] and he got some results in [2].

Definition 1.7 *A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling*

§2. Main Results

Sundaram, Ponraj and Somasundaram [6] have introduced the notion of prime cordial labeling.

Definition 2.1([6]) *A prime cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1,2,\ldots,|V|\}$ such that if each edge uv assigned the label 1 if $gcd(f(u), f(v)) = 1$ and 0 if $gcd(f(u), f(v)) > 1$, then the number of edges labeled with 1 and the number of edges labeled with 0 differ by at most 1.*

In [6], they have proved some graphs are prime cordial. Motivated by the concept of prime cordial labeling, we introduce a new special type of cordial labeling called divisor cordial labeling as follows.

Definition 2.2 *Let $G = (V, E)$ be a simple graph and $f : V \to \{1, 2, \ldots |V|\}$ be a bijection. For each edge uv, assign the label 1 if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 if $f(u) \nmid f(v)$. f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.*

A graph with a divisor cordial labeling is called a divisor cordial graph.

Example 2.3 Consider the following graph G.

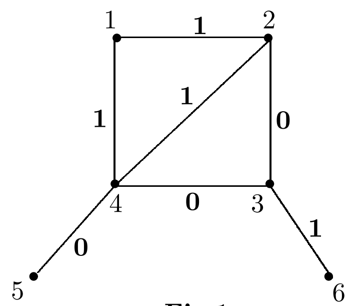

Fig.1

We see that $e_f(0)$= 3 and $e_f(1)$= 4. Thus $|e_f(0) - e_f(1)| = 1$ and hence G is divisor cordial.

Theorem 2.4 *The path P_n is divisor cordial.*

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n. Label these vertices in the following order.

$$\begin{array}{ccccc} 1, & 2, & 2^2, & \ldots, & 2^{k_1}, \\ 3, & 3 \times 2, & 3 \times 2^2, & \ldots, & 3 \times 2^{k_2}, \\ 5, & 5 \times 2, & 5 \times 2^2, & \ldots, & 5 \times 2^{k_3}, \\ \cdots & \cdots & \cdots & \cdots & \ldots, \\ \cdots & \cdots & \cdots & \cdots & \ldots. \end{array}$$

where $(2m - 1)2^{k_m} \leq n$ and $m \geq 1$, $k_m \geq 0$. We observe that $(2m - 1)2^a$ divides $(2m - 1)2^b$ $(a < b)$ and $(2m - 1)2^{k_i}$ does not divide $2m + 1$.

In the above labeling, we see that the consecutive adjacent vertices having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge. Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge.

Thus, $e_f(1) = \frac{n}{2}$ and $e_f(0) = \frac{n-2}{2}$ if n is even and $e_f(1) = e_f(0) = \frac{n-1}{2}$ if n is odd. Hence $|e_f(0) - e_f(1)| \leq 1$. Thus, P_n is divisor cordial. □

Theorem 2.4 can be illustrated in the following example.

Example 2.5 (1) n is even. Particularly, let $n = 12$.

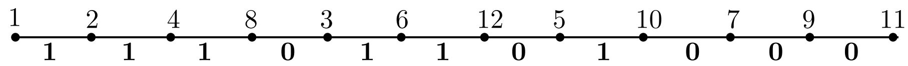

Fig.2

Here $e_f(1) = 6$ and $e_f(0) = 5$. Hence $|e_f(0) - e_f(1)| = 1$.

(2) n is odd. Particularly, let $n = 11$.

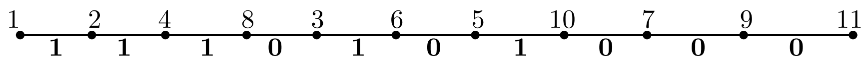

Fig.3

Here $e_f(0) = e_f(1) = 5$ and $|e_f(0) - e_f(1)| = 0$.

Observation 2.6 In the above labeling of path,

(1) the labels of vertices v_1 and v_2 must be 1 and 2 respectively, for all n. and

(2) the label of last vertex is always an odd number for $n \geq 3$.

In particular, the label v_n is n or $n-1$ according as n is odd or even.

Theorem 2.7 *The cycle C_n is divisor cordial.*

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n. We follow the same labeling pattern as in the path, except by interchanging the labels of v_1 and v_2. Then it follows from the observation (2). Thus C_n is divisor cordial. □

Theorem 2.8 *The wheel graph $W_n = K_1 + C_{n-1}$ is divisor cordial.*

Proof Let v_1 be the central vertex and $v_2, v_3, \ldots, v_n$ be the vertices of C_{n-1}.

Case 1 n is odd.

Label the vertices $v_1, v_2, \ldots, v_n$ as in the labels of cycle C_n in the Theorem 2.7, with the same order.

Case 2 n is even.

Label the vertices $v_1, v_2, \ldots, v_n$ as in the labels of cycle in the Theorem 2.7, with the same order except by interchanging the labels of the vertices v_2 and v_3.

In both the cases, we see that $e_f(0) = e_f(1) = n-1$. Hence $|e_f(0) - e_f(1)| = 0$. Thus, W_n is divisor cordial. □

The labeling pattern in the Theorem 2.8 is illustrated in the following example.

Example 2.9 (1) n is odd. Particularly, let $n = 11$.

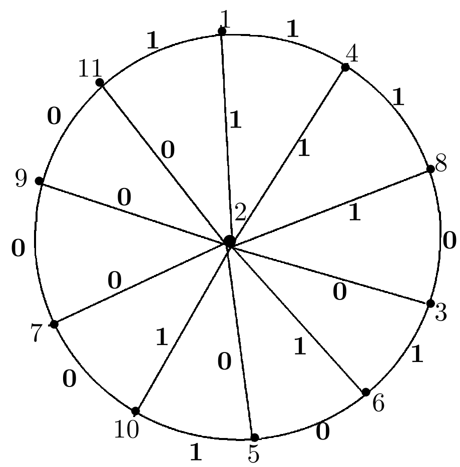

Fig.4

We see that $e_f(0) = e_f(1) = 10$.

(2) n is even. Particularly, let $n = 14$.

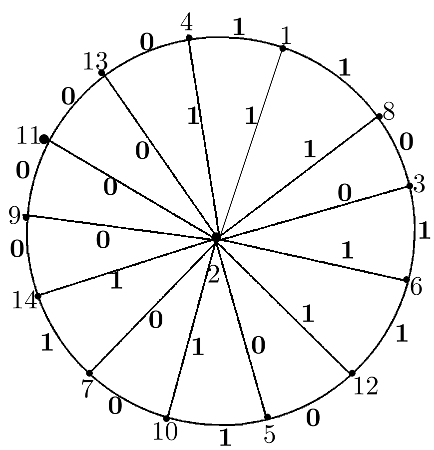

Fig.5

We see that $e_f(0) = e_f(1) = 13$.

Now we discuss the divisor cordiality of complete bipartite graphs.

Theorem 2.10 *The star graph $K_{1,n}$ is divisor cordial.*

Proof Let v be the central vertex and let $v_1, v_2, \ldots, v_n$ be the end vertices of the star $K_{1,n}$. Now assign the label 2 to the vertex v and the remaining labels to the vertices $v_1, v_2, \ldots, v_n$.

Then we see that

$$e_f(0) - e_f(1) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Thus $|e_f(0) - e_f(1)| \leq 1$ and hence $K_{1,n}$ is divisor cordial. □

Theorem 2.11 *The complete bipartite graph $K_{2,n}$ is divisor cordial.*

Proof Let $V = V_1 \cup V_2$ be the bipartition of $K_{2,n}$ such that $V_1 = \{x_1, x_2\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$. Now assign the label 1 to x_1 and the largest prime number p such that $p \leq n+2$ to x_2 and the remaining labels to the vertices $y_1, y_2, \ldots, y_n$. Then it follows that $e_f(0) = e_f(1) = n$ and hence $K_{2,n}$ is divisor cordial. □

Theorem 2.12 *The complete bipartite graph $K_{3,n}$ is divisor cordial.*

Proof Let $V = V_1 \cup V_2$ be the bipartition of V such that $V_1 = \{x_1, x_2, x_3\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$. Now define $f(x_1) = 1$, $f(x_2) = 2$, $f(x_3) = p$, where p is the largest prime number such that $p \leq n+3$ and the remaining labels to the vertices $y_1, y_2, \ldots, y_n$. Then clearly

$$e_f(0) - e_f(1) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $K_{3,n}$ is divisor cordial. □

Next we are trying to investigate the divisor cordiality of K_n. Obviously, K_1,K_2 and K_3 are divisor cordial. Now we consider K_4. The labeling of K_4 is given as follows.

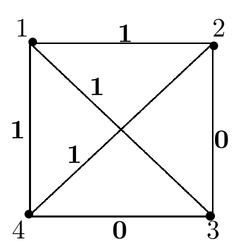

Fig.6

We see that $|e_f(0) - e_f(1)| = 2$ and hence K_4 is not divisor cordial. Next, we consider K_5.

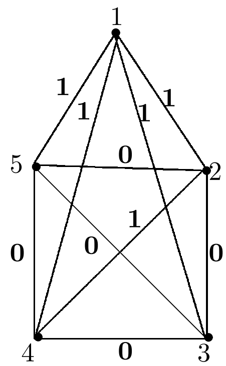

Fig.7

Here $|e_f(0) - e_f(1)| = 0$ and hence K_5 is divisor cordial. For the graph K_6, the labeling is given as follows.

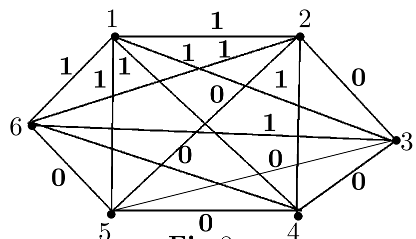

Fig.8

Here $|e_f(0) - e_f(1)| = 1$ and hence K_6 is divisor cordial. But K_n is not divisor cordial for $n \geq 7$, which will be proved in the following result.

Theorem 2.13 *K_n is not divisor cordial for $n \geq 7$.*

Proof If possible, let there be a divisor cordial labeling f for K_n. Let $v_1, \ldots, v_n$ be the vertices of K_n with $f(v_i) = i$. First we consider v_n. It contributes $d(n)$ and $(n-1) - d(n)$ respectively to $e_f(1)$ and $e_f(0)$. Consequently, the contribution of v_{n-1} to $e_f(1)$ and $e_f(0)$ are $d(n-1)$ and $n-2-d(n-1)$.

Proceeding likewise, we see that v_i contributes $d(i)$ and $i-1-d(i)$ to $e_f(1)$ and $e_f(0)$ respectively, for $i = n, n-1, \ldots, 2$. Then using Result 1.5, it follows that

$$\begin{aligned} |e_f(0) - e_f(1)| &= 2\{d(n) + \ldots + d(2)\} - \{(n-1) + \ldots + 1\} \\ &= 2\{D(n) - d(1)\} - \{\frac{(n-1)(n-2)}{2}\} \\ &= 2\{n \log n + n(2n-1) + \Delta(n) - 1\} - \{\frac{(n-1)(n-2)}{2}\} \\ &\geq 2 \end{aligned}$$

for $n \geq 7$. Thus, K_n is not divisor cordial. □

Theorem 2.14 *$S(K_{1,n})$, the subdivision of the star $K_{1,n}$, is divisor cordial.*

Proof Let $V(S(K_{1,n})) = \{v, v_i, u_i : \ 1 \leq i \leq n\}$ and let $E(S(K_{1,n})) = \{vv_i, v_iu_i : \ 1 \leq i \leq$

$n\}$. Define f by

$$f(v) = 1,$$
$$f(v_i) = 2i (1 \leq i \leq n)$$
$$f(u_i) = 2i + 1 \quad (1 \leq i \leq n).$$

Here $e_f(0) = e_f(1) = n$. Hence $S(K_{1,n})$ is divisor cordial. □

The following example illustrates this theorem.

Example 1.15 Consider $S(K_{1,7})$.

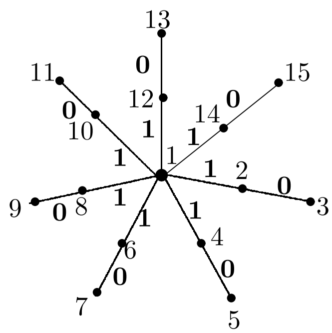

Fig.9

Here $e_f(0) = e_f(1) = 7$.

Theorem 2.16 *The bistar $B_{m,n}$ $(m \leq n)$ is divisor cordial.*

Proof Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(B_{m,n}) = \{uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Case 1 $m = n$.

Define f by

$$f(u) = 2,$$
$$f(u_i) = 2i + 1, (1 \leq i \leq n)$$
$$f(v) = 1,$$
$$f(v_j) = 2i + 2(1 \leq i \leq n).$$

Since $e_f(0) = e_f(1) = n$, it follows that f gives a divisor cordial labeling.

Case 2 $m > n$.

Subcase 1 $m + n$ is even.

Define f by

$$f(u) = 2,$$
$$f(u_i) = 2i + 1, (1 \leq i \leq \frac{m+n}{2}),$$
$$f(u_{\frac{m+n}{2}+i}) = 2n + 2 + 2i, (1 \leq i \leq \frac{m-n}{2}),$$
$$f(v) = 1,$$
$$f(v_j) = 2j + 2, 1 \leq j \leq n.$$

Since $e_f(0) = e_f(1) = \dfrac{m+n}{2}$, it follows that f is a divisor cordial labeling.

Subcase 2 $m + n$ is odd.

Define f by

$$f(u) = 2,$$
$$f(u_i) = 2i + 1, (1 \leq i \leq \frac{m+n+1}{2}),$$
$$f(u_{\frac{m+n+1}{2}+i}) = 2n + 2 + 2i, (1 \leq i \leq \frac{m-n-1}{2}),$$
$$f(v) = 1$$
$$f(v_j) = 2j + 2, (1 \leq j \leq n)$$

Since $e_f(0) = \dfrac{m+n+1}{2}$ and $e_f(1) = \dfrac{m+n-1}{2}$, $|e_f(0) - e_f(1)|= 1$. It follows that f is a divisor cordial labeling. □

The Case ii of the Theorem 2.16 is illustrated in the following example.

Example 2.17 (1) Consider $B_{10,6}$.

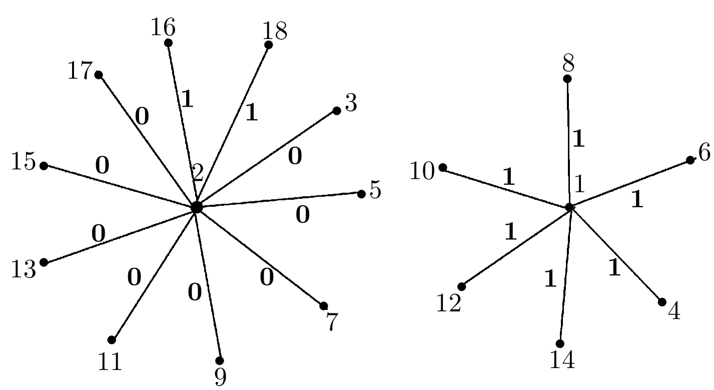

Fig.10

Here $e_f(0) = e_f(1) = 8$.

(2) Consider $B_{11,6}$. Here $e_f(0) = 9$, $e_f(1) = 8$.

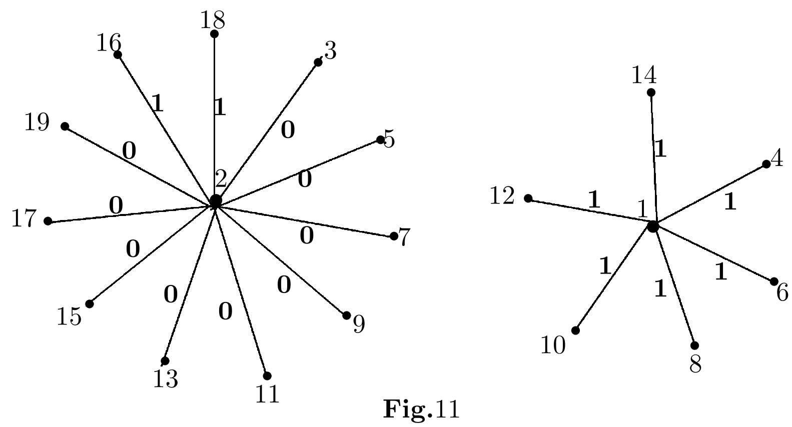

Fig.11

Theorem 2.18 *Let G be any divisor cordial graph of even size. Then the graph $G * K_{1,n}$ obtained by identifying the central vertex of $K_{1,n}$ with that labeled 2 in G is also divisor cordial.*

Proof Let q be the even size of G and let f be a divisor cordial labeling of G. Then it follows that, $e_f(0) = q/2 = e_f(1)$.

Let $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. Extend f to $G * K_{1,n}$ by assigning $f(v_i) = |V| + i (1 \leq i \leq n)$. In $G * K_{1,n}$, we see that $|e_f(0) - e_f(1)| = 0$ or 1 according to n is even or odd. Thus, $G * K_{1,n}$ is also divisor cordial. □

Theorem 2.19 *Let G be any divisor cordial graph odd size. If n is even, then the graph $G * K_{1,n}$ obtained by identifying the central vertex of $K_{1,n}$ with that labeled with* 2 *in G is also divisor cordial.*

Proof Let q be the odd size of G and let f be a divisor cordial labeling of G. Then it follows that, $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$.

Let $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. Extend f to $G * K_{1,n}$ by assigning $f(v_i) = |V| + i$ $(1 \leq i \leq n)$. Since n is even, the edges of $K_{1,n}$ contribute equal numbers to both $e_f(1)$ and $e_f(0)$ in $G * K_{1,n}$. Thus, $G * K_{1,n}$ is divisor cordial. □

§3. Conclusion

In the subsequent papers, we will prove that some cycle related graphs such as dragon, corona, wheel, wheel with two centres, fan, double fan, shell, books and one point union of cycles are divisor cordial. Also we will prove some special classes of graphs such as full binary trees, some star related graphs, $G * K_{2,n}$ and $G * K_{3,n}$ are also divisor cordial.

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