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Complete Fuzzy Graphs

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Abstract: In this paper, we provide three new operations on fuzzy graphs; namely direct product, semi-strong product and strong product.We give sufficient conditions for each one of them to be complete and we show that if any of these products is complete, then at least one factor is a complete fuzzy graph. Moreover, we introduce and study the notion of balanced fuzzzy graph and give necessary and sufficient conditions for the preceding products of two fuzzy balanced graphs to be balanced and we prove that any isomorphic fuzzy graph to a balanced fuzzy graph must be balanced.

Key Words: Neutrosophic set, fuzzy graph, complete fuzzy graph, balanced fuzzy graph.

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§1. Introduction

Graph theory has several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The concept of fuzzy relation which has a widespread application in pattern recognition was introduced by Zadeh [8] in his landmark paper "Fuzzy sets" in 1965. Fuzzy graph and several fuzzy analogs of graph theoretic concepts were first introduced by Rosenfeld [6] in 1975. Sense then, fuzzy graph theory is finding an increasing number of applications in modelling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems.

Mordeson and Peng [2] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. In [7], the definition of complement of a fuzzy graph was modified so that the complement of the complement is the original fuzzy graph, which agrees with the crisp graph case. Moreover some properties of self-complementary fuzzy graphs (fuzzy graphs that are isomorphic to their complements) and the complement of the operations of union, join and composition of fuzzy graphs that were introduced in [2] were studied. For more on the previous notions and the following ones, one can see [2]-[7].

A *neutrosophic set* based on neutrosophy, is defined for an element $x(T, I, F)$ belongs to the set if it is t true in the set, i indeterminate in the set, and f false, where t, i and f are

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real numbers taken from the sets T, I and F with no restriction on T, I, F nor on their sum $n = t + i + f$. Particularly, if $I = \emptyset$, we get the fuzzy set. Formally, a fuzzy subset of a non-empty set V is a mapping $\sigma : V \to [0,1]$ and a fuzzy relation μ on a fuzzy subset σ, is a fuzzy subset of $V \times V$. All throughout this paper, we assume that σ is reflexive, μ is symmetric and V is finite.

Definition 1.1([6]) *A fuzzy graph with V as the underlying set is a pair $G : (\sigma, \mu)$ where $\sigma : V \to [0,1]$ is a fuzzy subset and $\mu : V \times V \to [0,1]$ is a fuzzy relation on σ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$ and $\mu^* = \sup p(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}$.$H = (\sigma', \mu')$ is a* fuzzy subgraph *of G if there exists $X \subseteq V$ such that, $\sigma' : X \to [0,1]$ is a fuzzy subset and $\mu' : X \times X \to [0,1]$ is a fuzzy relation on σ' such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in X$.*

Definition 1.2([5]) *A fuzzy graph $G : (\sigma, \mu)$ is* complete *if $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.*

Next, we recall the following two results from [7].

Lemma 1.3 *Let $G : (\sigma, \mu)$ be a self-complementary fuzzy graph. Then $\sum_{x,y \in V} \mu(x,y) = (1/2) \sum_{x,y \in V} (\sigma(x) \wedge \sigma(y))$.*

Lemma 1.4 *Let $G : (\sigma, \mu)$ be a fuzzy graph with $\mu(x,y) = (1/2)(\sigma(x) \wedge \sigma(y))$ for all $x, y \in V$.Then G is self-complementary.*

Definition 1.5([1]) *Two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$ are* isomorphic *if there exists a bijection $h : V_1 \to V_2$ such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x,y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$.*

Lemma 1.6([3]) *Any two isomorphic fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ satisfy $\sum_{x \in V_1} \sigma_1(x) = \sum_{x \in V_2} \sigma_2(x)$ and $\sum_{x,y \in V_1} \mu_1(x,y) = \sum_{x,y \in V_2} \mu_2(x,y)$.*

In this paper, we provide three new operations on fuzzy graphs, namely direct product, semi-strong product and strong product. We give sufficient conditions for each one of them to be complete and show that if any one of these product of two fuzzy graphs is complete, then at least one of the two fuzzy graphs must be complete. Moreover, we introduce and study the notion of balanced fuzzy graph and show that this notion is weaker than complete and we give necessary and sufficient conditions for the direct product, semi-strong product and strong product of two balanced fuzzy graphs to be balanced. Finally we prove that given a balanced fuzzy graph G, then any isomorphic fuzzy graph to G must be balanced.

§2. Complete Fuzzy Graphs

We begin this section by defining three new fuzzy graphs products.

Definition 2.1 *The direct product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph G_1^* :*

(V_1, E_1) and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$,where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \sqcap G_2 : (\sigma_1 \sqcap \sigma_2, \mu_1 \sqcap \mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where

$$E = \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$$(\sigma_1 \sqcap \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v), \text{ for all } (u, v) \in V_1 \times V_2 \text{ and}$$
$$(\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1,u_2) \wedge \mu_2(v_1, v_2).$$

Definition 2.2 *The semi-strong product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$,where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \cdot G_2 : (\sigma_1 \cdot \sigma_2, \mu_1.\mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where*

$$E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$$(\sigma_1 \cdot \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v), \text{ for all } (u, v) \in V_1 \times V_2,$$
$$(\mu_1.\mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2) \quad \text{and}$$
$$(\mu_1.\mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1,u_2) \wedge \mu_2(v_1, v_2).$$

Definition 2.3 *The strong product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \otimes G_2 : (\sigma_1 \otimes \sigma_2, \mu \otimes \mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where*

$$E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, w)(u_2, w) : w \in V_2, (u_1, u_2) \in E_1\} \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$$(\sigma_1 \otimes \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v), \text{ for all } (u, v) \in V_1 \times V_2,$$
$$(\mu_1 \otimes \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2) \ ,$$
$$(\mu_1 \otimes \mu_2)((u_1, w)(u_2, w)) = \sigma_2(w) \wedge \mu_1(u_1, u_2) \text{ and}$$
$$(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1,u_2) \wedge \mu_2(v_1, v_2).$$

Next, we show that the direct product, the semi-strong product and the strong product of two complete fuzzy graphs are again fuzzy complete graphs.

Theorem 2.4 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $G_1 \sqcap G_2$ is complete.*

Proof If $(u_1, v_1)(u_2, v_2) \in E$,then since G_1 and G_2 are complete

$$\begin{aligned}(\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_1,u_2) \wedge \mu_2(v_1, v_2) \\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \\ &= (\sigma_1 \sqcap \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \sqcap \sigma_2)((u_2, v_2)).\end{aligned}$$

Hence, $G_1 \sqcap G_2$ is complete. □

Theorem 2.5 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $G_1 \bullet G_2$ is complete.*

Proof If $(u, v_1)(u, v_2) \in E$, then

$$\begin{aligned}(\mu_1 \bullet \mu_2)((u, v_1)(u, v_2)) &= \sigma_1(u) \wedge \mu_2(v_1, v_2)\\ &= \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \text{ (since } G_2 \text{ is complete)}\\ &= (\sigma_1 \bullet \sigma_2)((u, v_1)) \wedge (\sigma_1 \bullet \sigma_2)((u, v_2)).\end{aligned}$$

If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are complete

$$\begin{aligned}(\mu_1 \bullet \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2)\\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)\\ &= (\sigma_1 \bullet \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \bullet \sigma_2)((u_2, v_2)).\end{aligned}$$

Hence, $G_1 \bullet G_2$ is complete. □

Theorem 2.6 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $G_1 \otimes G_2$ is complete.*

Proof If $(u, v_1)(u, v_2) \in E$, then

$$\begin{aligned}(\mu_1 \otimes \mu_2)((u, v_1)(u, v_2)) &= \sigma_1(u) \wedge \mu_2(v_1, v_2)\\ &= \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \text{ (since } G_2 \text{ is complete)}\\ &= (\sigma_1 \otimes \sigma_2)((u, v_1)) \wedge (\sigma_1 \otimes \sigma_2)((u, v_2)).\end{aligned}$$

If $(u_1, w)(u_2, w) \in E$, then

$$\begin{aligned}(\mu_1 \otimes \mu_2)((u_1, w)(u_2, w)) &= \sigma_2(w) \wedge \mu_1(u_1, u_2)\\ &= \sigma_2(w) \wedge \sigma_1(u_1) \wedge \sigma_1(u_2) \text{ (since} G_1 \text{ is complete)}\\ &= (\sigma_1 \otimes \sigma_2)((u_1, w)) \wedge (\sigma_1 \otimes \sigma_2)((u_2, w)).\end{aligned}$$

If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are complete

$$\begin{aligned}(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2)\\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)\\ &= (\sigma_1 \otimes \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \otimes \sigma_2)((u_2, v_2)).\end{aligned}$$

Hence, $G_1 \otimes G_2$ is complete. □

An interesting property of complement is given next.

Theorem 2.7 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $\overline{G_1 \otimes G_2} \simeq \overline{G_1} \otimes \overline{G_2}$.*

Proof Let $G : (\sigma, \overline{\mu}) = \overline{G_1 \otimes G_2}$, $\overline{\mu} = \overline{\mu_1 \otimes \mu_2}$, $\overline{G^*} = (V, \overline{E})$, $\overline{G_1} : (\sigma_1, \overline{\mu_1})$, $\overline{G_1^*} = (V_1, \overline{E_1})$, $\overline{G_2} : (\sigma_2, \overline{\mu_2})$, $\overline{G_2^*} = (V_2, \overline{E_2})$and $\overline{G_1} \otimes \overline{G_2} : (\sigma_1 \otimes \sigma_2, \overline{\mu_1} \otimes \overline{\mu_2})$. We only need to show $\overline{\mu_1 \otimes \mu_2} = \overline{\mu_1} \otimes \overline{\mu_2}$. For any arc e joining nodes of V, we have the following cases:

Case 1 $e = (u, v_1)(u, v_2)$ where $(v_1, v_2) \in E_2$.Then as G is complete by Theorem 2.6, $\overline{\mu}(e) = 0$. On the other hand, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = 0$ since $(v_1, v_2) \notin \overline{E_2}$.

Case 2 $e = (u, v_1)(u, v_2)$ where $(v_1, v_2) \in E_2$ and $v_1 \neq v_2$. Since $e \in E$, $\mu(e) = 0$ and $\overline{\mu}(e) = \sigma(u, v_1) \wedge \sigma(u, v_2) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$ and as $(v_1, v_2) \in \overline{E_2}$, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = \sigma_1(u) \wedge \mu_2(v_1, v_2)$ and as $\overline{G_2}$ is complete, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$.

Case 3 $e = (u_1, w)(u_2, w)$ where $(u_1, u_2) \in E_1$. Since $e \in E$, $\overline{\mu}(e) = 0$ and as $(u_1, u_2) \notin \overline{E_1}$, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = 0$.

Case 4 $e = (u_1, w)(u_2, w)$ where $(u_1, u_2) \notin E_1$. Since $e \notin E$, $\mu(e) = 0$ and $\overline{\mu}(e) = \sigma(u_1, w) \wedge \sigma(u_2, w) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(w)$ and as $(u_1, u_2) \in \overline{E_1}$, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = \sigma_2(w) \wedge \overline{\mu_1}(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(w)$ since $\overline{G_1}$ is complete.

Case 5 $e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \notin E_1$ and $v_1 \neq v_2$. Since $e \in E$, $\overline{\mu}(e) = 0$ and as $(u_1, u_2) \in \overline{E_1}$, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = 0$.

Case 6 $e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \in E_1$ and $v_1 \neq v_2$. Since $e \notin E$, $\overline{\mu}(e) = 0$ and so $\overline{\mu}(e) = \sigma(u_1, w) \wedge \sigma(u_2, w) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$ and as $(u_1, u_2) \in \overline{E_1}$ and as $\overline{G_1}$ is complete, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = \overline{\mu_1}(u_1, u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) = \overline{\mu}(e)$.

Case 7 $e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \notin E_1$ and $(v_1, v_2) \notin E_2$. Since $e \notin E$, $\mu(e) = 0$ and $\overline{\mu}(e) = \sigma(u_1, w) \wedge \sigma(u_2, w) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$. As $(u_1, u_2) \in \overline{E_1}$ and if $v_1 = v_2$, then we have Case 4. If $(u_1, u_2) \in \overline{E_1}$ and $v_1 \neq v_2$, then we have Case 6.

In all cases $\overline{\mu_1 \otimes \mu_2} = \overline{\mu_1} \otimes \overline{\mu_2}$ and therefore, $\overline{G_1 \otimes G_2} \simeq \overline{G_1} \otimes \overline{G_2}$. □

By similar arguments to those in the preceding result and using Theorems 2.4 and 2.5, we can prove the following result.

Theorem 2.8 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$are fuzzy complete graphs, then $\overline{G_1 \sqcap G_2} \simeq \overline{G_1} \sqcap \overline{G_2}$ and $\overline{G_1 \bullet G_2} \simeq \overline{G_1} \bullet \overline{G_2}$.*

Next, we show that if the direct product, the semi-strong product or the strong product of two fuzzy graphs is complete, then at least one of the two fuzzy graphs must be complete.

Theorem 2.9 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are fuzzy graphs such that $G_1 \sqcap G_2$ is complete, then at least G_1 or G_2 must be complete.*

Proof Suppose that G_1 and G_2 are not complete. Then there exists at least one $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$ such that

$$\begin{aligned} \mu_1((u_1, v_1)) &< \sigma_1(u_1) \wedge \sigma_1(v_1)) \text{ and} \\ \mu_2((u_2, v_2)) &< \sigma_2(u_2) \wedge \sigma_2(v_2)) \end{aligned}$$

Now

$$\begin{aligned} (\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2) \\ &< \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \text{ (since } G_1 \text{ and } G_2 \text{ are complete).} \end{aligned}$$

But $(\sigma_1 \sqcap \sigma_2)((u_1, v_1)) = \sigma_1(u_1) \wedge \sigma_2(v_1)$ and $(\sigma_1 \sqcap \sigma_2)((u_2, v_2)) = \sigma_1(u_2) \wedge \sigma_2(v_2)$.Thus

$$\begin{aligned}(\sigma_1 \sqcap \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \sqcap \sigma_2)((u_2, v_2)) &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \\ &> (\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)).\end{aligned}$$

Hence, $G_1 \sqcap G_1$ is not complete, a contradiction. □

The next result can be proved in a similar manor to the preceding one.

Theorem 2.10 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are fuzzy graphs such that $G_1 \bullet G_2$ or $G_1 \otimes G_2$ is complete, then at least G_1 or G_2 must be complete.*

§3. Balanced Fuzzy Graphs

We begin this section by defining the density of a fuzzy graph and balanced fuzzy graphs. We then show that any complete fuzzy graph is balanced, but the converse need not be true.

Definition 3.1 *The density of a fuzzy graph $G : (\sigma, \mu)$ is*

$$D(G) = 2(\sum_{u,v \in V} \mu(u, v))/(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))).$$

G is balanced *if $D(H) \leq D(G)$ for all fuzzy non-empty subgraphs H of G.*

Theorem 3.2 *Any complete fuzzy graph is balanced.*

Proof Let G be a complete fuzzy graph. Then

$$\begin{aligned}D(G) &= 2(\sum_{u,v \in V} \mu(u, v))/(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))) \\ &= 2(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v)))/(\sum_{u,v \in V} \mu(u, v)) = 2\end{aligned}$$

.If H is a non-empty fuzzy subgraph of G,then

$$\begin{aligned}D(H) &= 2(\sum_{u,v \in V(H)} \mu(u, v))/(\sum_{u,v \in V(H)} (\sigma(u) \wedge \sigma(v))) \\ &\leq 2(\sum_{u,v \in V(H)} (\sigma(u) \wedge \sigma(v))/(\sum_{u,v \in V(H)} (\sigma(u) \wedge \sigma(v))) \\ & \quad 2(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))/(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))) \\ &= 2 = D(G).\end{aligned}$$

Thus G is balanced. □

The converse of the preceding result need not be true.

Example 3.3 The following fuzzy graph $G : (\sigma, \mu)$ is a balanced graph that is not complete.

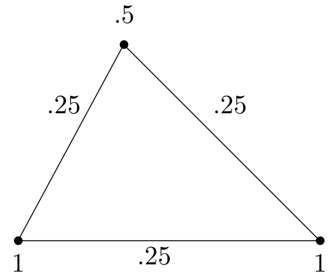

Fig.1

We next provide two types of fuzzy graphs each with density equals 1.

Theorem 3.4 *Every self-complementary fuzzy graph has density equals 1.*

Proof Let G be a self-complementary fuzzy graph. Then by Lemma 1.3,

$$\begin{aligned} D(G) &= 2(\sum_{u,v\in V}\mu(u,v))/(\sum_{u,v\in V}(\sigma(u)\wedge\sigma(v))) \\ &= 2(\sum_{u,v\in V}\mu(u,v))/(2\sum_{u,v\in V}\mu(u,v))) = 1. \end{aligned}$$

This completes the proof. □

The converse of the preceding result need not be true.

Example 3.5 The following fuzzy graph $G:(\sigma,\mu)$ has density equals 1, but it is not self-complementary.

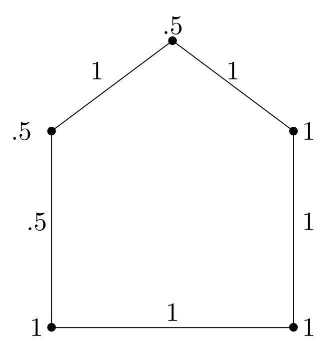

Fig.2

Theorem 3.6 *Let $G:(\sigma,\mu)$ be a fuzzy graph such that $\mu(u,v)=(1/2)(\sigma(u)\wedge\sigma(v))$, for all $u,v\in V$. Then $D(G)=1$.*

Proof Let $G:(\sigma,\mu)$ be a fuzzy graph such that $\mu(u,v)=(1/2)(\sigma(u)\wedge\sigma(v))$, for all $u,v\in V$. By Lemma 1.4, G is self-complementary and thus by the preceding Theorem $D(G)=1$. □

Next, we prove the following lemma that we use to give necessary and sufficient conditions for the direct product, semi-strong product and strong product of two fuzzy balanced graphs to be balanced.

Lemma 3.7 *Let G_1 and G_2 be fuzzy graphs. Then $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$ if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.*

Proof If $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$, then

$$\begin{aligned}
D(G_1) &= 2(\sum_{u_1,u_2\in V_1} \mu_1(u_1,u_2))/(\sum_{u_1,u_2\in V_1} (\sigma_1(u_1)\wedge\sigma_1(u_2))) \\
&\geq 2(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1(u_1,u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2))/(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1(u_1)\wedge\sigma_1(u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2))) \\
&= 2(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1(u_1,u_2)\wedge\mu_2(v_1,v_2))/(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1(u_1)\wedge\sigma_1(u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2))) \\
&= 2(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1\sqcap\mu_2((u_1,u_2)(v_1,v_2))/(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1\sqcap\sigma_2((u_1,u_2)(v_1,v_2))) \\
&= D(G_1\sqcap G_2).
\end{aligned}$$

Hence $D(G_1) \geq D(G_1 \sqcap G_2)$ and thus $D(G_1) = D(G_1 \sqcap G_2)$. Similarly, $D(G_2) = D(G_1 \sqcap G_2)$. Therefore, $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$. □

Theorem 3.8 *Let G_1 and G_2 be fuzzy balanced graphs. Then $G_1 \sqcap G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.*

Proof If $G_1 \sqcap G_2$ is balanced, then $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$ and by Lemma 3.7, $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.

Conversely, if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$ and H is a fuzzy subgraph of $G_1 \sqcap G_2$, then there exist fuzzy subgraphs H_1 of G_1 and H_2 of G_2. As G_1 and G_2 are balanced and $D(G_1) = D(G_2) = n_1/r_1$, then $D(H_1) = a_1/b_1 \leq n_1/r_1$ and $D(H_2) = a_2/b_2 \leq n_1/r_1$. Thus $a_1r_1 + a_2r_1 \leq b_1n_1 + b_2n_1$ and hence $D(H) \leq (a_1 + a_2)/(b_1 + b_2) \leq n_1/r_1 = D(G_1 \sqcap G_2)$. Therefore, $G_1 \sqcap G_2$ is balanced. □

By similar arguments to those in Lemma 3.7 and Theorem 3.8, we can prove the following result:

Theorem 3.9 *Let G_1 and G_2 be fuzzy balanced graphs. Then*

(1) *$G_1 \bullet G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \bullet G_2)$.*
(2) *$G_1 \otimes G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \otimes G_2)$.*

We end this section by showing that isomorphism between fuzzy graphs preserve balanced.

Theorem 3.10 *Let G_1 and G_2 be isomorphic fuzzy graphs. If G_2 is balanced, then G_1 is balanced.*

Proof Let $h : V_1 \to V_2$ be a bijection such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x,y) = \mu_2(h(x),h(y))$ for all $x,y \in V_1$. By Lemma 1.6, $\sum_{x\in V_1}\sigma_1(x) = \sum_{x\in V_2}\sigma_2(x)$ and $\sum_{x,y\in V_1}\mu_1(x,y) = \sum_{x,y\in V_2}\mu_2(x,y)$. If $H_1 = (\sigma_1^{'},\mu_1^{'})$ is a fuzzy subgraph of G_1 with underlying set W, then $H_2 = (\sigma_2^{'},\mu_2^{'})$ is a fuzzy subgraph of G_2 with underlying set $h(W)$ where $\sigma_2^{'}(h(x)) = \sigma_1^{'}(x)$ and $\mu_2^{'}(h(x),h(y)) = \mu_1^{'}(x,y)$ for all $x,y \in W$. Since G_2 is balanced, $D(H_2) \le D(G_2)$ and so

$$2(\sum_{x,y\in W}\mu_2^{'}(h(x),h(y)))/(\sum_{x,y\in W}(\sigma_2^{'}(x)\wedge\sigma_2^{'}(y))) \le 2(\sum_{x,y\in V_2}\mu_2(x,y))/(\sum_{x,y\in V_2}(\sigma_2(x)\wedge\sigma_2(y)))$$

and so

$$\begin{aligned} 2(\sum_{x,y\in W}\mu_1(x,y))/(\sum_{x,y\in W}(\sigma_2^{'}(x)\wedge\sigma_2^{'}(y))) &\le 2(\sum_{x,y\in V_1}\mu_1(x,y))/(\sum_{x,y\in V_2}(\sigma_2(x)\wedge\sigma_2(y))) \\ &\le D(G_1). \end{aligned}$$

Therefore, G_1 is balanced. □

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