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Separation for Triple-Harmonic Differential Operator in Hilbert Space

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Abstract: We separate the differential operator A of the form $Au(x) = -\Delta^3 u(x) + V(x)u(x)$ for all $x \in R^n$, in the Hilbert space $H = L_2(R^n, H_1)$ with the operator potential $V(x)$, where $L(H_1)$ is the space of all bounded operators on an arbitrary Hilbert space H_1, and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator on R^n. Moreover, we study the existence and uniqueness of the solution of the equation $Au(x) = -\Delta^3 u(x) + V(x)u(x) = f(x)$, in the Hilbert space H, where $f(x) \in H$.

Key Words: Separation, Tricomi differential operator, Hilbert space.

AMS(2010): 35K, 46K

§1. Introduction

The concept of separation for differential operators was first introduced by Everitt and Giertz [6, 7]. They have obtained the separation results for the Sturm Liouville differential operator

$$Ay(x) = -y''(x) + V(x)y(x), \quad x \in R, \tag{1}$$

in the space $L_2(R)$. They have studied the following question: What are the conditions on $V(x)$ such that if $y(x) \in L_2(R)$ and $Ay(x) \in L_2(R)$ imply that both of $y''(x)$ and $V(x)y(x) \in L_2(R)$? More fundamental results of separation of differential operators were obtained by Everitt and Giertz [8,9]. A number of results concerning the property referred to the separation of differential operators was discussed by Biomatov [2], Otelbaev [16], Zettle [21] and Mohamed etal [10-15]. The separation for the differential operators with the matrix potential was first studied by Bergbaev [1]. Brown [3] has shown that certain properties of positive solutions of disconjugate second order differential expressions imply the separation. Some separation criteria and inequalities associated with linear second order differential operators have been studied by many authors, see for examples [4,5,11,13,14,15,17,19].

Recently, Zayed [20] has studied the separation for the following biharmonic differential operator

$$Au(x) = \Delta\Delta u(x) + V(x)u(x), \quad x \in R^n, \tag{2}$$

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in the Hilbert space $H = L_2\left(R^n, H_1\right)$ with the operator potential $V\left(x\right) \in C^1(R^n, L\left(H_1\right))$ where $\Delta\Delta u\left(x\right)$ is the biharmonic differential operator, while $\Delta u\left(x\right) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2}$ is the Laplace operator in R^n.

The main objective of the present paper is to study the separation of the differential operator A of the form

$$Au\left(x\right) = -\Delta^3 u\left(x\right) + V\left(x\right) u\left(x\right). \tag{3}$$

We construct the coercive estimate for the differential operator (3). The existence and uniqueness Theorem for the solution of the differential equation

$$Au\left(x\right) = -\Delta^3 u\left(x\right) + V\left(x\right) u\left(x\right) = f\left(x\right) \tag{4}$$

in the Hilbert space $H = L_2\left(R^n, H_1\right)$ is also given, where $f\left(x\right) \in H$.

§2. Some Notations

In this section we introduce the definitions that will be used in the subsequent section. We consider the Hilbert space H_1 with the norm $\|.\|_1$and the inner product space $\langle u, v\rangle_1$ .We introduce the Hilbert space $H = L_2\left(R^n, H_1\right)$ of all functions $u(x), x \in R^n$ equipped with the norm

$$\|u\|^2 = \int_{R^n} \|u(x)\|_1^2 \, dx. \tag{5}$$

The symbol $\langle u, v\rangle$ where $u, v \in H$ denotes the scalar product in H which is defined by

$$\langle u, v\rangle = \int_{R^n} \langle u, v\rangle_1 \, dx. \tag{6}$$

Let $W_2^2\left(R^n, H_1\right)$ be the space of all functions $u\left(x\right)$, $x \in R$ which have generalized derivatives $D^\alpha u\left(x\right), \ \alpha \leq 2$ such that $u\left(x\right)$ and $D^\alpha u\left(x\right)$ belong to H. We say that the function $u\left(x\right)$ for all $x \in R$ belongs to $W_{2,\ loc}^2\left(R^n, H_1\right)$ if for all functions $Q\left(x\right) \in C_0^\infty\left(R^n\right)$ the functions $Q\left(x\right) y\left(x\right) \in W_{2,\ loc}^2\left(R^n, H_1\right)$.

§3. Main Results

Definition 3.1 *The operator A of the form $Au\left(x\right) = -\Delta^3 u\left(x\right) + V\left(x\right) u\left(x\right), \ x \in R^n$ is said to be separated in the Hilbert space H if the following statement holds:*

If $u(x) \in H \cap W_{2,\ loc}^2\left(R^n, H_1\right)$ and $Au\left(x\right) \in H$ imply that both $-\Delta^3 u\left(x\right)$ and $V\left(x\right) u\left(x\right) \in H$.

Theorem 3.1 *If the following inequalities are holds for all $x \in R^n$,*

$$\left\| V_0^{-\frac{1}{2}} \frac{\partial^3 v}{\partial x_i^3} V^{-1} V u \right\| \leq \sigma_1 \left\| V u \right\|, \tag{7}$$

$$\left\| V_0^{-\frac{1}{2}} \frac{\partial v}{\partial x_i} \frac{\partial^2 u}{\partial x_i^2} \right\| \leq \sigma_2 \left\| V u \right\|, \tag{8}$$

$$\left\| V_0^{-\frac{1}{2}} \frac{\partial u}{\partial x_i} \frac{\partial^2 v}{\partial x_i^2} \right\| \leq \sigma_3 \left\| Vu \right\|, \tag{9}$$

where σ_1, σ_2 and σ_3 are positive constants satisfy $\sigma_1 + \sigma_2 + \sigma_3 < \frac{2}{n}$, $V_0 = ReV$, then the coercive estimate

$$\|Vu\| + \left\|\Delta^3 u\right\| + \left\| \sum_{i=1}^{n} V_0^{\frac{1}{2}} \frac{\partial^3 u}{\partial x_i^3} \right\| \leq N \|Vu\|, \tag{10}$$

is valid, where

$$\begin{aligned} N &= 1 + 2\left[1 - \frac{n}{2\beta}(\sigma_1 + 2\sigma_2 + 3\sigma_3)\right]^{-1} \\ &+ \left[1 - \frac{n}{2\beta}(\sigma_1 + 2\sigma_2 + 3\sigma_3)\right]^{-\frac{1}{2}} \left[1 - \frac{n\beta}{2}(\sigma_1 + 2\sigma_2 + 3\sigma_3)\right]^{-\frac{1}{2}}, \end{aligned}$$

is a constant independent on $u(x)$ while β is given by $\frac{n}{2}(\sigma_1 + 2\sigma_2 + 3\sigma_3) < \beta < \frac{2}{n(\sigma_1 + 2\sigma_2 + 3\sigma_3)}$. Then the operator A given by (3) is separated in the Hilbert space H.

Proof From the definition of the inner product in the Hilbert space H, we can obtain $\left\langle \frac{\partial u}{\partial x_i}, v \right\rangle = -\left\langle u, \frac{\partial v}{\partial x_i} \right\rangle, i = 1, 2, 3, ..., n$ for all $u, v \in C_0^\infty(R^n)$.
Hence

$$\langle Au, Vu \rangle = \left\langle -\Delta^3 u + Vu, Vu \right\rangle = \left\langle -\Delta^3 u, Vu \right\rangle + \langle Vu, Vu \rangle$$

Setting $-\Delta^2 u = \Omega$, we have

$$\begin{aligned} \langle Au, Vu \rangle &= \langle \Delta\Omega, Vu \rangle + \langle Vu, Vu \rangle \\ &= \left\langle \sum_{i=1}^{n} \frac{\partial^2 \Omega}{\partial x_i^2}, Vu \right\rangle + \langle Vu, Vu \rangle \\ &= -\sum_{i=1}^{n} \left\langle \frac{\partial \Omega}{\partial x_i}, \frac{\partial (Vu)}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle \\ &= -\sum_{i=1}^{n} \left\langle \frac{\partial \Omega}{\partial x_i}, V\frac{\partial u}{\partial x_i} + u\frac{\partial V}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle \\ &= \sum_{i=1}^{n} \left\langle \Omega, \frac{\partial}{\partial x_i}(V\frac{\partial u}{\partial x_i}) \right\rangle + \sum_{i=1}^{n} \left\langle \Omega, \frac{\partial}{\partial x_i}(u\frac{\partial V}{\partial x_i}) \right\rangle + \langle Vu, Vu \rangle \\ &= \sum_{i=1}^{n} \left\langle \Omega, V\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i} \right\rangle + \sum_{i=1}^{n} \left\langle \Omega, u\frac{\partial^2 V}{\partial x_i^2} + \frac{\partial V}{\partial x_i}\frac{\partial u}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle \\ &= \sum_{i=1}^{n} \left\langle \Omega, V\frac{\partial^2 u}{\partial x_i^2} \right\rangle + \sum_{i=1}^{n} \left\langle \Omega, u\frac{\partial^2 V}{\partial x_i^2} \right\rangle + 2\sum_{i=1}^{n} \left\langle \Omega, \frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle. \end{aligned} \tag{11}$$

Setting $\Delta u = W$, we have $\Omega = -\Delta W$ and hence (11) reduces to

$$\begin{aligned}
\langle Au, Vu\rangle &= -\sum_{i=1}^{n}\left\langle \frac{\partial^2 W}{\partial x_i^2}, V\frac{\partial^2 u}{\partial x_i^2}\right\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial^2 W}{\partial x_i^2}, u\frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&\quad -2\sum_{i=1}^{n}\left\langle \frac{\partial^2 W}{\partial x_i^2}, \frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i}\right\rangle + \langle Vu, Vu\rangle \\
&= \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial xi}, \frac{\partial}{\partial x_i}(V\frac{\partial^2 u}{\partial x_i^2})\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_{1i}}, \frac{\partial}{\partial x_i}(u\frac{\partial^2 V}{\partial x_i^2})\right\rangle \\
&\quad +2\sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, \frac{\partial}{\partial x_i}(\frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i})\right\rangle + \langle Vu, Vu\rangle \\
&= \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, V\frac{\partial^3 u}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle \\
&\quad +3\sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial xi}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^i}\right\rangle + 3\sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle + \langle Vu, Vu\rangle \\
&= -\sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}(V\frac{\partial^3 u}{\partial x_i^3})\right\rangle - \sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}(u\frac{\partial^3 V}{\partial x_i^3})\right\rangle \\
&\quad -3\sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}(\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2})\right\rangle - 3\sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i})\right\rangle + \langle Vu, Vu\rangle \\
&= -\sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}V\frac{\partial^4 u}{\partial x_i^4}\right\rangle - \sum_{i=1}^{n}\left\langle W, u\frac{\partial^4 V}{\partial x_i^4}\right\rangle - 4\sum_{i=1}^{n}\left\langle W, \frac{\partial u}{\partial x_i}\frac{\partial^3 V}{\partial x_i^3}\right\rangle \\
&\quad -4\sum_{i=1}^{n}\left\langle W, \frac{\partial V}{\partial x_i}\frac{\partial^3 u}{\partial x_i^3}\right\rangle - 6\sum_{i=1}^{n}\left\langle W, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \langle Vu, Vu\rangle . \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, V\frac{\partial^4 u}{\partial x_i^4}\right\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, u\frac{\partial^4 V}{\partial x_i^4}\right\rangle - 4\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial u}{\partial x_i}\frac{\partial^3 V}{\partial x_i^3}\right\rangle \\
&\quad -4\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial V}{\partial x_i}\frac{\partial^3 u}{\partial x_i^3}\right\rangle - 6\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \langle Vu, Vu\rangle
\end{aligned} \tag{12}$$

Let us rewrite (12) as follows:

$$\langle Au, Vu\rangle = -I_1 - I_2 - 4I_3 - 4I_4 - 6I_5 + \langle Vu, Vu\rangle , \tag{13}$$

where

$$I_1 = \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, V\frac{\partial^4 u}{\partial x_i^4}\right\rangle = -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, V\frac{\partial^3 u}{\partial x_i^3}\right\rangle , \tag{14}$$

$$\begin{aligned}
I_2 &= \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, u\frac{\partial^4 V}{\partial x_i^4}\right\rangle = \sum_{i=1}^{n}\left\langle u\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^4 V}{\partial x_i^4}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(u\frac{\partial^2 u}{\partial x_i^2}), \frac{\partial^3 V}{\partial x_i^3}\right\rangle = -\sum_{i=1}^{n}\left\langle u\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^3 V}{\partial x_i^3}\right\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_i}, \frac{\partial^3 V}{\partial x_i^3}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_i}), \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n}\left\langle (\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial^2 V}{\partial x_i^2}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial u}{\partial x_i}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial V}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle - 2\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial V}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle - 2\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle, \qquad (15)
\end{aligned}$$

$$I_3 = \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^3 u}{\partial x_i^3}\frac{\partial V}{\partial x_i}\right\rangle = \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle, \qquad (16)$$

$$\begin{aligned}
I_4 &= \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial u}{\partial x_i}\frac{\partial^3 V}{\partial x_i^3}\right\rangle = \sum_{i=1}^{n}\left\langle \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^3 V}{\partial x_i^3}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i^2}), \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial u}{\partial x_i}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 V}{\partial x_i^2}\right\rangle - \sum_{i=1}^{n}\left\langle (\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial V}{\partial x_i}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + 2\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial V}{\partial x_i}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + 2\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle, \qquad (17)
\end{aligned}$$

$$\begin{aligned}
I_5 &= \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 V}{\partial x_i^2}\right\rangle = \sum_{i=1}^{n}\left\langle (\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial V}{\partial x_i}\right\rangle = -2\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial V}{\partial x_i}\right\rangle \\
&= -2\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle. \qquad (18)
\end{aligned}$$

Substitute(14)-(17) into (13) , we get

$$\begin{aligned}\langle Au,Vu\rangle &= \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, V\frac{\partial^3 u}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle + \\ & 2\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle + 3\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \langle Vu,Vu\rangle .\end{aligned} \tag{19}$$

Equating the real parts of the two sides of (19), we get

$$\begin{aligned}Re\,\langle Au,Vu\rangle &= \sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n} Re\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^3 V}{\partial x_i^3}V^{-1}Vu\right\rangle \\ & +2\sum_{i=1}^{n} Re\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle \\ & +3\sum_{i=1}^{n} Re\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \langle Vu,Vu\rangle .\end{aligned} \tag{20}$$

Since for any complex number Z,we have

$$-|Z| \le ReZ \le |Z| ,$$

then by the Cauchy- Schwarz inequality, we obtain

$$Re\,\langle Au,Vu\rangle \le |\langle Au,Vu\rangle| \le \|Au\|\,\|Vu\| . \tag{21}$$

With the help of (21) we have

$$\begin{aligned}Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^3 V}{\partial x_i^3}V^{-1}Vu\right\rangle &\ge -\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\left\|\sum_{i=1}^{n}V_0^{-\frac{1}{2}}\frac{\partial^3 V}{\partial x_i^3}V^{-1}Vu\right\| \\ &\ge -n\sigma_1\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\|Vu\| ,\end{aligned} \tag{22}$$

$$\begin{aligned}Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle &\ge -\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\left\|\sum_{i=1}^{n}V_0^{-\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\| \\ &\ge -n\sigma_2\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\|Vu\| ,\end{aligned} \tag{23}$$

$$\begin{aligned}Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle &\ge -\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\left\|\sum_{i=1}^{n}V_0^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\| \\ &\ge -n\sigma_3\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\|Vu\| .\end{aligned} \tag{24}$$

It is easy to see that for any $\beta > 0$ and $y_1,y_2 \in R^n$, the inequality

$$|y_1|\,|y_2| \le \frac{\beta\,|y_1|^2}{2} + \frac{|y_2|^2}{2\beta}. \tag{25}$$

is valid. Applying (25) to (22)-(24), we have

$$Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^3 V}{\partial x_i^3}V^{-1}Vu\right\rangle \geq -\frac{n\beta\sigma_1}{2}\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|^2 - \frac{n\sigma_1}{2\beta}\|Vu\|^2, \tag{26}$$

$$Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle \geq -\frac{n\beta\sigma_2}{2}\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|^2 - \frac{n\sigma_2}{2\beta}\|Vu\|^2, \tag{27}$$

$$Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle \geq -\frac{n\beta\sigma_3}{2}\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|^2 - \frac{n\sigma_3}{2\beta}\|Vu\|^2, \tag{28}$$

From (20),(21) and (26) - (28), we conclude that

$$\begin{aligned}&\left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2+3\sigma_3)\right]\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|^2 + \left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]\|Vu\|^2\\ \leq\ & \|Vu\|\,\|Au\|\end{aligned} \tag{29}$$

Choosing $\frac{n}{2}(\sigma_1+2\sigma_2+3\sigma_3) < \beta < \frac{2}{n}(\sigma_1+2\sigma_2+3\sigma_3)$, we obtain from (29) that

$$\|Vu\| \leq \left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-1}\|Au\|, \tag{30}$$

$$\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\| \leq \left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-\frac{1}{2}}\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-\frac{1}{2}}\|Au\|. \tag{31}$$

From (3) and (30) we get

$$\left\|\Delta^3 u\right\| \leq \|Vu\| + \|Au\| \leq \left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-1}\|Au\|. \tag{32}$$

Consequently, we deduce from (30)-(32) that

$$\|Vu\| + \left\|\Delta^3 u\right\| + \left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\| \leq N\|Au\| \tag{33}$$

where

$$\begin{aligned}N \;=\;& 1+2\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-1}\\ &+\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-\frac{1}{2}}\left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-\frac{1}{2}},\end{aligned}$$

and this completes the proof. □

Theorem 3.2 *If the operator A given by (3) is separated in the Hilbert space H and if there are positive functions $\psi(x)$ and $t(x) \in C^1(R^n)$ satisfying*

$$\left\|t^{-1}\left(\frac{\partial t}{\partial x_i}\right)V_0^{-\frac{1}{2}}\right\| \leq 2\sqrt{\sigma_1}, \tag{34}$$

$$\left\| \psi^{-1}(\frac{\partial \psi}{\partial x_i}) V_0^{-\frac{1}{2}} \right\| \leq 2\sqrt{\sigma_2}, \tag{35}$$

$$\left\| t^{\frac{1}{2}} \psi^{-\frac{1}{2}} (\frac{\partial u}{\partial x_i}) \right\| \leq 2\sqrt{\sigma_3} \left\| t^{\frac{1}{2}} \psi^{-\frac{1}{2}} V_0^{\frac{1}{2}} u \right\|, \tag{36}$$

$$\left\| t^{\frac{1}{2}} \psi^{-\frac{1}{2}} (\frac{\partial \Omega}{\partial x_i}) \right\| \leq \sqrt{\sigma_4} \left\| t^{\frac{1}{2}} \psi^{-\frac{1}{2}} \frac{\partial \psi}{\partial x_i} \right\|, \tag{37}$$

where $0 < \sigma_1 + \sigma_2 + \sigma_3 < \frac{\beta}{2n}$, *and* β *is defined in Theorem 1. Then the differential operator* $Au = -\Delta^3 u + Vu = f,\ f \in H$ *, has a unique solution on the Hilbert space* H.

Proof First, we prove the differential equation

$$Au = -\Delta^3 u + Vu = 0, \tag{38}$$

as the only zero solution $u(x) = 0$ for all $x \in R^n$. To this end we assume that $t(x)$ and $\psi(x)$ are two positive functions belonging to $C^1(R^n)$.

On setting $\Omega = -\Delta^2 u$, we get

$$\begin{aligned} \langle Vu, t\psi u \rangle &= \langle -\Delta\Omega, t\psi u \rangle = -\left\langle \sum_{i=1}^{n} \frac{\partial^2 \Omega}{\partial x_i^2}, t\psi u \right\rangle = \sum_{i=1}^{n} \left\langle \frac{\partial \Omega}{\partial x_i}, \frac{\partial}{\partial x_i}(t\psi u) \right\rangle \\ &= \sum_{i=1}^{n} \left\langle \frac{\partial \Omega}{\partial x_i}, t\psi \frac{\partial u}{\partial x_i} \right\rangle + \sum_{i=1}^{n} \left\langle \frac{\partial \Omega}{\partial x_i}, tu \frac{\partial \psi}{\partial x_i} \right\rangle + \sum_{i=1}^{n} \left\langle \frac{\partial \Omega}{\partial x_i}, u\psi \frac{\partial t}{\partial x_i} \right\rangle, \end{aligned} \tag{39}$$

Equating the real parts of both sides of (39), we have

$$\begin{aligned} \langle V_0 u, t\psi u \rangle &= \left\langle t^{\frac{1}{2}} \psi^{-\frac{1}{2}} V_0^{\frac{1}{2}} u, t^{\frac{1}{2}} \psi^{-\frac{1}{2}} V_0^{\frac{1}{2}} u \right\rangle \\ &= \sum_{i=1}^{n} Re \left\langle \frac{\partial \Omega}{\partial x_i}, t\psi \frac{\partial u}{\partial x_i} \right\rangle + \sum_{i=1}^{n} Re \left\langle \frac{\partial \Omega}{\partial x_i}, tu \frac{\partial \psi}{\partial x_i} \right\rangle \\ &\quad + \sum_{i=1}^{n} Re \left\langle \frac{\partial \Omega}{\partial x_i}, u\psi \frac{\partial t}{\partial x_i} \right\rangle. \end{aligned} \tag{40}$$

By using Cauchy- Schwarz inequality, we obtain

$$\begin{aligned} Re \left\langle \frac{\partial \Omega}{\partial x_i}, t\psi \frac{\partial u}{\partial x_i} \right\rangle &= Re \left\langle t^{\frac{1}{2}} \psi^{-\frac{1}{2}} \frac{\partial \Omega}{\partial x_i}, t^{\frac{1}{2}} \psi^{-\frac{1}{2}} \frac{\partial u}{\partial x_i} \right\rangle \\ &\leq \left\| t^{\frac{1}{2}} \psi^{-\frac{1}{2}} \frac{\partial \Omega}{\partial x_i} \right\| \left\| t^{\frac{1}{2}} \psi^{-\frac{1}{2}} \frac{\partial u}{\partial x_i} \right\|, \end{aligned} \tag{41}$$

$$\begin{aligned} Re \left\langle \frac{\partial \Omega}{\partial x_i}, tu \frac{\partial \psi}{\partial x_i} \right\rangle &= Re \left\langle t^{\frac{1}{2}} \psi^{-\frac{1}{2}} \frac{\partial \Omega}{\partial x_i}, t^{\frac{1}{2}} \psi^{-\frac{1}{2}} \left[\psi^{-1} \frac{\partial \psi}{\partial x_i} V_0^{-\frac{1}{2}} \right] V_0^{\frac{1}{2}} u \right\rangle \\ &\leq \left\| t^{\frac{1}{2}} \psi^{-\frac{1}{2}} \frac{\partial \Omega}{\partial x_i} \right\| \left\| t^{\frac{1}{2}} \psi^{\frac{-1}{2}} \left[\psi^{-1} \frac{\partial \psi}{\partial x_i} V_0^{-\frac{1}{2}} \right] V_0^{\frac{1}{2}} u \right\|, \end{aligned} \tag{42}$$

$$\begin{aligned} Re\left\langle \frac{\partial\Omega}{\partial x_i}, u\psi\frac{\partial t}{\partial x_i}\right\rangle &= Re\left\langle t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}, t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\left[t^{-1}\frac{\partial t}{\partial x_i}V_0^{-\frac{1}{2}}\right]V_0^{\frac{1}{2}}u\right\rangle \\ &\le \left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}\right\|\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\left[t^{-1}\frac{\partial t}{\partial x_i}V_0^{-\frac{1}{2}}\right]V_0^{\frac{1}{2}}u\right\|, \end{aligned} \tag{43}$$

From (25) in (41)- (43), we have

$$Re\left\langle \frac{\partial\Omega}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle \le \frac{\beta}{2}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}\right\|^2 + \frac{2}{\beta}\sigma_1\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2, \tag{44}$$

$$Re\left\langle \frac{\partial\Omega}{\partial x_i}, tu\frac{\partial \psi}{\partial x_i}\right\rangle \le \frac{\beta}{2}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}\right\|^2 + \frac{2}{\beta}\sigma_2\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2, \tag{45}$$

$$Re\left\langle \frac{\partial\Omega}{\partial x_i}, u\psi\frac{\partial t}{\partial x_i}\right\rangle \le \frac{\beta}{2}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}\right\|^2 + \frac{2}{\beta}\sigma_3\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2, \tag{46}$$

From (39) and (44)-(46) we have the following inequality:

$$\begin{aligned} \left[1-\frac{2n}{\beta}(\sigma_1+\sigma_2+\sigma_3)\right]\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2 &\le \frac{3\beta}{2}\sum_{i=1}^{n}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}\right\|^2 \\ &\le \frac{3\beta}{2}\sigma_4\sum_{i=1}^{n}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\psi}{\partial x_i}\right\|^2 \end{aligned} \tag{47}$$

Choosing $\psi(x) = 1$, for all $x \in R^n$, then if $0 < \frac{2n}{\beta}(\sigma_1+\sigma_2+\sigma_3) < 1$, we have

$$0 < \left[1-\frac{2n}{\beta}(\sigma_1+\sigma_2+\sigma_3)\right]\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2 \le 0, \tag{48}$$

and consequently, we have

$$0 < \left[1-\frac{2n}{\beta}(\sigma_1+\sigma_2+\sigma_3)\right]\int_{R^n}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|_1^2 dx \le 0. \tag{49}$$

The inequality (49) holds only for $u(x) = 0$. This proves that $u(x) = 0$ is the only solution of Eq.(38).

Second, We know that the linear manifold $M = \{f : Au = f, u \in C_0^\infty(R^n)\}$ is dense everywhere in H, so there exist a sequence of functions $\{u_r\} \in C_0^\infty(R^n)$, such that for all $f \in H, \|Au_r - f\| \longrightarrow 0$, as $r \longrightarrow \infty$.

By applying the coercive estimate (33), we find that

$$\|V(u_p-u_r)\| + \left\|\Delta^3(u_p-u_r)\right\| + \left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3(u_p-u_r)}{\partial x_i^3}\right\| \le N\,\|A(u_p-u_r)\|\,, \tag{50}$$

that $\|V(u_r-w_0)\|, \left\|\Delta^3(u_r-w_1)\right\|$ and $\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3(u_r-w_2)}{\partial x_i^3}\right\|$ are convergent to zero, as $r \longrightarrow \infty$. This implies that $u_r \longrightarrow V^{-1}w_0 = u$, $\Delta^3 u_r \longrightarrow \Delta^3 u$ and $\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u_r}{\partial x_i^3} \longrightarrow \sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}$ as $r \longrightarrow \infty$. Hence for any $f \in H$ there exist $u \in H \cap W_{2,loc}^2(R^n, H_1)$, such that $Au = f$.

Suppose that $\overline{u}$ is another solution of the equation $Au = f$, then $A(u - \overline{u}) = 0$. But $Au = 0$ has only zero solution, then $u = \overline{u}$ and the uniqueness is proved. Hence the proof of theorem 3.2 is completed. $\square$

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