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On k-Equivalence Domination in Graphs

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Abstract: Let $G = (V, E)$ be a graph. A subset S of V is called an *equivalence set* if every component of the induced subgraph $\langle S\rangle$ is complete. If further at least one component of $\langle V - S\rangle$ is not complete, then S is called a Smarandachely equivalence set. Let k be any nonnegative integer. An equivalence set $S \subseteq V$ is called a *k-equivalence set* if $\Delta(\langle S\rangle) \leq k$. A k-equivalence set which dominates G is called a *k-equivalence dominating set* of G. In this paper we introduce some parameters using the just defined notion and discuss their relations with other graph theoretic parameters.

Key Words: Domination, irredundance, Smarandachely equivalence set, k-equivalence set, k-equivalence domination, k-equivalence irredundance.

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In this paper we consider only finite undirected simple graphs. For graph theoretic terminology we rely on [5]. Throughout this article, let G be a graph with vertex set V and edge set E.

One of the dominant areas in graph theory is the study of domination and related notions such as independence, irredundance, covering and matching. (In this connection see [9-10].)

Let $v \in V$. The open neighbourhood of v denoted by $N(v)$ and the closed neighbourhood of v denoted by $N[v]$ are defined by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of V is said to be an *independent set* if no two vertices in S are adjacent. A subset S of V is called a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The cardinality of a minimum dominating set is called the *domination number* and it is denoted by $\gamma(G)$.

There are many variations of domination in graphs. In the book by Haynes et al. [9] it is proposed that a type of domination is "fundamental" if every connected nontrivial graph has a dominating set of this type and this type of dominating set S is defined in terms of some "natural" property of the subgraph induced by S. Examples include total domination, independent domination, connected domination and paired domination. In this paper we introduce

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the concept of k-equivalence domination, which is a fundamental concept in the above sense.

An *equivalence graph* is a vertex disjoint union of complete graphs. An *equivalence covering* of a G is a family of equivalence subgraphs of G such that every edge of G is an edge of at least one member of the family. The *equivalence covering number* of G is the cardinality of a minimum equivalence covering of G. The equivalence covering number was first studied in [6]. Interesting bounds for the equivalence covering number in terms of maximal degree of the complement were obtained in [2]. The computation of the equivalence covering number of split graphs was considered in [4].

An important concept which uses equivalence graph is subcoloring studied in [1,8,11]. A *subcoloring* of G is a partition of its vertex set into subsets $X_1, X_2, \ldots, X_k$, where for each $i \leq k$ the induced subgraph $\langle X_i \rangle$ is an equivalence graph. The order of a minimum subcoloring is called the *subchromatic number* of G. The notion of subchromatic number is a natural generalization of the well studied chromatic number since for any independent set S, the induced subgraph $\langle S \rangle$ is trivially an equivalence graph.

The concept of equivalence graph also arises naturally in the study of domination in claw-free graphs, as shown by the following theorem proved in [7].

Theorem 1([7]) *Any minimal dominating set of a $K_{1,3}$-free graph is a collection of disjoint complete subgraphs.*

Motivated by these observations, we have introduced the concept of equivalence set and equivalence domination number in [3].

Definition 2 *A subset S of V is called an equivalence set if every component of the induced subgraph $\langle S \rangle$ is complete. A dominating set of G which is also an equivalence set is called an equivalence dominating set of G. The equivalence domination number $\gamma_e(G)$ is defined to be the cardinality of a minimum equivalence dominating set of G. An equivalence set S is called a Smarandachely equivalence set if at least one component of $\langle V - S \rangle$ is not complete.*

In this paper we introduce the concept of k-equivalence set and several parameters using this concept and investigate their relation with the six basic parameters of the domination chain. (For details see [9, §3.5].)

Definition 3 *Let k be any nonnegative integer. A subset S of V is called a k-equivalence set if every component of the induced subgraph $\langle S \rangle$ is complete—i.e., if S is an equivalence set of G—and $\Delta(\langle S \rangle) \leq k$.*

The concept of k-equivalence set is a natural generalization of the concept of independence, since every independent set is obviously 0-equivalence set. Also every $(k-1)$-equivalence set is a k-equivalence set and k-equivalence is a hereditary property. Hence a k-equivalence set S is a maximal k-equivalence set if and only if $S \cup \{v\}$ is not a k-equivalence set for all $v \in V - S$. Thus a k-equivalence set $S \subseteq V$ is maximal if and only if for every $v \in V - S$, there exists a clique C in $\langle S \rangle$ such that v is adjacent to a vertex in C and v is not adjacent to a vertex in C or there exist two cliques C_1 and C_2 in $\langle S \rangle$ such that v is adjacent to a vertex in C_1 and to a vertex in C_2 or there exists a clique C in $\langle S\rangle$ such that $|C| = k+1$ and v is adjacent to all vertices in C.

Definition 4 *The k-equivalence number $\beta_e^k(G)$ and the lower k-equivalence number $i_e^k(G)$ are defined as follows.*

$$\beta_e^k(G) = \max\{|S| : S \textit{ is a maximal } k\textit{-equivalence set of } G\} \textit{ and}$$
$$i_e^k(G) = \min\{|S| : S \textit{ is a maximal } k\textit{-equivalence set of } G\}.$$

Clearly $i_e^k(G) \leq \beta_e^k(G)$ and $\beta_0(G) \leq \beta_e^k(G)$.

Definition 5 *A dominating set S of V which is also a k-equivalence set is called a k-equivalence dominating set of G. The k-equivalence domination number $\gamma_e^k(G)$ and the upper k-equivalence domination number $\Gamma_e^k(G)$ are defined by*

$$\gamma_e^k(G) = \min\{|S| : S \textit{ is a minimal } k\textit{-equivalence dominating set of } G\} \textit{ and}$$
$$\Gamma_e^k(G) = \max\{|S| : S \textit{ is a minimal } k\textit{-equivalence dominating set of } G\}.$$

Since every maximal k-equivalence set is a dominating set of G and every maximal independent set is a minimal k-equivalence dominating set, the parameters $\gamma_e^k(G)$ and $\Gamma_e^k(G)$ fit into the domination chain, thus leading to the following extended domination chain: $ir(G) \leq \gamma(G) \leq \gamma_e^k(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_e^k(G) \leq \Gamma(G) \leq IR(G)$.

Definition 6 *An irredundant set which is also a k-equivalence set is called a k-equivalence irredundant set. The k-equivalence irredundance number $ir_e^k(G)$ and the upper k-equivalence irredundance number $IR_e^k(G)$ are defined by*

$$ir_e^k(G) = \min\{|I| : I \textit{ is a maximal } k\textit{-equivalence irredundant set of } G\} \textit{ and}$$
$$IR_e^k(G) = \max\{|I| : I \textit{ is a maximal } k\textit{-equivalence irredundant set of } G\}.$$

Remark 7 Let S be a minimal k-equivalence dominating set of G. Since S is a minimal dominating set, it is a maximal irredundant set. Thus S is a maximal k-equivalence irredundant set of G. Thus we have the following: Any minimal k-equivalence dominating set is a maximal k-equivalence irredundant set.

For any G, we have $ir_e^k(G) \leq \gamma_e^k(G) \leq \Gamma_e^k(G) \leq IR_e^k(G)$ and $ir_e^k(G) \leq \gamma_e^k(G) \leq i_e^k(G) \leq \beta_e^k(G)$.

Lemma 8 *If D is a minimal k-equivalence dominating set of G, then D is both a minimal dominating set and a minimal $(k+1)$-equivalence dominating set of G.*

Proof Assume that D is a minimal k-equivalence dominating set of G, and let $x \in D$. Then $D - \{x\}$ is not a k-equivalence dominating set and $\Delta(\langle D - \{x\}\rangle) \leq \Delta(\langle D\rangle) \leq k$. Therefore D is a minimal dominating set of G and D is a $(k+1)$-equivalence set. □

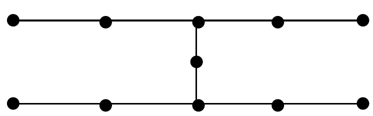
Corollary 9 *For every nonnegative integer k, $\gamma_e^{k+1}(G) \leq \gamma_e^k(G)$ and $\Gamma_e^k(G) \leq \Gamma_e^{k+1}(G)$.*

The proof of the next result is similar to that of Theorem 3.2 in [3].

Theorem 10 *For any graph G, $\gamma(G) \leq 2ir_e^k(G)$.*

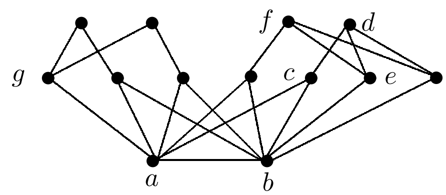
Proof Let $I = \{x_1, x_2, \ldots, x_k\}$ be an ir_e^k-set of G. Let y_i be a private neighbor of x_i with respect to I and let $A = I \cup \{y_1, y_2, \ldots, y_k\}$. If there exists a vertex x in $V - A$ such that $N(x) \cap (V - A) = \emptyset$, then $B = I \cup \{x\}$ is an k-equivalence set of G and x is an isolated vertex in $\langle B \rangle$. Further for each i, y_i is a private neighbor of x_i with respect to B; therefore B is a k-equivalence irredundant set— a contradiction. Whence A is a dominating set of G; therefore $\gamma(G) \leq 2ir_e^k(G)$. □

Let k be any integer ≥ 2. Consider the graphs H_1, H_2 displayed in Figure 1 and Figure 2 respectively. Obviously $ir_e^k(H_1) = 4$ and $\gamma(H_1) = 5$. Since $\{a, b, c\}$ is a maximal equivalence irredundant set in H_2, $ir_e^k(H_2) = 3$. Since $\{a, e, f, g\}$ is a maximal irredundant set in H_2, $ir(H_2) = 4$. Now for the graph $H_3 = P_3 \circ 2K_1$, we have $ir(H_3) = 3$, $ir_e^k(H_3) \geq 4$ and $\gamma(H_3) = 3$. From these information, it is clear that the parameters ir and ir_e^k and the parameters γ and ir_e^k are not comparable. It is not difficult to show that the just mentioned statement holds when $k \leq 1$.



H_1

Figure 1



H_2

Figure 2

For the complete bipartite graph $H_4 = K_{2,r}, r \geq 3$, we have $i_e^k(H_4) = 2$ and $\beta_0(H_4) = \Gamma_e^k(H_4) = \Gamma(H_4) = IR_e^k(H_4) = \beta_e^k(H_4) = r$. Also $i_e^k(K_n) = \beta_e^k(K_n) = k + 1 \leq n$ whereas $i(K_n) = \beta_0(K_n) = \Gamma_e^k(K_n) = \Gamma(K_n) = IR_e^k(K_n) = IR(K_n) = 1$. Further $i(K_n \circ 2K_1) = 2n - 1$ and $i_e^k(K_n \circ 2K_1) = 2n - (k+1)$. Hence i_e is not comparable with any of $IR, IR_e^k, \Gamma, \Gamma_e^k, i(G)$ and β_0. For the graph H_5 obtained from $K_{4,4,4} \circ K_1$, by adding edges in such a way that the subgraph induced by the set of all pendant vertices of the latter is a cycle, we have $\Gamma(H_5) = IR(H_5) = 12$ and $\beta_e^k(H_5) < 12$. Thus β_e^k is not comparable with IR and Γ.

Let H_6 be the graph obtained from the path $P_6 := (a_1, a_2, a_3, a_4, a_5, a_6)$ and the complete graph K_6 with $V(K_6) = \{b_1, b_2, b_3, b_4, b_5, b_6\}$ by adding the edges a_1b_1, a_2b_2, a_4b_4, a_5b_5 and a_6b_6. At least one vertex of $V(K_6)$ belongs to every dominating set of H_6 whence $\Gamma(H_6) = 4$. Since $\{b_1, b_2, b_4, b_5, b_6\}$ is an equivalence irredundant set of H_6, $IR_e^k(H_6) > \Gamma(H_6)$, when $k \geq 4$. It is not difficult to show that the just mentioned statement holds when $k \leq 3$. Also for the graph $H_7 = C_5 \Box K_2$, we have $\Gamma(H_7) = 5$ and $IR_e^k(H_7) = 4$. Thus Γ and IR_e^k are not comparable.

The following Hasse diagram summarizes the relationship between the various parameters for the graph G.

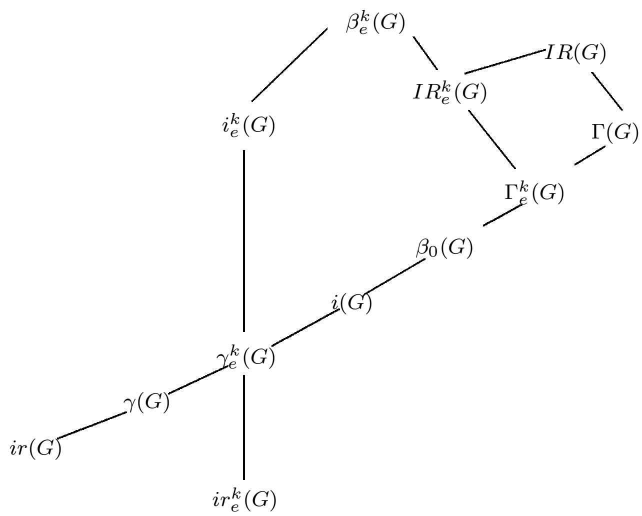


Figure 3. Relationship between parameters

Remark 11 It is easy to show that $\gamma_e^k(G) \leq i(G) \leq |V(G)| - \Delta(G)$.

Proposition 12 If G is connected, then

$$\gamma_e^k(G) \leq n - \left\lfloor \frac{2(\text{diam}(G)+1)}{3} \right\rfloor.$$

Proof Consider an arbitrary induced path P of length diam(G) in a connected graph G. Every interior vertex in diametrical path dominates at least 3 vertices in G and also there exists maximal k-equivalence set in $\langle V - P \rangle$. Therefore

$$\gamma_e^k(G) \leq n - (\text{diam}(G)+1) + \left\lfloor \frac{\text{diam}(G)+1}{3} \right\rfloor = n - \left\lfloor \frac{2}{3}(\text{diam}(G)+1) \right\rfloor.$$

Also this bound is sharp when $G \cong P_n$, where $n \equiv 2 \pmod 3$. □

Theorem 13 *If $\Delta(G) \geq 3$ and k is an integer such that $0 \leq k \leq \Delta - 3$, then $\gamma_e^k(G) \leq (\Delta - k - 1)\gamma_e(G) - (k+1)(\Delta - k - 2)$.*

Proof Let D be a γ_e-set of G. If D is k-equivalence set, then $\gamma_e^k(G) = \gamma_e(G)$. Assume $\Delta(\langle D \rangle) \geq k+1$. Let $x \in D$ such that $\deg_{\langle D \rangle}(x) \geq k+1$ and let $Q = N(x) \cap (V - D)$. Let P be the set of all private neighbors of x with respect to D. Clearly $P \neq \emptyset$. Let R be a minimum k-equivalence dominating set of $\langle P \rangle$ and let $D' = (D - \{x\}) \cup R$. Now $|R| \leq |Q| \leq \Delta - (k+1)$. It follows that the set D' is an equivalence dominating set of G and $\langle D' \rangle$ has fewer vertices of degree at least $k+1$ than $\langle D \rangle$. Let E be a minimal equivalence dominating set of G such that $E \subseteq D'$. Then

$$|E| \leq |D'| = |D| - 1 + |R| = \gamma_e(G) - 1 + |R| \leq \gamma_e(G) + \Delta - k - 2.$$

Continue to repeat the above process until no more vertices of degree larger than k exist in the resultant set. (Note that the number of such repetitions is at most $|D|-(k+1)$.) Hence

$$\begin{aligned}\gamma_e^k(G) &\le |D| \le \gamma_e(G) + (|D|-(k+1))(\Delta-k-2)\\ &= (\Delta-k-1)\gamma_e(G) - (k+1)(\Delta-k-2).\end{aligned}$$

The above bound is attained when $G = K_3 \circ 2K_1$. Here $\gamma_e(G) = \gamma_e^2(G) = 3, \gamma_e^1(G) = 4$. □

Theorem 14 *If k is an integer such that $0 \le k \le \omega - 3$, then $\gamma_e^k(G) \le \left(\frac{\omega-k}{2}\right)\gamma_e^{k+1}(G)$.*

Proof Let D be a γ_e^{k+1}-set of G. If D is k-equivalence set, then $\gamma_e^k(G) = \gamma_e^{k+1}(G)$. Let k be any nonnegative integer not more than $\omega - 3$. Suppose D is not a k-equivalence set. Let X be a subset of D such that for all x, $\deg_{\langle D\rangle}(x) = k+1$ and let Y be a minimum independent set of $\langle X\rangle$. Since every vertex of $X - Y$ has at least one of its $(k+1)$ neighbors in Y, $D - Y$ is a k-equivalence set. Note that there are $|Y|(k+1)$ edges between Y and $D-Y$. Since D is $(k+1)$-equivalence set, $|Y|(k+1) \le |D-Y|(k+1)$. Thus $|Y| \le \frac{1}{2}|D|$.

Let P be the set of all private neighbors of Y with respect to D and R be a minimum k-equivalence dominating set of $\langle P\rangle$. Then R dominates P and $D-Y$ dominates $V-P$. Therefore $R \cup (D-Y)$ is a k-equivalence dominating set and there are no edges between $D-Y$ and R. Since $|R| \le |P| \le |Y|(\omega-k-1)$, we obtain

$$\begin{aligned}\gamma_e^k(G) &\le |D| - |Y| + |R| \le |D| - |Y| + |Y|(\omega-k-1) = |D| + |Y|(\omega-k-2)\\ &\le |D| + \frac{|D|}{2}(\omega-k-2) = \left(\frac{\omega-k}{2}\right)\gamma_e^{k+1}(G).\end{aligned}$$

□

Theorem 15 *If $\gamma_e^k(G) \ge 2$, then $m \le \left\lfloor \frac{1}{2}(n-\gamma_e^k(G))(n-\gamma_e^k(G)+2)\right\rfloor$, where n and m are respectively, the order and the size of the graph G.*

Proof We prove this result by induction on number of vertices. We can assume that $n > 2$ for otherwise the proof is obvious; we can also assume that the result holds for any graph whose order is less than n. If $\gamma_e^k(G) = 2$, then also the conclusion holds. So assume that $\gamma_e^k(G) \ge 3$. Let $v \in V(G)$ with $\deg(v) = \Delta(G)$. Then by Remark 11, $|N(v)| = \Delta(G) \le n - \gamma_e^k(G)$; i.e., $\Delta(G) = n - \gamma_e^k(G) - r$ where $0 \le r \le n - \gamma_e^k(G)$. Let $S = V - N[v]$. Then $|S| = \gamma_e^k(G) + r - 1$. If $u \in N(v)$, then $(S - N(u)) \cup \{u, v\}$ is a dominating set of G and $\gamma_e^k(G) \le |S - N(u)| + 2$. Thus $\gamma_e^k(G) \le \gamma_e^k(G) + r - 1 - |S \cap N(u)| + 2$ and so $|S \cap N(u)| \le r+1$ for all $u \in N(v)$. Hence the number of edges between $N(v)$ and S, say ℓ_1, is at most $\Delta(G)(r+1)$.

Further, if D is a γ_e^k-set of $\langle S\rangle$, then $D \cup \{v\}$ is a k-equivalence dominating set of G. Hence $\gamma_e^k(G) \le |D \cup \{v\}|$, implying that $\gamma_e^k(\langle S\rangle) \ge \gamma_e^k(G) - 1 \ge 2$. Let ℓ_2 be the size of $\langle S\rangle$. By the inductive hypothesis,

$$\begin{aligned}\ell_2 &\le \left\lfloor \frac{1}{2}(|S| - \gamma_e^k(\langle S\rangle))(|S| - \gamma_e^k(\langle S\rangle) + 2)\right\rfloor\\ &\le \left\lfloor \frac{1}{2}(\gamma_e^k(G) + r - 1 - \gamma_e^k(G) + 1)(\gamma_e^k(G) + r - 1 - \gamma_e^k(G) + 1 + 2)\right\rfloor\\ &= \frac{1}{2}r(r+2).\end{aligned}$$

Let $\ell_3 = |E\langle N[v]\rangle|$. Note that for each u in $N(v)$ there are at most $r+1$ vertices in S which are adjacent to u. Therefore,

$$\begin{aligned}
|E| &= \ell_1 + \ell_2 + \ell_3 \\
&\leq \Delta(G)\cdot(r+1) + \frac{1}{2}r\cdot(r+2) + \Delta(G) + \frac{1}{2}\Delta(G)(\Delta(G)-r-2) \\
&= \frac{1}{2}(n-\gamma_e^k(G))(n-\gamma_e^k(G)+2) - \frac{1}{2}\Delta(G)(n-\gamma_e^k(G)-\Delta(G)) \\
&\leq \frac{1}{2}(n-\gamma_e^k(G))(n-\gamma_e^k(G)+2).
\end{aligned}$$

□

Concluding Remarks

We have proved that the decision problem corresponding to the parameters γ_e and Γ_e are NP-complete in [3]. Therefore the computations of γ_e^k and Γ_e^k are also NP-complete. The problem of designing efficient algorithms for computing the parameters in connection with a notion of k-equivalence for special classes of graphs is an interesting direction for further research. In particular one can attempt the design of such algorithms for families of graphs with bounded tree-width.

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