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On the Basis Number of the Wreath Product of Wheels with Stars

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Abstract: A cycle basis $\mathcal{B}$ of G is called a *Smarandachely* (k,d)*-fold* for integers $,d,k,\ d-k\geq 0$ if each edge of G occurs in at least k and at most d of the cycles in $\mathcal{B}$. Particularly, a Smarandachely $(0,d)$-fold basis is abbreviated to a d-fold basis. The basis number of a graph G is defined to be the least integer d such that G has a d-fold basis for its cycle space. In this work, the basis number for the wreath product of wheels with stars is investigated.

Key Words: Wreath product, Smarandachely (k,d)-fold basis, basis number.

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§1. Introduction

For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. The set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X\oplus Y=(X\backslash Y)\cup(Y\backslash X)$ and scalar multiplication $1\cdot X=X$ and $0\cdot X=\varnothing$ for all $X,Y\in\mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E},\oplus,\cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that for a connected graph G the dimension of the cycle space is the *cyclomatic number* or the *first Betti number,* $\dim\mathcal{C}(G)=|E(G)|-|V(G)|+1$.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a *Smarandachely* (k,d)*-fold* for integers $,d,k,\ d-k\geq 0$ if each edge of G occurs in at least k and at most d of the cycles in $\mathcal{B}$. Particularly, a Smarandachely $(0,d)$-fold basis is abbreviated to a d-fold basis. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The first important use of the basis number goes bask to 1937 when MacLane proved the following result (see [17]):

Theorem 1.1 (MacLane) *The graph G is planar if and only if $b(G)\leq 2$.*

Later on, Schmeichel [17] proved the existence of graphs that have arbitrary large basis number. In fact he proved that for any integer r there exists a graph with basis greater than

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or equal to r. Also, he proved that for $n \geq 5$, $b(K_n) = 3$ where K_n is the complete graph of n vertices. There after, Banks and Schmeichel [8] proved that $b(Q_n) = 4$ where Q_n is the n-cube. For the completeness, it should be mentioned that a basis $\mathcal{B}$ of the cycle space $\mathcal{C}(G)$ of a graph G is Smarandachely if each edge of G occurs in at least 2 of the cycles in $\mathcal{B}$. The following result will be used frequently in the sequel [15]:

Lemma 1.2 (Jaradat, et al.) *Let A, B be sets of cycles of a graph G, and suppose that both A and B are linearly independent, and that $E(A) \cap E(B)$ induces a forest in G (we allow the possibility that $E(A) \cap E(B) = \varnothing$). Then $A \cup B$ is linearly independent.*

From 1982 more attention has given to address the problem of finding the basis number in graph products. In the literature there are a lot of graph products. In fact, there are more than 256 different kind of products, we mention out of these product the most common ones, The Cartesian, the direct, the strong the lexicographic, semi-composite and the wreath product. The first four of the above products were extensively studied by many authors, we refer the reader to the following articles and references cited there in: [2], [4], [5], [6], [7], [9], [10], [11], [13], [14], [15] and [16]. In contrast to the first four products, a very little is known about the basis number of the wreath products, ρ, of graphs. Schmeichel [18] proved that $b(P_2 \rho N_m) \leq 4$. Ali [1] proved that $b(K_n \rho N_m) \leq 9$. Al-Qeyyam and Jaradat [3] proved that $b(S_n \rho P_m), b(S_n \rho S_m) \leq 4$. In this paper, we investigate the basis number of the wreath product of wheel graphs with stars.

For completeness we give the definition of the following products: Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. (1) The Cartesian product $G \Box H$ has the vertex set $V(G \Box H) = V(G) \times V(H)$ and the edge set $E(G \Box H) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$. (2) The Lexicographic product $G_1[G_2]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)\}$. (3) The wreath product $G \rho H$ has the vertex set $V(G \rho H) = V(G) \times V(H)$ and the edge set $E(G \rho H) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2$ and $v_1v_2 \in H$, or $u_1u_2 \in G$ and there is $\alpha \in$Aut(H) such that $\alpha(v_1) = v_2\}$.

In the rest of this paper, we let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of the wheel W_n(the star S_n), with $d_{W_n}(u_1) = n - 1$ ($d_{S_n}(u_1) = n - 1$) and $\{v_1, v_2, \ldots, v_m\}$ be the vertex set S_m with $d_{S_m}(v_1) = m - 1$. Also, $C_{n-1} = u_2u_3 \ldots u_nu_2$ and N_{m-1} is the null graph with vertex set $\{v_2, v_3, \ldots, v_m\}$. Wherever they appear a, b, c, d and l stand for vertices. Also, $f_B(e)$ stands for the number of elements of B containing the edge e, and $E(B) = \cup_{C \in B} E(C)$ where $B \subseteq \mathcal{C}(G)$.

§2. The Basis Number of $W_n \rho S_m$

Throughout this work, we set the following sets of cycles:

$$\mathcal{H}_{ab} = \{(a, v_j)(b, v_i)(a, v_{j+1})(b, v_{i+1})(a, v_j) \mid 2 \leq i, j \leq m - 1\},$$

$$\mathcal{E}_{cab} = \left\{\mathcal{E}_{cab}^{(j)} = (c, v_2)(a, v_j)(b, v_m)(a, v_{j+1})(c, v_2) \mid 2 \leq j \leq m - 1\right\},$$

$$\mathcal{G}_{ab} = \left\{ \mathcal{G}_{ab}^{(j)} = (a,v_1)(a,v_j)(b,v_2)(a,v_{j+1})(a,v_1) \mid 2 \le j \le m-1 \right\},$$

$$\mathcal{W}_{cab} = \{(c,v_1)(c,v_2)(a,v_2)(b,v_m)(b,v_1)(a,v_1)(c,v_1)\},$$

and

$$\mathcal{S}_{ab} = \{(a,v_1)(a,v_2)(b,v_2)(b,v_1)(a,v_1)\}.$$

Note that $\mathcal{H}_{ab}$ is the Schemeichel's 4-fold basis of $\mathcal{C}(ab\rho N_{m-1})$ (see Theorem 2.4 in [18]). Moreover, (1) if $e = (a,v_2)(b,v_m)$ or $e = (a,v_m)(b,v_2)$ or $e = (a,v_2)(b,v_2)$ or $e = (a,v_m)(b,v_m)$, then $f_{\mathcal{H}_{ab}}(e) = 1$; (2) if $e = (a,v_2)(b,v_i)$ or $(a,v_j)(b,v_2)$ or $(a,v_m)(b,v_i)$ or $(a,v_j)(b,v_m)$, then $f_{\mathcal{H}_{ab}}(e) \le 2$; and (3) if $e \in E(ab\rho N_{m-1})$ and it is not of the above forms, then $f_{\mathcal{H}_{ab}}(e) \le 4$.

Lemma 2.1 *Every linear combination of cycles of $\mathcal{E}_{cab}$ contains at least one edge of the form $(b,v_i)(a,v_m)$ where $2 \le i \le m$ and at least one edge of the form $(c,v_2)(b,v_i)$ where $2 \le i \le m$.*

Proof Note that $E(\mathcal{E}_{cab}) \subseteq \{(b,v_m)(a,v_j),(c,v_2)(a,v_j)|j = 2,3,\dots,m\}$. Since each of $\{(b,v_m)(a,v_j)|j = 2,3,\dots,m\}$ and $\{(c,v_2)(a,v_j)|j = 2,3,\dots,m\}$ forms an edge set of a star and since any linear combination of cycles is a cycle or an edge disjoint union of cycles, any linear combination of cycles of $\mathcal{E}_{cab}$ must contains at least one edge of $\{(b,v_m)(a,v_j)|j = 2,3,\dots,m\}$ and at least one edge of $\{(c,v_2)(a,v_j)|j = 2,3,\dots,m\}$. □

Using the same argument as in Lemma 2.1, we get the following result.

Lemma 2.2 *Every linear combination of cycles of $\mathcal{G}_{ab}$ contains at least one edge of the form $(a,v_1)(a,v_i)$ where $2 \le i \le m$ and at least one of the form $(b,v_2)(a,v_i)$ where $2 \le i \le m$.* □

We now set the following cycles:

$$\mathcal{U}_{lab}^{(j)} = (l,v_j)(a,v_j)(b,v_j)(l,v_j),\ j = 1,2,\dots,m$$
$$\mathcal{U}_{ab} = (a,v_1)(a,v_2)(b,v_m)(b,v_1)(a,v_1).$$

Let

$$\mathcal{O}_{labc} = \mathcal{H}_{bc} \cup \mathcal{G}_{cb} \cup \{\mathcal{U}_{lbc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{E}_{cba}$$

Lemma 2.3 *$\mathcal{O}_{labc}$ is linearly independent.*

Proof Note that $\mathcal{H}_{bc}$ is isomorphic to the Schemeichel's 4-fold basis of $bc\rho N_{m-1}$. Thus, $\mathcal{H}_{bc}$ is a linearly independent set. By Lemma 3.2.3 of [3], each of $\mathcal{G}_{cb}$, and $\mathcal{E}_{cba}$ is linearly independent. The cycle $\mathcal{U}_{bc}$ contains the edge $(b,v_2)(c,v_m)$ which does not occur in $\mathcal{U}_{lbc}^{(1)}$, thus $\{\mathcal{U}_{lbc}^{(1)}, \mathcal{U}_{bc}\}$ is linearly independent. It is easy to see that any linear combination of cycles of $\{\mathcal{U}_{lbc}^{(1)}, \mathcal{U}_{bc}\}$ contains either $(b,v_1)(c,v_1)$ or $(l,v_1)(b,v_1)$ and non of them occurs in any cycle of $\mathcal{H}_{bc}$, thus $\{\mathcal{U}_{l,bc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{H}_{bc}$ is linearly independent. By Lemma 2.1, any linear combination of cycles of $\mathcal{E}_{cba}$ contains an edge of the form $(b,v_i)(a,v_m)$ for $2 \le i \le m$ which does not occur in any cycle of $\{\mathcal{U}_{l,bc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{H}_{bc}$. Thus $\{\mathcal{U}_{l,bc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{H}_{bc} \cup \mathcal{E}_{cba}$ is linearly independent. Finally, by Lemma 2.2, any linear combination of cycles of $\mathcal{G}_{cb}$ contains an edge of the form $(c,v_1)(c,v_i)$

for $2 \leq i < m$ which appears in no cycle of $\{\mathcal{U}_{lbc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{H}_{bc} \cup \mathcal{E}_{cba}$. Thus $\mathcal{O}_{labc}$ is linearly independent. □

The coming result follows from being that

$$E(\mathcal{O}_{labc}) - \{(b, v_i)(c, v_j) | 2 \leq i, j \leq m\} \cup \{(b, v_1)(c, v_1)\}$$

forms and edge set of a tree and the fact that any linear combination of an independent set of cycles is a cycle or an edge disjoint union of cycles.

Lemma 2.4 *Any linear combination of cycles of $\mathcal{O}_{labc}$ must contain an edge of $\{(b, v_i)(c, v_j)|$*

$2 \leq i, j \leq m\} \cup \{(b, v_1)(c, v_1)\}$. □

Now, we let

$$\mathcal{O}^{*}_{lbc} = \mathcal{H}_{bc} \cup \mathcal{G}_{cb} \cup \{\mathcal{U}_{lbc}^{(1)}, \mathcal{U}_{bc}\} \text{ and } \mathcal{O}^{\circledast}_{labcd} = \mathcal{O}_{labc} \cup \mathcal{E}_{dcb}.$$

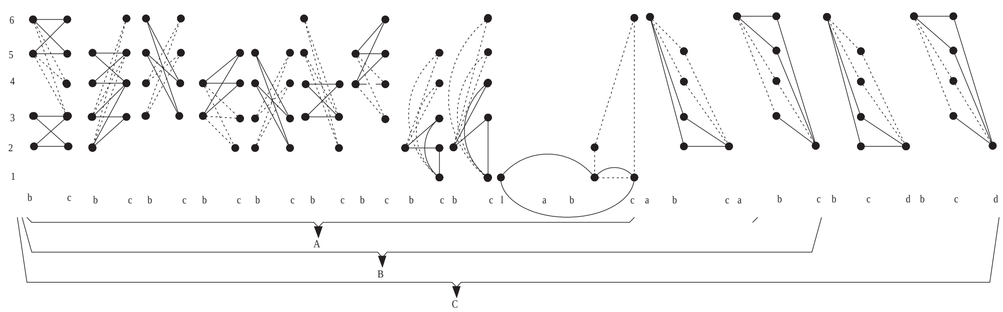

Figure 1: A represents the cycles of $\mathcal{O}^{*}_{lbc}$, B represents the cycles of $\mathcal{O}_{labc}$ and C represents the cycles of $\mathcal{O}^{\circledast}_{labcd}$ for $m = 6$.

Remark 2.5 Let $e \in E(\mathcal{O}^{\circledast}_{labcd})$. From the definitions of $\mathcal{O}^{*}_{lab}, \mathcal{O}_{labc}$ and $\mathcal{O}^{\circledast}_{labcd}$ and by the aid of Figure 1 below, one can easily see the following:

(1) If $e = (l, v_1)(b, v_1)$ or $(l, v_1)(c, v_1)$or $(b, v_1)(b, v_2)$ or $(c, v_1)(c, v_2)$ or $(c, v_1)(c, v_2)$, then $f_{\mathcal{O}^{*}_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 1$. (2) If $e = (c, v_1)(c, v_j)$ such that $3 \leq j \leq m$ or $e = (b, v_1)(c, v_1)$, then $f_{\mathcal{O}^{*}_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 2$. (3) If $e = (a, v_m)(b, v_j)$ such that $3 \leq j \leq m - 1$, then $f_{\mathcal{O}^{*}_{lab}}(e) = 0$ and $f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 2$. (4) If $e = (c, v_j)(d, v_2)$ such that $3 \leq j \leq m - 1$, then $f_{\mathcal{O}^{*}_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = 0$ and $f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 2$. (5) If $e = (a, v_m)(b, v_m)$ or $(a, v_m)(b, v_2)$ or $(c, v_m)(d, v_2)$, then $f_{\mathcal{O}^{*}_{lab}}(e) = 0$and $f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 1$. (6) If $e = (b, v_2)(c, v_2)$, then $f_{\mathcal{O}^{*}_{lab}}(e) = 2$ and $f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 3$. (7) If $e = (b, v_m)(c, v_2)$, then $f_{\mathcal{O}^{*}_{lab}}(e) = 1$,$f_{\mathcal{O}_{labc}}(e) = 2$ and $f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 2$. (8) If $e = (b, v_j)(c, v_2)$ such that $3 \leq j \leq m - 1$, then $f_{\mathcal{O}^{*}_{lab}}(e) = 2$and $f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 4$. (9) If $e = (b, v_m)(c, v_j)$ such that $3 \leq j \leq m - 1$, then $f_{\mathcal{O}^{*}_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = 2$ and $f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 4$. (10) If $e = (b, v_2)(c, v_m)$, then $f_{\mathcal{O}^{*}_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 3$. (10) If $e = (b, v_j)(c, v_k)$ such that $2 \leq j, k \leq m$ and e is not as in (1)-(10), then $f_{\mathcal{O}^{*}_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) \leq 4$.

The automorphism group of S_m is isomorphic to the symmetric group on the set $\{v_2, v_3, \ldots, v_m\}$ with $\alpha(v_1) = v_1$ for any $\alpha \in Aut(G)$. Therefore, for any two vertices v_i, v_j $(2 \leq i, j \leq m)$, there is $\alpha \in Aut(G)$ such that $\alpha(v_i) = v_j$. Hence, $W_n\rho S_m$ is decomposable into $S_n\rho S_m \cup C_{n-1}[N_{m-1}] \cup \{(u_j, v_1)(u_{j+1}, v_1)|2 \leq j < n\} \cup \{(u_n, v_1)(u_2, v_1)\}$. Thus

$$|E(W_n\rho S_m)| = |E(S_n\rho S_m)| + (n-1)(m-1)^2 + (n-1) = |E(S_n\rho S_m)| + (n-1)(m^2 - 2m + 2).$$

Hence,

$$\dim \mathcal{C}(W_n\rho S_m) = \dim \mathcal{C}(S_n\rho S_m) + (n-1)(m^2 - 2m + 2).$$

By Theorem 3.2.5 of [3], we have

$$\dim \mathcal{C}(S_n\rho S_m) = (n-1)(m^2 - 2m + 1). \tag{1}$$

Therefore,

$$\dim \mathcal{C}(W_n\rho S_m) = (n-1)(2m^2 - 4m + 3). \tag{2}$$

Lemma 2.6 *The set* $\mathcal{O} = \mathcal{O}^*_{u_1u_2u_3} \cup (\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}) \cup \mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}$ *is linearly independent subset of* $\mathcal{C}(W_n\rho S_m)$.

Proof By Lemmas 2.3, $\mathcal{O}^*_{u_1u_2u_3}$ is linearly independent. Note that, $\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3} = \mathcal{O}_{u_1,u_{n-1}u_nu_2} \cup \mathcal{E}_{u_1,u_3u_2u_n}$. By Lemma 2.1, any linear combination of cycles of $\mathcal{E}_{u_3u_2u_n}$ contains an edge of $\{(u_3, v_2)(u_2, v_j)|2 \leq j \leq m\}$ which is not in any cycle of $\mathcal{O}_{u_1u_{n-1}u_nu_2}$. Thus, $\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}$ is linearly independent. We now use mathematical induction on n to show that $\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}$ is linearly independent. If $n = 4$, then $\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}} = \mathcal{O}_{u_1u_2u_3u_4}$. And so, the result is followed from Lemma 2.3. Assume that n is greater than 3 and it is true for less than n. Note that $\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}} = \cup_{i=3}^{n-2} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}} \cup \mathcal{O}_{u_1u_{n-2}u_{n-1}u_n}$. By Lemma 2.3 and the inductive step, each of $\cup_{i=3}^{n-2} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}$ and $\mathcal{O}_{u_1u_{n-2}u_{n-1}u_n}$ is linearly independent. Since any linear combination of cycles of $\mathcal{O}_{u_1u_{n-2}u_{n-1}u_n}$ contains an edge of $\{(u_n, v_i)(u_{n-1}, v_j)|2 \leq i, j \leq m\} \cup \{(u_n, v_1)(u_{n-1}, v_1)\}$ (Lemma 2.4) which does not occur in any cycle of $\cup_{i=3}^{n-2} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}$, $\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}$ is linearly independent. Also, since $E(\mathcal{O}^*_{u_1u_2u_3}) \cap E(\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}) = \{(u_3, v_1)(u_1, v_1), (u_3, v_1)(u_3, v_2)\} \cup \{(u_2, v_m)(u_3, v_i) \mid 2 \leq i \leq m\}$ which is an edge set of a tree, $\mathcal{O}^*_{u_1u_2u_3} \cup (\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}})$ is linearly independent by Lemma 1.2. Similarly, $E(\mathcal{O}^*_{u_1u_2u_3} \cup (\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}})) \cap E(\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}) = \{(u_1, v_1)(u_2, v_1), (u_2, v_1)(u_2, v_2), (u_1, v_1)(u_n, v_1), (u_n, v_1)(u_n, v_2)\} \cup \{(u_{n-1}, v_m)(u_n, v_i), (u_3, v_2)(u_2, v_i) \mid 2 \leq i \leq m\}$ which is an edge set of a tree. Thus, $\mathcal{O}$ is linearly independent by Lemma 1.2. □

Now, let

$$\mathcal{L}_{ab} = \mathcal{H}_{ab} \cup \mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab} \text{ and } \mathcal{Y}_{cab} = \mathcal{E}_{cab} \cup \mathcal{H}_{ca} \cup \mathcal{G}_{ca} \cup \mathcal{W}_{cab}.$$

Theorem 2.7 *For any wheel* W_n *of order* $n \geq 4$ *and star* S_m of order $m \geq 3$,

$$3 \leq b(W_n\rho S_m) \leq 4.$$

Proof To prove the first inequality, we assume that $W_n\rho S_m$ has a 2-fold basis for $n \geq 4$ and $m \geq 3$, say $\mathcal{B}$. Since the girth of $W_n\rho S_m$ is 3, we have $4|\mathcal{B}| \leq 3|E(W_n\rho S_m)|$. Hence,

$3(n-1)(2m^2-4m+3) \leq 2[(n-1)(2m^2-4m+3)+nm]$, which implies that $n(2m^2-6m+3) - 2m^2+4m-3 \leq 0$. But $n \geq 3$, thus, $4(2m^2-6m+3)-2m^2+4m-3 \leq 0$, that is $m \leq \frac{20}{6} - \frac{9}{m}$. Therefore, $m < 4$.

To prove the second inequality, it is enough to exhibit a 4-fold basis. Define $\mathcal{B}(W_n \rho S_m) = B(S_n \rho S_m) \cup \mathcal{O}$ where $B(S_n \rho S_m) = (\cup_{i=2}^{n-1} \mathcal{Y}_{u_{i+1}u_1u_i}) \cup \mathcal{L}_{u_1u_2}$ is the cycle basis of $S_n \rho S_m$ (Theorem 3.2.5 of [3]). By Lemma 2.6 $\mathcal{O}$ is linearly independent. Since

$$E(\mathcal{B}(S_n \rho S_m)) \cap E(\mathcal{O}) = E((N_{n-1} \Box S_m) \cup (S_n \Box v_1)) \tag{3}$$

which is an edge set of a tree, $\mathcal{B}(W_n \rho S_m)$ is linearly independent by Lemma 1.2**.** Note that,

$$|\mathcal{H}_{ab}| = (m-2)^2 \text{ and } |\mathcal{G}_{ba}| = |\mathcal{E}_{cba}| = (m-2). \tag{4}$$

Thus by (4),

$$\begin{aligned} |\mathcal{O}^*_{u_1u_2u_3}| &= |\mathcal{O}^*_{lab}| = |\mathcal{H}_{ab}| + |\mathcal{G}_{ba}| + 2 \\ &= (m-2)^2 + (m-2) + 2 \\ &= m^2 - 3m + 4, \end{aligned} \tag{5}$$

and so

$$\begin{aligned} |\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}| &= |\mathcal{O}_{labc}| = |\mathcal{O}^*_{lab}| + |\mathcal{E}_{cba}| \\ &= m^2 - 3m + 4 + (m-2) \\ &= m^2 - 2m + 2, \end{aligned} \tag{6}$$

$$\begin{aligned} |\mathcal{O}^{\circledast}_{u_{n-1}u_nu_2u_3}| &= |\mathcal{O}^{\circledast}{}_{labcd}| = |\mathcal{O}_{labc}| + |\mathcal{E}_{dcb}| \\ &= m^2 - 2m + 2 + (m-2) \\ &= m^2 - m. \end{aligned} \tag{7}$$

Hence (5), (6) and (7), imply

$$\begin{aligned} |\mathcal{O}| &= |\mathcal{O}^*_{u_1u_2u_3}| + \sum_{i=3}^{n-1} |\mathcal{O}_{u_{i-1}u_iu_{i+1}}| + |\mathcal{O}^{\circledast}_{u_{n-1}u_nu_2u_3}| \\ &= m^2 - 3m + 4 + (n-3)(m^2-2m+2) + m^2 - m \\ &= (n-1)(m^2-2m+2). \end{aligned} \tag{8}$$

Thus (1), (2) and (8), give

$$\begin{aligned} |\mathcal{B}(W_n \rho S_m)| &= |\mathcal{B}(S_n \rho S_m)| + |\mathcal{O}| \\ &= (n-1)(m^2-2m+1) + (n-1)(m^2-2m+2). \\ &= (n-1)(2m^2-4m+3) \\ &= \dim \mathcal{C}(W_n \rho S_m) \end{aligned}$$

Therefore, $\mathcal{B}(W_n \rho S_m)$ forms a basis for $\mathcal{C}(W_n \rho S_m)$. To this end, we show that $f_{\mathcal{B}(W_n \rho S_m)}(e) \leq 4$ for each edge $e \in E(W_n \rho S_m)$. To do that we count the number of cycles of $\mathcal{B}(W_n \rho S_m)$ that contain the edge e. To this end, and according to (3) we split our work into two cases.

Case 1. $e \in E(W_n\rho S_m) - E((N_{n-1}\Box S_m) \cup (S_n\Box v_1))$.

Then we have the following:

Subcase 1.1. $e \in E(u_2u_3\rho S_m) - (E(S_n\rho S_m) \cup E(C_{n-1}\Box v_1))$. Then e occurs only in cycles of $\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3} \cup \mathcal{O}^{*}_{u_1u_2u_3} \cup \mathcal{O}_{u_1u_2u_3u_4}$. By the help of Remark 2.5, we have the following: (1) If $e = (u_3, v_2)(u_2, v_j)$ such that $2 \le j \le m-1$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1u_2u_3}}(e) + f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e) \le 2+2$. (2) If $e = (u_3, v_2)(u_2, v_m)$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1u_2u_3}}(e) + f_{\mathcal{O}_{u_1u_2u_3u_4}}(e) + f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e) \le 1+1+2$. (3) If $e = (u_2, v_m)(u_3, v_j)$ such that $3 \le j \le m$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1u_2u_3}}(e) + f_{\mathcal{O}_{u_1u_2u_3u_4}}(e) \le 1+1+2$. (4) If $e = (u_2, v_j)(u_3, v_k)$ such that $2 \le j, k \le m$ and e is not as in in (1) or (2) or (3), then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1u_2u_3}}(e) \le 4$.

Subcase 1.2. $e \in E(u_iu_j\rho S_m) - (E(S_n\rho S_m) \cup E(C_{n-1}\Box v_1))$ such that $3 \le j \le n-2$. Then e occurs only in cycles of $\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}} \cup \mathcal{O}_{u_1u_iu_{i+1}u_{i+2}}$. By the help of Remark 3.5, we have the following: (1) If $e = (u_i, v_m)(u_{i+1}, v_j)$ such that $2 \le j \le m$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}}(e) + f_{\mathcal{O}_{u_1u_iu_{i+1}u_{i+2}}}(e) \le 2+2$. (2) If $e = (u_i, v_j)(u_{i+1}, v_k)$ such that $2 \le j, k \le m$ and e is not as in (1), then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}}(e) \le 4$.

Subcase 1.3. $e \in E(u_{n-1}u_n\rho S_m) - (E(S_n\rho S_m) \cup E(C_{n-1}\Box v_1))$. Then e occurs only in cycles of $\mathcal{O}_{u_1u_{n-2}u_{n-1}u_n} \cup \mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}$. By the help of Remark 3.5, we have the following: (1) If $e = (u_{n-1}, v_m)(u_n, v_j)$ such that $2 \le j \le m$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}}(e) + f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e) \le 2+2$. (2) If $e = (u_i, v_j)(u_{i+1}, v_k)$ such that $2 \le j, k \le m$ and e is not as in (1), then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}}(e) \le 4$.

Subcase 1.4. $e \in E(u_nu_2\rho S_m) - (E(S_n\rho S_m) \cup E(C_{n-1}\Box v_1))$. Then e occur only in cycles of $\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}$. By Remark 2.5, $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e) \le 4$.

Subcase 1.5. $e \in C_{n-1}\Box v_1$. By Remark 2.5, we have the following: (1) If $e = (u_2, v_1)(u_3, v_1)$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1u_2u_3}}(e) = 2$. (2) If $e = (u_i, v_1)(u_{i+1}, v_1)$ such that $3 \le i \le n-1$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}}(e) = 2$. (3) If $e = (u_2, v_1)(u_n, v_1)$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e) = 2$.

Subcase 1.6. $e \in (S_n\rho S_m) - E((N_{|V(S_n - \{u_1\})|}\Box S_m) \cup (S_n\Box v_1))$. Then e occurs only in cycles of $\mathcal{B}(S_n\rho S_m)$. Thus by Theorem 3.2.5 of [3], $f_{\mathcal{B}(W_n\rho S_m)}(e) \le f_{\mathcal{B}(S_n\rho S_m)} \le 4$.

Case 2. $e \in E((N_{|V(S_n - \{u_1\})|}\Box S_m) \cup (S_n\Box v_1))$.

Then by the aid of Remark 2.5 and Theorem 3.2.5 of [3] we have the following.

Subcase 2.1. $e \in \cup_{i=4}^{n}(u_i\Box S_m)$.

Then e occurs only in cycles of $\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}} \cup \mathcal{B}(S_n\rho S_m)$. Thus, $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}}(e) + f_{\mathcal{B}(S_n\rho S_m)} \le 2+2$.

Subcase 2.2. $e \in (u_3\Box S_m)$. Then e occurs only in cycles of $\mathcal{O}^{*}_{u_1u_2u_3} \cup \mathcal{B}(S_n\rho S_m)$. Thus, $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1u_2u_3}}(e) + f_{\mathcal{B}(S_n\rho S_m)} \le 2+2$.

Subcase 2.3. $e \in (u_2\Box S_m)$. Then e occurs only in cycles of $\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3} \cup \mathcal{B}(S_n\rho S_m)$. Thus, $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e) + f_{\mathcal{B}(S_n\rho S_m)} \le 2+2$.

Subcase 2.4. $e \in S_n\Box v_1$. Then we have the following: (1) If $e = (u_1, v_1)(u_2, v_1)$, then

$f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^*_{u_1u_2u_3}}(e)+f_{\mathcal{O}_{u_1u_2u_3u_4}}+f_{\mathcal{B}(S_n\rho S_m)}$. (2) If $e=(u_1,v_1)(u_3,v_1)$, then $f_{\mathcal{O}^*_{u_1u_2u_3}}(e)+f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e)+f_{\mathcal{B}(S_n\rho S_m)} \leq 1+1+2$. (3) If $e=(u_1,v_1)(u_i,v_1)$ such that $3\leq i\leq n-1$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}}(e)+f_{\mathcal{O}_{u_1u_iu_{i+1}u_{i+2}}}(e)+f_{\mathcal{B}(S_n\rho S_m)} \leq 1+1+2$. (4) If $e=(u_1,v_1)(u_n,v_1)$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{n-2}u_{n-1}u_n}}(e)+f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e)+f_{\mathcal{B}(S_n\rho S_m)} \leq 1+1+1$. □

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