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On $(\in vq)$-Fuzzy Bigroup

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Abstract: In this paper, we introduce the concept of fuzzy singleton to bigroup, and uses it to define $(\in v\ q)$- fuzzy bigroup and discuss its properties. We investigate whether or not the fuzzy point of a bigroup will belong to or quasi coincident with its fuzzy set if the constituent fuzzy points of the constituting subgroups both belong to or quasi coincident with their respective fuzzy sets, and vise versa. We also prove that a fuzzy bisubset μ is an $(\in vq)$-fuzzy subbigroup of the bigroup G if its constituent fuzzy subsets are $(\in vq)$-fuzzy subgroups of their respective subgroups among others.

Key Words: Bigroups, fuzzy bigroups, fuzzy singleton on bigroup, $(\in vq)$- fuzzy subgroups, $(\in vq)$- fuzzy bigroup

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§1. Introduction

Fuzzy set was introduced by Zadeh[14] in 1965. Rosenfeld [9] introduced the notion of fuzzy subgroups in 1971. Ming and Ming [8] in 1980 gave a condition for fuzzy subset of a set to be a fuzzy point, and used the idea to introduce and characterize the notions of quasi coincidence of a fuzzy point with a fuzzy set. Bhakat and Das [2,3] used these notions by Ming and Ming to introduce and characterize another class of fuzzy subgroup known as $(\in vq)$- fuzzy subgroups. This concept has been further developed by other researchers. Recent contributions in this direction include those of Yuan et al [12,13].

The notion of bigroup was first introduced by P.L.Maggu [5] in 1994. This idea was extended in 1997 by Vasantha and Meiyappan [10]. These authors gave modifications of some results earlier proved by Maggu. Among these results was the characterization theorems for sub-bigroup. Meiyappan [11] introduced and characterized fuzzy sub-bigroup of a bigroup in 1998.

In this paper, using these mentioned notions and with emphases on the elements that are both in G_1 and G_2 of the bigroup G, we define the notion of $(\in, \in vq)$ - fuzzy sub bigroups as an extension of the notion of $(\in, \in vq)$- fuzzy subgroups and discuss its properties. Apart from this section that introduces the work, section 2 presents the major preliminary results that are useful for the work. In section 3, we define a fuzzy singleton on a bigroup. Using this definition,

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we investigate whether or not the fuzzy point of a bigroup will belong to or quasi coincident with its fuzzy set if the constituent fuzzy points of the constituting subgroups both belong to or quasi coincident with their respective fuzzy sets, and vise versa. Theorems 3.4 and 3.5 give the results of these findings. In the same section, we define $(\in vq)$- fuzzy subgroup and prove that a fuzzy bisubset μ is an $(\in vq)$- fuzzy sub bigroup of the bigroup G if its constituent fuzzy subsets are $(\in vq)$- fuzzy subgroups of their respective subgroups.

§2. Preliminary Results

Definition 2.1([10,11]) *A set $(G, +, \cdot)$ with two binary operations " + " and " $\cdot$ " is called a bi-group if there exist two proper subsets G_1 and G_2 of G such that*

(i) $G = G_1 \cup G_2$;

(ii) $(G_1, +)$ *is a group;*

(iii) $(G_2, \cdot)$ *is a group.*

Definition 2.2([10]) *A subset $H(\neq 0)$ of a bi-group $(G, +, \cdot)$ is called a sub bi-group of G if H itself is a bi-group under the operations of " + " and " $\cdot$ " defined on G.*

Theorem 2.3([10]) *Let $(G, +, \cdot)$ be a bigroup. If the subset $H \neq 0$ of a bigroup G is a sub bigroup of G, then $(H, +)$ and $(H, \cdot)$ are generally not groups.*

Definition 2.4([14]) *Let G be a non empty set. A mapping $\mu : G \to [0, 1]$ is called a fuzzy subset of G.*

Definition 2.5([14]) *Let μ be a fuzzy set in a set G. Then, the level subset μ_t is defined as: $\mu_t = \{x \in G : \mu(x) \geq t\}$ for $t \in [0, 1]$.*

Definition 2.6([9]) *Let μ be a fuzzy set in a group G. Then, μ is said to be a fuzzy subgroup of G, if the following hold:*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ $\forall\ x, y \in G$;

(ii) $\mu(x^{-1}) = \mu(x)$ $\forall\ x \in G$.

Definition 2.7([9]) *Let μ be a fuzzy subgroup of G. Then, the level subset μ_t, for $t \in Im\mu$ is a subgroup of G and is called the level subgroup of G.*

Definition 2.8 ([8]) *A fuzzy subset μ of a group G of the form*

$$\mu(y) = \begin{cases} t(\neq 0) & if\ y = x, \\ 0 & if\ y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.9([11]) *Let μ_1 be a fuzzy subset of a set X_1 and μ_2 be a fuzzy subset of a set X_2, then the fuzzy union of the sets μ_1 and μ_2 is defined as a function $\mu_1 \cup \mu_2 : X_1 \cup X_2 \longrightarrow [0, 1]$ given by:*

$$(\mu_1 \cup \mu_2)(x) = \begin{cases} max(\mu_1(x), \mu_2(x)) & if \ x \in X_1 \cap X_2, \\ \mu_1(x) & if \ x \in X_1, x \notin X_2 \\ \mu_2(x) & if \ x \in X_2 \& x \notin X_1 \end{cases}$$

Definition 2.10([11]) *Let $G = (G_1 \cup G_2, +, \cdot)$ be a bigroup. Then $\mu : G \to [0,1]$ is said to be a fuzzy sub-bigroup of the bigroup G if there exist two fuzzy subsets $\mu_1(of G_1)$ and $\mu_2(of G_2)$ such that:*

(i) $(\mu_1, +)$ *is a fuzzy subgroup of* $(G_1, +)$*,*

(ii) $(\mu_2, \cdot)$ *is a fuzzy subgroup of* $(G_2, \cdot)$*, and*

(iii) $\mu = \mu_1 \cup \mu_2$*.*

Definition 2.11([8]) *A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set μ , written as $x_t \in \mu$ (resp. $x_t \ q \ \mu$) if $\mu(x) \geq t$(resp. $\mu(x) + t > 1$).*
"$x_t \ \in \mu$ or $x_t \ q \ \mu$" will be denoted by $x_t \ \in vq \ \mu$).

Definition 2.12([2,3]) *A fuzzy subset μ of G is said to be an ($\in vq$)- fuzzy subgroup of G if for every $x, y \ \in G$ and $t, r \ \in (0,1]$:*

(i) $x_t \ \in \mu, y_r \ \in \mu \Rightarrow (xy)_{M(t,r)} \ \in vq \ \mu$

(ii) $x_t \ \in \mu \Rightarrow (x^{-1})_t \ \in vq \ \mu$.

Theorem 2.13([3]) *(i) A necessary and sufficient condition for a fuzzy subset μ of a group G to be an ($\in, \in vq$)-fuzzy subgroup of G is that $\mu(xy^{-1}) \geq M(\mu(x), \mu(y), 0.5)$ for every $x, \ y \ \in G$.*
(ii). Let μ be a fuzzy subgroup of G. Then $\mu_t = \{x \ \in G : \mu(x) \geq t\}$ is a fuzzy subgroup of G for every $0 \leq t \leq 0.5$. Conversely, if μ is a fuzzy subset of G such that μ_t is a subgroup of G for every $t \in (0, 0.5]$, then μ is an fuzzy ($\in, \in vq$)-fuzzy subgroup of G.

Definition 2.14([3]) *Let X be a non empty set. The subset $\mu_t = \{x \ \in X : \mu(x) \geq t\}$ or $\{\mu(x) + t > 1\} = \{x \ \in \ X : x_t \ \in vq \ \mu\}$ is called ($\in vq$)- level subset of X determined by μ and t.*

Theorem 2.15([3]) *A fuzzy subset μ of G is a fuzzy subgroup of G if and only if μ_t is a subgroup for all $t \in (0,1]$.*

§3. Main Results

Definition 3.1 *Let $G = G_1 \cup G_2$ be a bi-group. Let $\mu = \mu_1 \cup \mu_2$ be a fuzzy bigroup. A fuzzy subset $\mu = \mu_1 \cup \mu_2$ of the form:*

$$\mu(x) = \begin{cases} M(t,s) \neq 0 & if \ x = y \in G, \\ 0 & if \ x \neq y \end{cases}$$

where $t, s \ \in [0,1]$ such that

$$\mu_1(x) = \begin{cases} t \neq 0 & if x = y \in G_1, \\ 0 & if \ x \neq y \end{cases}$$

and

$$\mu_2(x) = \begin{cases} s \neq 0 & if x = y \in G_2, \\ 0 & if\ x \neq y \end{cases}$$

is said to be a fuzzy point of the bi-group G with support x and value $M(t,s)$ and is denoted by $x_{M(t,s)}$.

Theorem 3.2 *Let $x_{M(t,s)}$ be a fuzzy point of the bigroup $G = G_1 \cup G_2$. Then:*
(i) $x_{M(t,s)} = x_t \Leftrightarrow x \in G_1 \cap G_2^c$ *or* $t > s$
(ii) $x_{M(t,s)} = x_s \Leftrightarrow x \in G_1^c \cap G_2$ *or* $t < s$
$\forall\ t, s\ \in [0,1]$, *where x_t, x_s are fuzzy points of the groups G_1 and G_2 respectively.*

Proof (i) Suppose $x_{M(t,s)} = x_t$. Then $M(t,s) = t \Rightarrow t\ > s$. And $t\ >\ s \Rightarrow 0 \leq s < t$. Hence, if $s = 0$ then $x\ \in\ G_1 \cap G_2^c$.
Conversely, suppose

$$x\ \in G_1 \cap G_2^c\ ,\ then\ x\ \in G_1\ and\ x\ \notin G_2,$$

which implies that $x_s = 0$. Therefore $x_{M(t,s)} = x_t$. Also, if $t > 0$, $x_{M(t,s)} = x_t$. Hence the proof.
(ii) Similar to that of (i). □

Definition 3.3 *A fuzzy point $x_{M(t,s)}$ of the bigroup $G = G_1 \cup G_2$, is said to belong to (resp. be quasi coincident with) a fuzzy subset $\mu = \mu_1 \cup \mu_2$ of G, written as $x_{M(t,s)} \in \mu$[resp. $x_{M(t,s)} q\mu$] if $\mu(x) \geq M(t,s)$(resp. $\mu(x) + M(t,s) > 1$). $x_{M(t,s)}\ \in \mu$ or $x_{M(t,s)}\ q\mu$ will be denoted by $x_M(t,s)\ \in vq\mu$.*

Theorem 3.4 *Let $G = G_1 \cup G_2$ be a bigroup. Let μ_1 and μ_2 be fuzzy subsets of G_1 and G_2 respectively. Suppose that x_t and x_s are fuzzy points of the groups G_1 and G_2 respectively such that $x_t \in vq\mu_1$ and $x_s \in vq\mu_2$, then $x_{M(t,s)} \in vq\mu$ where $x_{M(t,s)}$ is a fuzzy point of the bigroup G, and $\mu : G \to [0,1]$ is such that $\mu = \mu_1 \cup \mu_2$.*

Proof Suppose that

$$x_t \in vq\mu_1\ and\ x_s \in vq\mu_2,$$

then we have that

$$\mu_1(x) \geq t\ or\ \mu_1(x) + t > 1,$$

and

$$\mu_2(x) \geq s\ or\ \mu_2(x) + s > 1.$$

$$\mu_1(x) \geq t\ and\ \mu_2(x) \geq s \Rightarrow Max[\mu_1(x), \mu_2(x)] \geq M(t,s).$$

This means that

$$(\mu_1 \cup \mu_2)(x) \geq M(t,s)\ since\ x \in G_1 \cap G_2$$

That is

$$\mu(x) \geq M(t,s) \qquad (1)$$

Similarly,

$$\mu_1(x) + t > 1 \ and \ \mu_2(x) + s > 1$$

imply that

$$\mu_1(x) + t \ + \ \mu_2(x) + s > 2$$

$$\Rightarrow 2Max[\mu_1(x), \mu_2(x)] + 2M[t, s] > 2$$

$$\Rightarrow Max[\mu_1(x), \mu_2(x)] + M[t, s] > 1$$

$$\Rightarrow (\mu_1 \cup \mu_2)(x) + M(t, s) > 1$$

$$\Rightarrow \mu(x) + M(t, s) > 1 \qquad (2)$$

Combining (1) and (2), it follows that:

$$\mu(x) \geq M(t, s) \ \ or \ \ \mu(x) + M(t, s) > 1$$

which shows that

$$x_{M(t,s)} \in vq\mu$$

hence the proof. □

Theorem 3.5 *Let $G = G_1 \cup G_2$ be a bigroup. $\mu = \mu_1 \cup \mu_2$ a fuzzy subset of G, where μ_1, μ_2 are fuzzy subsets of G_1 and G_2 respectively. Suppose that $x_{M(t,s)}$ is a fuzzy point of the bigroup G then $x_{M(t,s)} \in vq\mu$ does not imply that $x_t \in vq\mu_1$ and $x_s \in vq\mu_2$, where x_t and x_s are fuzzy points of the groups G_1 and G_2, respectively.*

Proof Suppose that $x_{M(t,s)} \in vq\mu$, then

$$\mu(x) \geq M(t, s) \ \ or \ \ \mu(x) \ + \ M(t, s) \ > \ 1$$

By definition 2.9, this implies that

$$(\mu_1 \cup \mu_2)(x) \geq M(t, s) \ or \ (\mu_1 \cup \mu_2)(x) \ + \ M(t, s) \ > \ 1$$

$$\Rightarrow Max[\mu_1(x), \mu_2(x)] \geq M(t, s) \ or \ Max[\mu_1(x), \mu_2(x)] \ + \ M(t, s) \ > \ 1$$

Now, suppose that $t \ > \ s$, so that $M(t, s) = t$, we then have that

$$Max[\mu_1(x), \mu_2(x)] \geq t \ or \ (Max[\mu_1(x), \mu_2(x)] \ + \ t) \ > \ 1$$

which means that $x_t \in vqMax[\mu_1, \mu_2]$, and by extended implication, we have that
$x_s \in vqMax[\mu_1, \mu_2]$.
If we assume that $Max[\mu_1, \mu_2] \ = \ \mu_1$, then we have that

$$x_t \in vq\mu_1 \ and \ x_s \in vq\mu_1,$$

and since $0 \leq \ s < t < 1$, we now need to show that at least $x_s \in vq\mu_2$
since by assumption, $\mu_1 > \mu_2$. To this end, let the fuzzy subset μ_2 and the fuzzy singleton x_s

be defined in such a way that $\mu_2 < s < 0.5$, then it becomes a straight forward matter to see that $x_s \bar{\in} vq\mu_2$. Even though, $x_{M(t,s)} \in vq\mu$ still holds. And the result follows accordingly. □

Corollary 3.6 *Let $G = G_1 \cup G_2$ be a bigroup. $\mu = \mu_1 \cup \mu_2$ a fuzzy subset of G, where μ_1, μ_2 are fuzzy subsets of G_1 and G_2 respectively. Suppose that $x_{M(t,s)}$ is a fuzzy point of the bigroup G then $x_{M(t,s)} \in vq\mu$ imply that $x_t \in vq\mu_1$ and $x_s \in vq\mu_2$, if and only if*

$$0.5 < min[t,s] \leq min[\mu_1(x), \mu_2(x)] < 1$$

where x_t and x_s are fuzzy points of the groups G_1 and G_2 respectively.

Definition 3.7 *A fuzzy bisubset μ of a bigroup G is said to be an $(\in v\ q)$- fuzzy sub bigroup of G if for every $x,\ y \in G$ and $t_1,\ t_2,\ s_1,\ s_2,\ t,\ s, \in [0,1]$,*
(i) $x_{M(t_1,t_2)} \in \mu,\ y_{M(s_1,s_2)} \in \mu \Rightarrow (xy)_{M(t,s)} \in vq\mu$
(ii) $x_{M(t_1,t_2)} \in \mu \Rightarrow (x^{-1})_{M(t_1,t_2)} \in vq\mu$
where $t = M(t_1,t_2)$ *and* $s = M(s_1,s_2)$.

Theorem 3.8 *Let $\mu = \mu_1 \cup \mu_2 :\ G = G_1 \cup G_2 \rightarrow [0,1]$ be a fuzzy subset of G. Suppose that μ_1 is an $(\in vq)$-fuzzy subgroup of G_1 and μ_2 is an $(\in vq)$-fuzzy subgroup of G_2, then μ is an $(\in vq)$-fuzzy subgroup of G.*

Proof Suppose that μ_1 is an $(\in vq)$-fuzzy subgroup of G_1 and μ_2 is an $(\in vq)$-fuzzy subgroup of G_2. Let $x,\ y, \in G_1,\ G_2$ and $t_1,\ t_2,\ s_1,\ s_2 \in [0,1]$ for which

$$x_{t_1} \in \mu_1\ ,\ y_{s_1} \in \mu_1 \Rightarrow (xy)_{M(t_1,s_1)} \in vq\mu_1$$

and

$$x_{t_2} \in \mu_2\ ,\ y_{s_2} \in \mu_2 \Rightarrow (xy)_{M(t_2,s_2)} \in vq\mu_2.$$

This implies that

$$\mu_1(xy) \geq M(t_1,s_1)\ or\ \mu_1(xy) + M(t_1,s_1) > 1,$$

and

$$\mu_2(xy) \geq M(t_2,s_2)\ or\ \mu_2(xy) + M(t_2,s_2) > 1.$$

$$\Rightarrow \mu_1(xy) + \mu_2(xy) \geq M(t_1,s_1) + M(t_2,s_2)$$

$$or\ \mu_1(xy) + \mu_2(xy) + M(t_1,s_1) + M(t_2,s_2) > 2$$

$$\Rightarrow 2Max[\mu_1(xy), \mu_2(xy)] \geq 2M[t,s]$$

$$or\ 2Max[\mu_1(xy), \mu_2(xy)] + 2M[t,s] > 2$$

$$\Rightarrow Max[\mu_1(xy), \mu_2(xy)] \geq M[t,s]$$

$$or\ Max[\mu_1(xy), \mu_2(xy)] + M[t,s] > 1$$

$$\Rightarrow \mu_1 \cup \mu_2(xy) \geq M[t,s]\ or\ \mu_1 \cup \mu_2(xy) + M[t,s] > 1$$

$$\Rightarrow \mu(xy) \geq M[t,s] \ or \ \mu(xy) + M[t,s] \ > \ 1$$

$$\Rightarrow (xy)_{M(t,s)} \in vq\mu.$$

To conclude the proof, we see that

$$x_{t_1} \in \mu_1 \Rightarrow (x^{-1})_{t_1} \in vq\mu_1, \ and \ x_{t_2} \in \mu_2 \ \Rightarrow (x^{-1})_{t_2} \in vq\mu_2$$

And since it is a straightforward matter to see that

$$(x^{-1})_{t_1} \in vq\mu_1, \ and \ (x^{-1})_{t_2} \in vq\mu_2 \ \Rightarrow \ (x^{-1})_{M(t_1,t_2)} \in vq \ \mu,$$

then, the result follows accordingly. □

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