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Connectivity of Smarandachely Line Splitting Graphs

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Abstract: Let $G(V, E)$ be a graph. Let $U \subseteq V(G)$ and $X \subseteq E(G)$. For each vertex $u \in U$, a new vertex u' is taken and the resulting set of vertices is denoted by $V_1(G)$. The *Smarandachely splitting graph* $S^U(G)$ of a graph G is defined as the graph having vertex set $V(G) \bigcup V_1(G)$ with two vertices adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex u' of V_1 and the other to a vertex w of G where w is in $N_G(u)$. Particularly, if $U = V(G)$, such a Smarandachely splitting graph $S^{V(G)}(G)$ is abbreviated to *Splitting graph* of G and denoted by $S(G)$. The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i. For each edge $e_i \in X$, a new vertex e_i' is taken and the resulting set of vertices is denoted by $E_1(G)$. The *Smarandachely line splitting graph* $L_s^X(G)$ of a graph G is defined as the graph having vertex set $E(G) \bigcup E_1(G)$ and two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of E_1 and the other to an element e_j of $E(G)$ where e_j is in $N_G(e_i)$. Particularly, if $X = E(G)$, such a Smarandachely line splitting graph $L_S^{V(G)}(G)$ is abbreviated to *Line Splitting Graph* of G and denoted by $L_S(G)$. In this paper, we study the connectivity of line splitting graphs.

Key Words: Line graph, Smarandachely splitting graph, splitting graph, Smarandachely line splitting graph, line splitting graph.

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§1. Introduction

By a graph, we mean a finite, undirected graph without loops or multiple edges. Definitions not given here may be found in [2]. For a graph G, $V(G)$ and $E(G)$ denote its vertex set and edge set respectively.

A vertex-cut in a graph G is a set S of vertices of G such that $G \setminus S$ is disconnected. Similarly, an edge-cut in a graph G is a set X of edges of G such that $G \setminus X$ is disconnected.

The open neighborhood $N(u)$ of a vertex u in $V(G)$ is the set of vertices adjacent to u. $N(u) = \{v/uv \in E(G)\}$.

Let $U \subseteq V(G)$ and $X \subseteq E(G)$. For each vertex $u \in U$, a new vertex u' is taken and the resulting set of vertices is denoted by $V_1(G)$. The *Smarandachely splitting graph* $S^U(G)$ of a

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graph G is defined as the graph having vertex set $V(G)\bigcup V_1(G)$ with two vertices adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex $u^{'}$ of V_1 and the other to a vertex w of G where w is in $N_G(u)$. Particularly, if $U = V(G)$, such a Smarandachely splitting graph $S^{V(G)}(G)$ is abbreviated to *Splitting graph* of G and denoted by $S(G)$. The concept of Splitting Graph was introduced by Sampathkumar and Walikar in [4].

The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i. $N(e_i) = \{e_j/e_i, e_j \text{ are adjacent in } E(G)\}$.

For each edge $e_i \in X$, a new vertex $e_i^{'}$ is taken and the resulting set of vertices is denoted by $E_1(G)$. The *Smarandachely line splitting graph* $L_s^X(G)$ of a graph G is defined as the graph having vertex set $E(G)\bigcup E_1(G)$ and two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an element $e_i^{'}$ of E_1 and the other to an element e_j of $E(G)$ where e_j is in $N_G(e_i)$. Particularly, if $X = E(G)$, such a Smarandachely line splitting graph $L_s^{V(G)}(G)$ is abbreviated to *Line Splitting Graph* of G and denoted by $L_s(G)$. The concept of Line splitting graph was introduced by Kulli and Biradar in [3].

We first make the following observations.

Observation 1. The graph G is an induced subgraph of $S(G)$. The line graph $L(G)$ is an induced subgraph of $L_s(G)$.

Observation 2. If $G = L_s(H)$ for some graph H, then $G = S(L(H))$.

The following will be useful in the proof of our results.

Theorem A([1]) *If a graph G is m-edge connected, $m \geq 2$, then its line graph $L(G)$ is m-connected.*

Theorem B([2]) *A graph G is n-connected if and only if every pair of vertices are joined by at least n vertex disjoint paths.*

§2. Main Results

Theorem 1 *Let G be a (p, q) graph. Then $L_s(G)$ is connected if and only if G is a connected graph with $p \geq 3$.*

Proof Let G be a connected graph with $p \geq 3$ vertices. Let $V(L_s(G)) = V_1 \bigcup V_2$ where $< V_1 >= L(G)$ and V_2 is the set of all newly introduced vertices, such that $v_1 \to v_2$ is a bijective map from V_1 onto V_2 satisfying $N(v_2) = N(v_1)\bigcap V_1$ for all $v_1 \in V_1$. Let $a, b \in V(L_s(G))$. We consider the following cases.

Case 1. $a, b \in V_1$. Since G is a connected graph with $p \geq 3$, $L(G)$ is a nontrivial connected graph. Since $L(G)$ is an induced subgraph of $L_s(G)$, there exists an $a - b$ path in $L_s(G)$.

Case 2. $a \in V_1$ and $b \in V_2$. Let $v \in V_1$ be such that $N(b) = N(v)\bigcap V_1$. Choose $w \in N(b)$. Since a and $w \in V_1$, as in Case 1, a and w are joined by a path in $L_s(G)$. Hence a and b are connected by a path in $L_s(G)$.

Case 3. $a, b \in V_2$. As in Case 2, there exist w_1 and w_2 in V_1 such that $w_1 \in N(a)$ and

$w_2 \in N(b)$. Consequently, $w_1a, w_2b \in E(L_s(G))$. Also w_1 and w_2 are joined by a path in $L_s(G)$. Hence a and b are connected by a path in $L_s(G)$.

In all the cases, a and b are connected by a path in $L_s(G)$. Thus $L_s(G)$ is connected.

Conversely, if G is disconnected or $G = K_2$, then obviously $L_s(G)$ is disconnected. □

Theorem 2 *For any graph G, $\kappa(L_s(G)) = min\{2\kappa(L(G)), \delta_e(G) - 2\}$.*

Proof By Whitney's result, $\kappa(L_s(G)) \leq \lambda(L_s(G)) \leq \delta(L_s(G))$. Also, $\kappa(L(G)) \leq \lambda(L(G)) \leq \delta(L(G))$. Since $L(G)$ is an induced subgraph of $L_s(G)$, $\kappa(Ls(G)) \geq \kappa(L(G))$. We have the following cases.

Case 1. If $\kappa(L(G)) = 0$, then obviously $\kappa(L_s(G)) = 0$.

Case 2. If $\kappa(L(G)) = 1$, then $L(G) = K_2$ or it is connected with a cut-vertex e_i.

We consider the following subcases.

Subcase 2.1. $L(G) = K_2$, then $L_s(G) = P_4$. Consequently, $\kappa(L_s(G)) = \delta(L(G)) = 1$.

Subcase 2.2. $L(G)$ is connected with a cut-vertex e_i. Let e_j be a pendant vertex of $L(G)$ which is adjacent to e_i. Then $e_j^{'}$ is a pendant vertex of $L_s(G)$ and e_i is also a cut-vertex of $L_s(G)$. Hence $\kappa(L_s(G)) = \delta(L(G))$. If $\delta(L(G)) \geq 2$, then the removal of a cut-vertex e_i of $L(G)$ and its corresponding vertex $e_i^{'}$ from $L_s(G)$ results in a disconnected graph. Hence $\kappa(Ls(G)) = 2\kappa(L(G))$.

Now suppose $\kappa(L(G)) = n$. Then $L(G)$ has a minimum vertex-cut $\{e_l : 1 \leq l \leq n\}$ whose removal from $L(G)$ results in a disconnected graph. There are two types of vertex-cuts in $L_s(G)$ depending on the structure of $L(G)$. Among these, one vertex-cut contains exactly $2n$ vertices, e_l's and $e_l^{'}$'s of $L_s(G)$ whose removal increases the components of $L_s(G)$ and the other is $\delta(L(G))$-vertex-cut. Thus we have

$$\kappa(L_s(G)) = \begin{cases} 2n, \quad \text{if} \quad n \leq \frac{\delta(L(G))}{2} = \frac{\delta_e(G)-2}{2}; \\ \\ \delta(L(G)) = \delta_e(G) - 2, \quad \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \kappa(L_s(G)) &= min\{2\kappa(L(G)), \delta(L(G))\} \\ &= min\{2\kappa(L(G)), \delta_e(G) - 2\}. \end{aligned}$$ □

Theorem 3 *For any graph G, $\lambda(L_s(G)) = min\{3\lambda(L(G)), \delta_e(G) - 2\}$.*

Proof Since $\delta(L_s(G)) = \delta(L(G))$, by Whitney's result $\kappa(L_s(G)) \leq \lambda(L_s(G)) \leq \delta(L(G))$. Since $L(G)$ is an induced subgraph of $L_s(G), \lambda(L_s(G)) \geq \lambda(L(G))$.

We consider the following cases.

Case 1. If $\lambda(L(G)) = 0$, then obviously $\lambda(L_s(G)) = 0$.

Case 2. If $\lambda(L(G)) = 1$, then $L(G) = K_2$ or it is connected with a bridge $x = e_ie_j$.

We have the following subcases of this case.

Subcase 2.1. $L(G) = K_2$, then $L_s(G) = P_4$. Consequently, $\lambda(L_s(G)) = \delta(L(G)) = 1$.

Subcase 2.2. $L(G)$ is connected with a bridge e_ie_j. If e_i is a pendant vertex, then $L_s(G)$ is connected with the some pendant vertex $e_i^{'}$. There is only one edge incident with $e_i^{'}$ whose removal disconnects it. Thus $\lambda(L_s(G)) = \delta(L(G)) = 1$. If neither e_i nor e_j is a pendant vertex and $\delta(L(G)) = 2$, then $\delta(L_s(G)) = 2$ and let e_k be a vertex of $L_s(G)$ with $deg_{L_s(G)}e_k = 2$. In $L_s(G)$, there are only two edges incident with e_k and the removal of these disconnects $L_s(G)$. So $\lambda(L_s(G)) = \delta(L(G))$. If $\delta(L(G)) \geq 3$, then the removal of edges $e_ie_j, e_i^{'}e_j$ and $e_ie_j^{'}$ from $L_s(G)$ results in a disconnected graph. Hence $\lambda(L_s(G)) = 3\lambda(L(G))$.

Now suppose $\lambda(L(G)) = n$. Then $L(G)$ has a minimum edge-cut $\{e_l = u_lv_l : 1 \leq l \leq n\}$ whose removal from $L(G)$ results in a disconnected graph. As above, there are two types of edge-cuts in $L_s(G)$ depending on the structure of $L(G)$. Among these, one edge-cut contains exactly 3n edges $\{u_lv_l, u_l^{'}v_l, u_lv_l^{'}, 1 \leq l \leq n\}$ whose removal increases the components of $L_s(G)$ and the other is $\delta(L(G))-$edge-cut. Thus we have

$$\lambda(L_s(G)) = \begin{cases} 3n, & \text{if} \quad n \leq \frac{\delta(L(G))}{3} = \frac{\delta_e(G)-2}{3}; \\ \\ \delta(L(G)) = \delta_e(G) - 2, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \lambda(L_s(G)) &= min\{3\lambda(L(G)), \delta(L(G))\} \\ &= min\{3\lambda(L(G)), \delta_e(G) - 2\} \end{aligned}$$

□

Theorem 4 *If a graph G is n-edge connected, $n \geq 2$, then $L_s(G)$ is n-connected.*

Proof Let G be a n-edge connected graph, $n \geq 2$. Then by Theorem A, $L(G)$ is n-connected. We show that there exist n-disjoint paths between any two vertices of $L_s(G)$. Let x and y be two distinct vertices of $L_s(G)$. We consider the following cases.

Case 1. Let $x, y \in E(G)$. Then by Theorem B, x and y are joined by n-disjoint paths in $L(G)$. Since $L(G)$ is an induced subgraph of $L_s(G)$, there exist n-disjoint paths between x and y in $L_s(G)$.

Case 2. Let $x \in E(G)$ and $y \in E_1(G)$. Since $\lambda(G) \leq \delta(G) < 2\delta(G) \leq \delta_e(G)$, there are at least n edges adjacent to x. Let $x_i, i = 1, 2, ..., n$ be edges of G, adjacent to x. Then the vertices $x_i^{'}, i = 1, 2, ..., n$ are adjacent to the vertex x in $L_s(G)$, where $x_i^{'} \in E_1(G), i = 1, 2, ..., n$. It follows from Case 1, that there exist n-disjoint paths from x to x_i, $i = 1, 2, ..., n$ in $L_s(G)$. Since $y \in E_1(G)$, we have $N(y) = N(w) \bigcap E$, for some $w \in E(G)$. Since $|N(w)| \geq n$, let $y_1, y_2, ..., y_n \in E(G)$ such that $y_i \in N(w), i = 1, 2, ..., n$. So $y_i \in N(y), i = 1, 2, ..., n$. Also, since x and $y_i \in E(G)$, i=1,2,...,n, as in Case 1, there exist n-disjoint paths in $L_s(G)$ between x and $y_i, i = 1, 2, ..., n$. Hence x and y are joined by n-disjoint paths in $L_s(G)$.

Case 3. Let $x, y \in E_1(G)$. As in Case 2, $x_i \in N(x), i = 1, 2, ..., n$ and $y_i \in N(y), i = 1, 2, ..., n$ for some $x_i, y_i \in E(G), i = 1, 2, ..., n$. Consequently, x_ix and $y_iy \in E(L_s(G)), i = 1, 2, ..., n$.

Also by Case 1, every pair of x_i and y_i are joined by n-disjoint paths in $L_s(G)$. Hence x and y are joined by n-disjoint paths in $L_s(G)$.

Thus it follows from Theorem B that $L_s(G)$ is n-connected. □

However, the converse of the above Theorem is not true. For example, in Figure 1, $L_s(G_1)$ is 2-connected, whereas G_1 is edge connected.

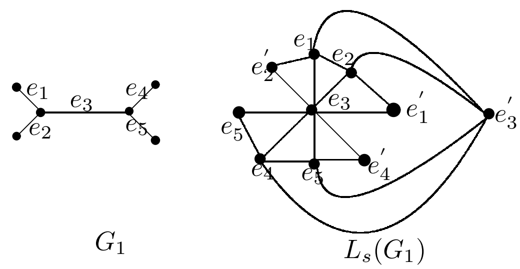

Figure 1

Corollary 5 *If a graph G is n-connected, $n \geq 2$, then $L_s(G)$ is n-connected.*

The converse of above corollary is not true. For instance, In Figure 2, $L_s(G_2)$ is 2-connected, but G_2 is connected.

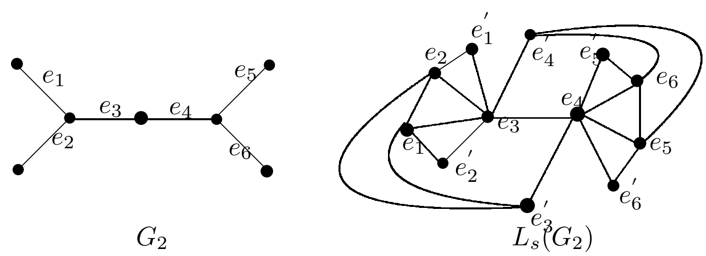

Figure 2

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