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Classification of Differentiable Graph

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Abstract: We will classify the differentiable graph representing the solution of differential equation. Present new types of graphs. Theorems govern these types are introduced. Finally the effect of step size h on the differentiable graph is discussed.

Key Words: Differentiable, graph, numerical methods.

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§1. Definitions and Background

Definition 1([2]) *A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints*

Definition 2([1,2,4,7,8]) *A loop is an edge whose endpoints are equal. Multiple edges are edges have the same pair of endpoints.*

Definition 3([2,6]) *A simple graph is a graph having no loops or multiple edges . We specify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge e with end points u and v.*

Definition 4([2]) *A directed graph or digraph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices. the first vertex of the ordered pair is the tail of the edge, and the second is the head; together, they are the endpoints. We say that an edge is an edge from its tail to its head.*

Definition 5([2]) *A digraph is simple if each ordered pair is the head and tail of at most one edge. In a simple digraph, we write uv for an edge with tail u and head v. If there is an edge from u to v, then v is a successor of u, and u is a predecessor of v. We write $u \to v$ for "there is an edge from u to v".*

Definition 6([7,8]) *A null graph is a graph containing no edges.*

Definition 7([2]) *The order of a graph G, written $n(G)$, is the number of vertices in G. An n-vertex graph is a graph of order n. The size of a graph G, written $e(G)$, is the number of*

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edges in G for $n \in \mathbb{N}$.

Definition 8 *Let $f\ (x,y)$ be areal valued function of two variable defined for $a \leq x \leq b$ and all real y, then*

$$\begin{aligned} y' &= f\ (x,y), \qquad x \in S = [0,T] \subseteq R \qquad (1) \\ y(x_0) &= y_0 \qquad\qquad\qquad\qquad\qquad\quad (2) \end{aligned} \tag{1.1}$$

is called initial value problem (I.V.P.), where (1) is called ordinary differential equation (O.D.E) of the first order and equation (2) is called the initial value.

Definition 9([3,6]) *For the problem* (1.1) *where the function $f\ (x,y)$ is continuous on the region $(0 \leq x \leq T, |y| \leq R)$ and differentiable with respect to x such that $\left|\frac{df}{dy}\right| \leq L, L = const.$Divide the segment $[0,T]$ into n equal parts by the points $x_i = ih, h = \frac{T}{n}$ is called a step size, $(i = \overline{0,n})$ such that $x_0 = 0 < x_1 < ... < x_{n-1} < x_n = T$ the approximate numerical solutions for this problem at the mesh points $x = x_i$ will be denoted by y_j.*

Definition 10([3]) *Numerical answers to problems generally contain errors. Truncation error occurs as a result of truncating an infinite process to get a finite process.*

Definition 11 *For Riemannian manifolds M and N (not necessarily of the same dimension), a map $f : M \to N$ is said to be a topological folding of M into N if, for each piecewise geodesic path $\gamma : I \to M(I = [0,1] \subseteq R)$, the induced path $f \circ \gamma : I \to N$ is piecewise geodesic. If, in addition, $f : M \to N$ preserves lengths of paths, we call f an isometric folding of M into N. Thus an isometric folding is necessarily a topological folding* [9]. *Some applications are introduced in* [5].

§2. Main Results

We will introduce several types of approximate differentiable graph which represent the solution of initial value problems **I**.**V**.**P**.

$$\begin{aligned} y' &= f(x,y), \\ y(x_0) &= y_0. \end{aligned} \tag{2.1}$$

According to the used methods for solving these problems.

Definition 12 *We can study the solution of ordinary differential equation $y' = f\ (x,y)$ using differentiable graph which is a smooth graph with vertex set $\{(x, y(x)) : x, y \in R\}$ and edge set $d((x_i, y(x_i)), (x_{i+1}, y(x_{i+1})))$ where d represent the distance function. A differentiable graph is a smooth graph represent the solution of ordinary differential equation $y' = f\ (x,y), x \in S$ whose vertices are $(x, y(x)), \forall x \in S$ and its edges are the distance between any two consequent points. In this graph the number of vertices is ∞,the number of edges is so.*

Since the finite difference methods which solve $(I.V.P.)$ divided into the following:

(i) general multi-step methods (implicit-explicit).

(ii) general single-step methods (implicit-explicit).

So we have the following new types of differentiable graph:

Type [1]: Single-Compound digraph H_{N_1}

Definition 13 *A numerical digraph G_N is a simple differentiable digraph consists of numerical vertices V_N^j which represent the numerical solutions y_j of (I.V.P.), and ordered numerical edge set $E_N = \{e_N^1, e_N^2, ..., e_N^n\}$ where $e_N^{j+1} = |(x_{j+1}, y_{j+1}) - (x_j, y_j)| = \left|v_N^{j+1} - v_N^j\right|, v_N^j$ is the tail of the edge, and $v_N^{j+1}$is the head.*

Definition 14 *A compound graph (digraph) H is a graph (digraph) whose vertex set consists of a set of graphs (digraphs) i.e. $V(G) = \{G_1, G_2, ...\}$ and an edge set of unordered (ordered) pairs of this graphs i.e. $E(G) = \{(G_1, G_2), (G_2, G_3), \cdots\}$.*

Corollary 1 *The compound digraph H of a numerical digraph is numerical digraph H_N.*

Definition 15 *A single-compound digraph H_{N_1} is a compound digraph H_N has one null graph is the tail of digraph.*

Theorem 1 *The single-step methods (implicit) due to a single-compound digraph H_{N_1}.*

Proof The basis of many simple numerical technique for solving the differential equation

$$y' = f(x, y), y(x_0) = y_0, a \le x \le b \tag{2.2}$$

is to find some means of expressing the solution at $x + h$ i.e., $y(x + h)$ in terms of $y(x)$.where $(x, y(x))$ represent a vertex in the differentiable graph, $(x + h, y(x + h))$ is the next vertex, the initial value (x_0, y_0) is called the source of graph. An approximate solution can be generated at the discrete points $x_0 + h, x_0 + 2h, ...$representing the vertices of the induced differentiable graph.

All these methods where y_{n+1} is given in terms of y_n alone, $n = 0, 1, 2,...$, are called single step methods. The general linear single step method is given by

$y_{n+1} + \alpha_1 y_n = h[\beta_0 f\ (x_{n+1}, y_{n+1}) + \beta_1 f\ (x_n, y_n)]$ where $\alpha_1, \beta_0, \beta_1$ are constants. If $\beta_0 = 0$ then the method gives y_{n+1} explicitly otherwise it is given implicitly. The trapezium method $y_{n+1} = y_n + \frac{h}{2}[f\ (x_{n+1}, y_{n+1}) + f\ (x_n, y_n)]$ is implicit. In general this equation would be solved by using the iteration method i.e.,

$\{y_{n+1}\}^{r+1} = y_n + \frac{h}{2}[f\ (x_{n+1}, y_{n+1}) + f\ (x_n, \{y_n\}^r)]$, $r = 0, 1, 2, ...$, where $\{y_{n+1}\}^0$ can be obtained from a single -step method and represents a source of numerical digraph G_{N+1}in the vertex V_{n+1} of compound graph H_N. Finally we get A single-compound digraph H_{N_1}. As shown in Figure 1. □

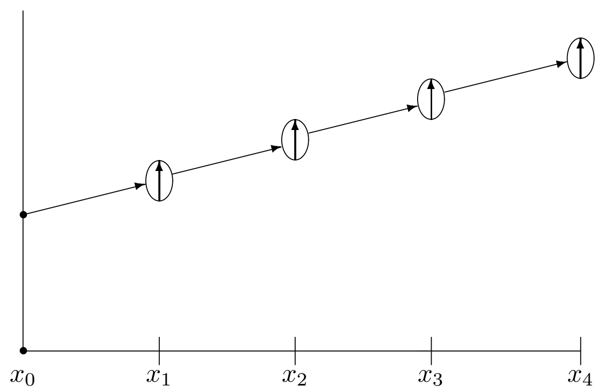

Figure 1. Single Compound Graph H_{N_1}

Definition 16 *A single-compound digraph H_{N_1} is a Compound numerical digraph has a unique null graph which is the source of graph.*

Type [2]: A simple numerical digraph G_N

Definition 17 *A numerical digraph G_N is a simple differentiable digraph consists of numerical vertices V_N^j which represent the numerical solutions y_j of (I.V.P.) and ordered numericaledge set $E_N = \{e_N^1, e_N^2, ..., e_N^n\}$ where $e_N^{j+1} = |(x_{j+1}, y_{j+1}) - (x_j, y_j)| = \left|v_N^{j+1} - v_N^j\right|, v_N^j$ is the tail of the edge, and $v_N^{j+1}$is the head.*

Theorem 2 *The explicit single-step method get a simple numerical digraph G_N.*

Proof The general single step given by

$$y_{n+1} = y_n + h\phi(x_n, y_n, h), x_n = x_0 + nh, y(x_0) = y_0.$$

For example, Euler's method has $\phi(x, y, h) = f(x, y)$,then

$y_{n+1} = y_n + hf(x_n, y_n)$, and for differential equation (2.1) give the following differentiable digraph (Figure 2)

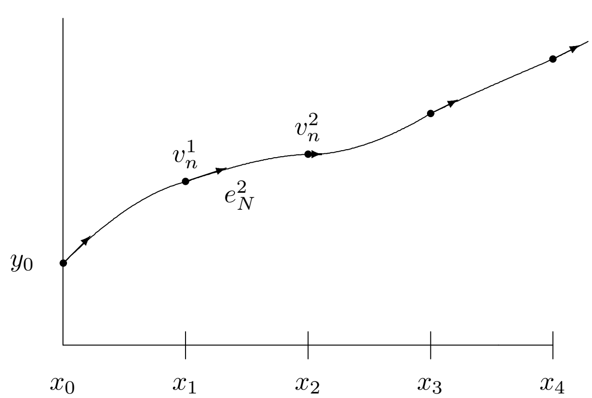

Figure 2: Simple numerical graph

where (x_n, y_n) represent the set of vertices $\left\{v_N^j\right\}, j = 0, 1, ...,$and $|(x_{j+1}, y_{j+1}) - (x_j, y_j)|$ represent the set of edges $\left\{e_N^{j+1}\right\}$.The initial value y_0 represent the source of simple numerical digraph G_N. □

Type [3]: **Multi-Compound Digraph** H_{N_m}

Definition 18 *A multi-compound digraph* H_{N_m} *is a compound digraph* H_N *has m null graphs are the tail of digraph.*

Theorem 3 *The implicit multi-step method give a multi-compound digraph* H_{N_m}.

Proof The general multi-step method is defined to be

$$y_{n+1} + \alpha_1 y_n + ... + \alpha_m y_{n-m+1} = h[\beta_0 f_{n+1} + \beta_1 f_n + ... + \beta_m f_{n-m+1}], \tag{2.3}$$

where f_p is used to denote $f(x_p, y_p), n = m-1, m-2, ...$. To apply this general method we need m steps which represent m null graphs $G_{N_0}, G_{N_1}, ..., G_{N_{m-1}}$ in a multi-compound digraph H_{N_m} as indicate in the following example. If $\beta_0 = 0$ then the method (2.3) gives y_{n+1} explicitly otherwise it is given implicitly, when $m = 1$ equation (2.3) reduce to the single step method.□

Example 1 Find the differentiable graph of $y' = y^2, y(0) = 1$ using a 3-step method.

Solution 1. by using

$$y_{n+1} - y_n = h[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]/24, h = 0.1, \tag{2.4}$$

then $n = 2, 3, ..., \Rightarrow \{y_3\}^{r+1} - y_2 = 0.1[9\{f_3\}^r + 19f_2 - 5f_1 + f_0]/24, r = 0, 1, 2, ...,$ so the iterative vertex $(x_3, \{y_3\}^{r+1})$ depend on the vertex $(x_3, \{y_3\}^0)$ which can be determined from an explicit 3-srep method say

$$y_{n+1} - y_n = h[23f_n - 16f_{n-1} + 5f_{n-2}]l12, \tag{2.5}$$

at $n = 2 \Rightarrow y_3 - y_2 = h[23f(x_2, y_2) - 16f(x_1, y_1) + 5f(x_0, y_0)]l12$, where $V_0 = (x_0, y_0), V_1 = (x_1, y_1), V_2 = (x_2, y_2)$ represent three null graphs $G_{N_0}, G_{N_1}, G_{N_2}$ in the induced compound digraph by predictor method (2.5) we get the vertex $v_{N_3} = (x_3, \{y_3\}^0)$which is the tail of the digraph G_{N_3} in the compound digraph H_{N_3}then correct $\{y_3\}^0$using equation (2.4) until we get the fixed vertex $v_{N_3}^f$. This gives a numerical digraph $G_{N_3} = V_3$ and similarly we get the other vertices (simple digraphs) $V_4 = G_{N_4}, ..., V_l = G_{N_l}, l$ is a + ve integer. Finally we get bounded compound digraph H_{N_3} as shown in Figure 3.

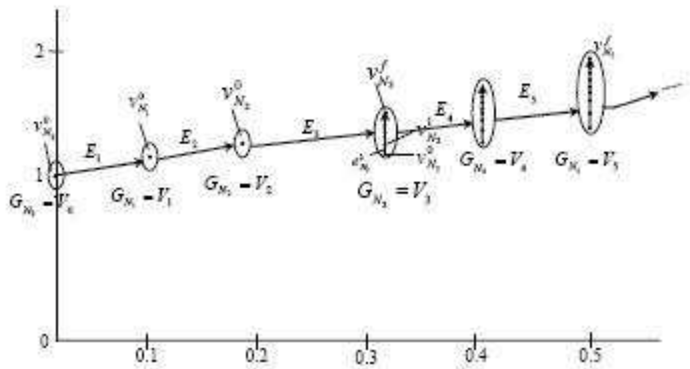

Figure 3: Multi-compound digraph

Definition 19 *A fixed vertex V_N^f is a numerical vertex which all next vertices coincided on it.*

Corollary 2 *The multi-compound digraph H_{N_m} must have null graphs.*

Type [4]: <u>Nonhomogeneous Numerical Digraph G_{N_m}</u>

Definition 20 *A nonhomogeneous graph G is a graph whose vertices divided into multi-groups such that each one has a specific character.*

Definition 21 *A nonhomogeneous numerical digraph G_{N_m} is a numerical digraph whose vertices divided into multi-groups such that each one has a specific character.*

Theorem 4 *The explicit multi -step method give nonhomogeneous numerical digraph G_{N_m}.*

Proof The general explicit multi-step method

$y_{n+1} + \alpha_1 y_n + ... + \alpha_m y_{n-m+1} = h[\beta_1 f_n + \beta_2 f_{n-1} + ... + \beta_m f_{n-m+1}]$, i.e., to determine the vertex (x_{n+1}, y_{n+1}) we need know m vertices begin from (x_0, y_0) up to (x_n, y_n).

for example: The difference method

$y_{n+1} - y_n = h[23f_n - 16f_{n-1} + 5f_{n-2}]/12, n = 2, 3, ...$, is 3-step method, the group of vertices $(x_3, y_3), (x_4, y_4), ...$, are given by this multi-step method whenever the group $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ are gotten from single-step method .See Figure 3. $\square$

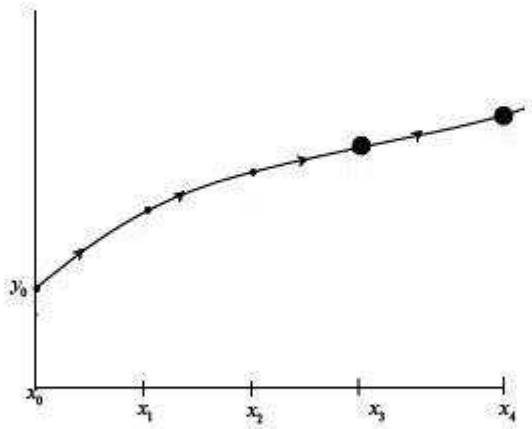

Figure 4: Nonhomogenous Numerical Digraph G_{N_m}

There is an important role to the step size h in the all types of numerical digraphs.

Definition 22 *The initial tight graph (digraph) T is a package of graphs (digraph) which have one source.*

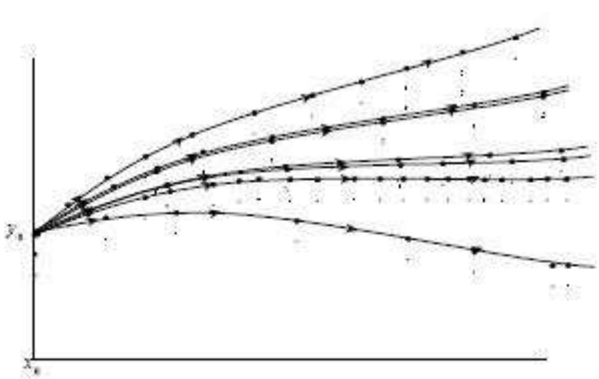

Figure 5: Initial Tight Digraph T

Theorem 5 *As the order of numerical digraph in bounded interval* $\rightarrow \infty$ *the consistent digraph is obtained.*

Proof Since the local error of the approximate solution of (I.V.P.)(1.1) depends on the step size h s.t. $\sup|E(h,x)| \leq Mh^k, where\ M, k$ is a positive integers [4], for all sufficiently small h, the order of bounded numerical digraph$\rightarrow \infty$, and then the difference method is said to be consistent of order k. □

Theorem 6 *The limit of foldings F_j of initial tight graph give a convergent numerical graph.*

Proof Let $F_i : T \to T$ be a folding map of an initial tight graph T s.t., $F_i(G_N^j) = G_N^m$, where order of $(G_N^j) \leq$ order of (G_N^m), then $\lim_{i\to\infty} F_i =$ The highest order numerical digraph, which is required. As shown in Figure 6. □

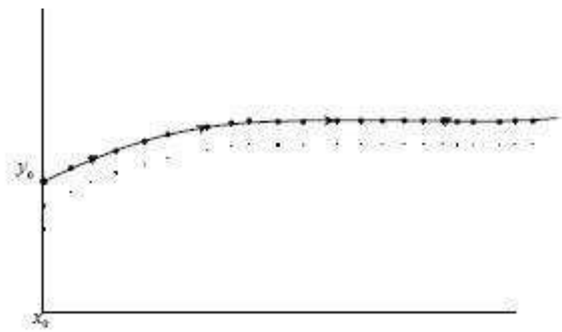

Figure 6: Limit of Foldings F_j

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