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Smarandachely Bondage Number of a Graph

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Abstract: A dominating set D of a graph G is called a *Smarandachely dominating s-set* if for an integer s, each vertex v in $V-D$ is adjacent to a vertex $u \in D$ such that $\deg u + s = \deg v$. The minimum cardinality of Smarandachely dominating s-set in a graph G is called the *Smarandachely dominating s-number* of G, denoted by $\gamma_S^s(G)$. Such a set with minimum cardinality is called a *Smarandachely dominating s-set*. The *Smarandachely bondage s-number* $b_S^s(G)$ of a graph G is defined to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_S^s(G-E') > \gamma_S^s(G)$. Particularly, the set with minimum Smarandachely bondage s-number for all integers $s \geq 0$ or $s \leq 0$ is called the *strong* or *weak dominating number* of G, denoted by $\gamma_s(G)$ or $\gamma_w(G)$, respectively. In this paper, we present some bounds on $b_s(G)$ and $b_w(G)$ and give exact values for $b_s(G)$ and $b_w(G)$ for complete graphs, paths, wheels and bipartite complete graphs. Some general bounds are also given.

Key Words: Smarandachely dominating s-set, Smarandachely dominating s-number, Smarandachely bondage s-number, strong or weak bondage numbers.

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§1. Introduction

In this paper, we follow the notation of [6,7]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $D \subseteq V$ is a dominating set of G if every vertex v in $V - D$ there exists a vertex u in D such that u and v are adjacent in G. The domination number of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. The concept of domination in graphs, with its many variations, is well studied in graph theory. A thorough study of domination appears in [6,7]. Let $uv \in E$. Then, u and v dominate each other. A dominating set D of a graph G is called a *Smarandachely dominating s-set* if for an integer s, each vertex

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v in $V-D$ is adjacent to a vertex $u \in D$ such that $\deg u + s = \deg v$. The minimum cardinality of Smarandachely dominating s-set in a graph G is called the *Smarandachely dominating s-number* of G, denoted by $\gamma_S^s(G)$. Such a set with minimum cardinality is called a *Smarandachely dominating s-set*. The *Smarandachely bondage s-number* $b_S^s(G)$ of a graph G is defined to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_S^s(G-E') > \gamma_S^s(G)$. Particularly, the set with minimum Smarandachely bondage s-number for all integers $s \geq 0$ or $s \leq 0$ is called the *strong* or *weak dominating number* of G, denoted by $\gamma_s(G)$ or $\gamma_w(G)$, respectively.

As a special case of Smarandachely bondage number, the strong (weak) domination was introduced by E. Sampathkumar and L.Pushpa Latha in [8]. For any undefined term, we refer Harary [4]. By definition, the bondage number $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ for which $\gamma(G-E') > \gamma(G)$. Thus, the bondage number of G is the smallest number of edges whose removal renders every minimum dominating set of G a nondominating set in the resulting spanning subgraph. Since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$, the bondage number of a nonempty graph is well defined. This concept was introduced by Bauer, Harary, Nieminen and Suffel [1] and has been further studied by Fink, Jacobson, Kinch and Roberts [2], Hartnell and Rall [5], etc. The strong bondage number of G, denoted $b_s(G)$, as the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_s(G-E') > \gamma_s(G)$. This concept was introduced by J. Ghoshal, R. Laskar, D. Pillone and C. Wallis [3].

We define the weak bondage number of G, denoted $b_w(G)$, as the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_w(G-E') > \gamma_w(G)$, and we deal with the strong bondage number of a nonempty graph G.

§2. Exact Values for $b_s(G)$ and $b_w(G)$

We begin our investigation of the strong and weak bondage numbers by computing its value for several well known classes of graphs. In several instances we shall have cause to use the ceiling function of a number x. This is denoted $\lceil x \rceil$ and takes the value of the least integer greater than or equal to x. We begin with a rather straightforward evaluation of the strong and weak bondage numbers of the complete graph of order n.

Proposition 2.1 *The strong bondage number of the complete graph K_n $(n \geq 2)$ is*

$$b_s(K_n) = \lceil n/2 \rceil.$$

Proof. Let $u_1, u_2, ..., u_n$ be the n vertices of degree $n-1$. Then clearly removal of fewer than $n/2$ edges results in a graph H having maximum degree $n-1$. Hence $b_s(K_n) \geq \lceil n/2 \rceil$. Now we consider the following cases.

Case 1. If n is even, then the removal of $n/2$ independent edges $u_1u_2, u_3u_4, ..., u_{n-1}u_n$ results in a graph H' regular of degree $n-2$. Hence $b_s(K_n) = n/2$.

Case 2. If n is odd, then the removal of $(n-1)/2$ independent edges $u_1u_2, u_3u_4, ..., u_{n-2}u_{n-1}$

yields a graph H'' containing exactly one vertex u_n of degree $n-1$. Thus by removing an edge incident with u_n we obtain a graph H''' with maximum degree $n-2$. Hence $b_s(K_n) = (n-1)/2+1$.

Combining cases (1) and (2) it follows that $b_s(K_n) = \lceil n/2 \rceil$. □

Proposition 2.2 *The weak bondage number of the complete graph K_n $(n \geq 2)$ is*

$$b_w(K_n) = 1.$$

Proof If H is a spanning subgraph of K_n that is obtained by removing any edge from K_n, then H contains two vertices of degree $n-2$. Whence $\gamma_w(H) = 2 > 1 = \gamma_w(K_n)$. Hence $b_w(K_n) = 1$. □

If G is a regular graph, then $\gamma(G) = \gamma_s(G)$ because in a regular graph, the degrees of all the vertices are equal. We next consider paths P_n and cycles C_n on n vertices and find that $\gamma(C_n) = \gamma_s(C_n)$ because C_n is a regular graph. Also $\gamma(P_n) = \gamma_s(P_n)$ since we can choose from all the γ sets of P_n, one which dose not include either end vertex. Such a γ set is also a γ_s set and hence we get $\gamma(P_n) \geq \gamma_s(P_n)$ but since $\gamma(G) \geq \gamma_s(G)$ for all graphs G, which follows

Lemma 2.3 *The strong domination number of the n-cycle and the path of order n are respectively*

(*i*) $\gamma_s(C_n) = \lceil n/3 \rceil$ *for* $n \geq 3$ *and*

(*ii*) $\gamma_s(P_n) = \lceil n/3 \rceil$ *for* $n \geq 2$.

Lemma 2.4 *The weak domination number of the $n - cycle$ and the path of order n are respectively*

(*i*) $\gamma_w(C_n) = \lceil n/3 \rceil$ *for* $n \geq 3$ *and*

(*ii*)

$$\gamma_w(P_n) = \begin{cases} \lceil n/3 \rceil & \text{if } n \equiv 1 \text{ (mod3)}, \\ \lceil n/3 \rceil + 1 & \text{otherwise.} \end{cases}$$

Proof (*i*) Since C_n is a regular graph, so $\gamma_w(C_n) = \gamma(C_n)$ and proof techniques in [2].

(*ii*) $\gamma_w(P_n) = \lceil (n-4)/3 \rceil + 2 = \gamma(P_{n-4}) + 2$, the proof is the same as in [2]. □

Theorem 2.5 *The strong bondage number of the n-cycle (with $n \geq 3$) is*

$$b_s(C_n) = \begin{cases} 3 & \text{if } n \equiv 1 \text{ (mod3)}, \\ 2 & \text{otherwise.} \end{cases}$$

Proof Since $\gamma_s(C_n) = \gamma_s(P_n)$ for $n \geq 3$, we see that $b_s(C_n) \geq 2$. If $n \equiv 1$ (mod3) the removal of two edges from C_n leaves a graph H consisting of two paths P and Q. If P has order n_1 and Q has order n_2, then either $n_1 \equiv n_2 \equiv 2$ (mod3), or, without loss of generality,

$n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 1 \pmod 3$. In the former case,

$$\begin{aligned}\gamma_s(H) &= \gamma_s(P)+\gamma_s(Q) = \lceil n_1/3\rceil + \lceil n_2/3\rceil \\ &= (n_1+1)/3+(n_2+1)/3 = (n_1+n_2+2)/3 = (n+2)/3 = \lceil n/3\rceil = \gamma_s(C_n).\end{aligned}$$

In the latter case.

$$\gamma_s(H) = \gamma_s(P)+\gamma_s(Q) = n_1/3+(n_2+2)/3 = (n+2)/3 = \lceil n/3\rceil = \gamma_s(C_n).$$

In either case, when $n \equiv 1 \pmod 3$ we have $b_s(C_n) \geq 3$. Now we consider two cases.

Case 1 Suppose that $n \equiv 0, 2 \pmod 3$. The graph H obtained removing two adjacent edges from C_n consist of an isolated vertex and a path of order $n-1$. Thus

$$\gamma_s(H) = \gamma_s(P_1)+\gamma_s(P_{n-1}) = 1+\lceil (n-1)/3\rceil = 1+\lceil n/3\rceil = 1+\gamma_s(C_n),$$

Whence $b_s(C_n) \leq 2$ in this case. Combining this with the upper strong bondage obtained earlier, we have $b_s(C_n) = 2$ if $n \equiv 0, 2 \pmod 3$.

Case 2 Suppose now that $n \equiv 1 \pmod 3$. The graph H resulting from the deletion of three consecutive edges of C_n consists of two isolated vertices and a path of order $n-2$. Thus,

$$\gamma_s(H) = 2+\lceil (n-2)/3\rceil = 2+(n-1)/3 = 2+(\lceil n/3\rceil - 1) = 1+\gamma_s(C_n),$$

So that $b_s(C_n) \leq 3$. With the earlier inequality we conclude that $b_s(C_n) = 3$ when $n \equiv 1 \pmod 3$. □

Theorem 2.6 *The weak bondage number of the n-cycle (with $n \geq 3$) is*

$$b_w(C_n) = \begin{cases} 2 & \text{if } n \equiv 1 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

Proof Assume $n \not\equiv 1 \pmod 3$ since $\gamma_w(P_n) = \lceil n/3\rceil + 1 = \gamma_w(C_n)+1 > \gamma_w(C_n)$. Hence $b_w(C_n) = 1$. Now assume $n \equiv 1 \pmod 3$ since $\gamma_w(C_n) = \gamma_w(P_n)$ it follows that $b_w(C_n) \geq 2$.

Let H be the graph obtained by the removal of two edges from C_n such that P_3 and P_{n-3} are formed. Then $\gamma_w(H) = \gamma_w(P_3)+\gamma_w(P_{n-3}) = 2+\lceil (n-3)/3\rceil = 2+\lceil n/3\rceil - 1 = \lceil n/3\rceil + 1 > \gamma_w(C_n)$. Hence $b_w(C_n) \leq 2$ thus $b_w(C_n) = 2$. □

As an immediate Corollary to Theorem 2.5 we have the following.

Corollary 2.7 *The strong bondage number of the path (with $n \geq 3$) is given by*

$$b_s(P_n) = \begin{cases} 2 & \text{if } n \equiv 1 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2.8 *The weak bondage number of the path (with $n \geq 3$) is*

$$b_w(P_n) = \begin{cases} 2 & \textit{if } n = 3, 5, \\ 1 & \textit{otherwise.} \end{cases}$$

Proof It is easy to verify that $b_w(P_n) = 2$ for $n = 3, 5$.

Let H be the graph obtained by the removal of one edge from P_n such that P_3 and P_{n-3} are formed. Then $\gamma_w(H) = \gamma_w(P_3) + \gamma_w(P_{n-3})$. Now we consider the following cases.

Case 1 If $n \equiv 1$ (mod3) then $\gamma_w(H) = \gamma_w(P_3) + \gamma_w(P_{n-3}) = 2 + \lceil (n-3)/3 \rceil = 2 + \lceil n/3 \rceil - 1 = \lceil n/3 \rceil + 1$ then $\gamma_w(H) > \gamma_w(P_n)$. Hence $b_w(P_n) = 1$.

Case 2 If $n \not\equiv 1$ (mod3) we have $\gamma_w(H) = 2 + \lceil (n-3)/3 \rceil + 1 = 2 + \lceil n/3 \rceil - 1 + 1 = 2 + \lceil n/3 \rceil > \gamma_w(P_n)$ then $\gamma_w(H) > \gamma_w(P_n)$. Hence $b_w(P_n) = 1$. □

Lemma 2.9 *The strong and weak domination numbers of the wheel W_n (with $n \geq 4$) are*

(*i*) $\gamma_s(W_n) = 1$;

(*ii*) $\gamma_w(W_n) = \lceil (n-1)/3 \rceil$.

Proof (*i*) Since $\gamma(W_n) = \gamma_s(W_n)$ so proof techniques same in [2].

(*ii*) Since $\gamma_w(W_n) = \gamma(C_{n-1}) = \lceil (n-1)/3 \rceil$ so proof techniques same in [2]. □

Proposition 2.10 *The strong bondage number of the wheel W_n (with $n \geq 4$) is $b_s(W_n) = 1$.*

Proof Let x be the vertex of maximum degree of W_n. Let v be a vertex of W_n such that *deg* $v <$ *deg* x. Let H be the graph obtained from W_n by removing edge xv. Then no one vertex strongly dominates H. So $\gamma_s(W_n - xv) > \gamma_s(W_n)$. Hence $b_s(W_n) = 1$. □

Proposition 2.11 *The weak bondage number of W_n (with $n \geq 4$) is given by*

$$b_w(W_n) = \begin{cases} 2 & \text{if } n \equiv 2 \text{ (mod3)}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof Assume $n \equiv 0, 1$ $(mod 3)$, let e be an edge on the $(n-1)$-cycle. Then $\gamma_w(W_n - e) = \lceil (n-5)/3 \rceil + 2 = \lceil (n-2)/3 \rceil + 1 = \lceil (n-1)/3 \rceil + 1 > \lceil (n-1)/3 \rceil = \gamma_w(W_n)$, whence $b_w(W_n) = 1$.

Now assume $n \equiv 2$ $(mod 3)$, the removal of any one edge from W_n will not alter $\gamma_w(W_n)$. So when $n \equiv 2$ $(mod 3)$ we have $b_w(W_n) \geq 2$.

Let H be the graph obtained by the removal of two adjacent edges from W_n such that these edges are not incident with the vertex of maximum degree. Then $\gamma_w(H) = \lceil (n-6)/3 \rceil + 3 = \lceil n/3 \rceil + 1 = \lceil (n-1)/3 \rceil + 1 > \lceil (n-1)/3 \rceil = \gamma_w(W_n)$, whence $b_w(W_n) = 2$. □

Lemma 2.12 *The strong and weak domination numbers of the $K_{r,t}$ are*

(*i*)

$$\gamma_s(K_{r,t}) = \begin{cases} 2 & \text{if } 2 \leq r = t, \\ \text{r} & \text{if } 1 \leq r < t. \end{cases}$$

(ii)

$$\gamma_w(K_{r,t}) = \begin{cases} \text{t} & \text{if } 1 \leq r < t, \\ 2 & \text{if } 2 \leq r = t. \end{cases}$$

Proof (i) see [3].

(ii) Note that the vertices in the second partite set have the smallest degree. If $1 \leq r < t$, then to weakly dominate these vertices, we need include all of them in any wd-set and these suffice to weakly dominate the rest. If $r = t \geq 2$, we claim $\gamma_w = 2$. Since $t \geq 2$, none of the vertices in the graph are of full degree hence γ_w in this case is greater than 1. Now to demonstrate a wd-set of cardinality 2, we can take one vertex from the first partite set which weakly dominate the rest of the vertices in the first partite set, we use a vertex from the second partite set. Note that a vertex from the second partite set has equal degree as the vertices in the first set since $r = t$. □

The next theorem establishes the strong and weak bondage numbers of the complete bipartite graph $K_{r,t}$.

Theorem 2.13 *Let $K_{r,t}$ be a complete bipartite graph, where $4 \leq r \leq t$, then*

$$b_s(K_{r,t}) = \begin{cases} \text{2r} & \text{if } t = r + 1, \\ \text{r} & \text{otherwise.} \end{cases}$$

Proof Let $V = V_1 \cup V_2$ be the vertex set of $K_{r,t}$ such that $|V_1| = r$ and $|V_2| = t$. We consider the following cases.

Case 1 Suppose $t = r + 1$ and $v \in V_2$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r,t-1}$. Hence
$\gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + 2 < r = \gamma_s(K_{r,t})$. Now let $v \in V_2$ and $u \in V_1$ be a vertex of $K_{r,t}$, then by removing all edges incident to both u and v, we obtain a graph H containing two components $2K_1$ and $K_{r-1,t-1}$, thus

$$\gamma_s(H) = 2\gamma_s(K_1) + \gamma_s(K_{r-1,t-1}) = 2 + r - 1 = r + 1 > r = \gamma_s(K_{r,t}).$$

Hence

$$b_s(K_{r,t}) = deg\ u + deg\ v - 1 = |V_2| + |V_1| - 1 = t + r - 1 = 2r$$

for $t = r + 1$.

Case 2 Suppose $r = t$, then by Lemma 2.12, $\gamma_s(K_{r,t}) = 2$. Let $v \in V_2$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r,t-1}$, thus
$\gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + t - 1 = t = r > 2 = \gamma_s(K_{r,t})$. Hence $b_s(K_{r,t}) = deg\ v = |V_1| = r$ for $r = t$.

Case 3 Suppose $r+1 < t$, then by Lemma 2.12, $\gamma_s(K_{r,t}) = r$. Let $v \in V_2$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r,t-1}$. Hence

$\gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + r > r = \gamma_s(K_{r,t})$. Thus $b_s(K_{r,t}) = deg\ v = |V_1| = r$ for $r + 1 < t$. □

Theorem 2.14 *Let $K_{r,t}$ be a complete bipartite graph, where $1 \le r \le t$, then $b_w(K_{r,t}) = t$.*

Proof Let $V = V_1 \cup V_2$ be the vertex set of $K_{r,t}$ where $|V_1| = r$ and $|V_2| = t$. Let $v \in V_1$ and $r = t \ge 2$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r-1,t}$. Hence

$\gamma_w(H) = \gamma_w(K_1) + \gamma_w(K_{r-1,t}) = 1 + t > 2 = \gamma_w(K_{r,t})$. Thus

$b_w(K_{r,t}) = deg\ v = |V_2| = t$.

Now suppose $r < t$ and $v \in V_1$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r-1,t}$. Hence

$\gamma_w(H) = \gamma_w(K_1) + \gamma_w(K_{r-1,t}) = 1 + t > t = \gamma_w(K_{r,t})$. Thus

$$b_w(K_{r,t}) = \text{deg} v = |V_2| = t.$$ □

§3. The Strong and Weak Bondage Numbers of a Tree

We now consider the strong and weak bondage numbers for a tree T. Define a support to be a vertex in a tree which is adjacent to an end-vertex (see [3]).

Proposition 3.1 *Every tree T with $(n \ge 4)$ has at least one of the following characteristics.*

(1) A support adjacent to at least 2 end-vertex.

(2) A support is adjacent to a support of degree 2.

(3) A vertex is adjacent to 2 support of degree 2.

(4) The support of a leaf and the vertex adjacent to the support are both of degree 2.

Proof See [3] for the proof. □

Theorem 3.2 *If T is a nontrivial tree then $b_s(T) \le 3$.*

Proof See [3] for the proof. □

Proposition 3.3 *If any vertex of tree T is adjacent with two or more end-vertices, then $b_s(T) = 1$.*

Proof Let u be a cut vertex adjacent two or more end-vertices. Then u belongs to every minimum strong dominating set of T. Let v be an end-vertex adjacent to u. Then $T - uv$ contains an isolated vertex and a tree T' of order $n - 1$. Therefore $\gamma_s(T - uv) = \gamma_s(T') + 1 > \gamma_s(T)$. Hence $b_s(T) = 1$. □

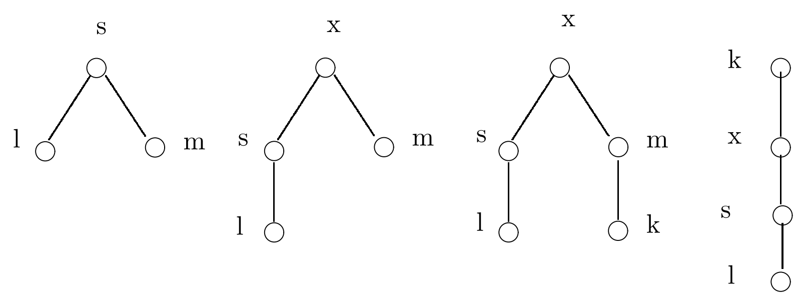

Fig.1: End characteristics of trees in Case 2 of the Proof of Theorem 3.4

Theorem 3.4 *If T is a nontrivial tree, then $b_w(T) \leq \Delta(T)$.*

Proof The statement is obviously true for trees order 2 or 3, so we shall suppose that T has at least 4 vertices. Now we consider the following cases.

Case 1 Suppose T has a support vertex s that is adjacent to two (and possibly more) end-vertex, that dose not belong to a weak dominating set. Let E_s denote the set of edges incident with s. And let D be a minimum weak dominating set for $T - E_s$. Then s is in D and $D \setminus \{s\}$ is a weak dominating set for T. Hence $\gamma_w(T - E_s) > \gamma_w(T)$ thus $b_w(T) \leq |E_s| = deg\ s \leq \Delta(T)$.

Case 2 Suppose a support vertex is adjacent to a support vertex of degree 2. Delete the edge (s, l). The vertex x then has two end-vertices an adjacent to s and m. Let D be wd-set of $T - \{(s, l)\}$. Then s is in D and $D \setminus \{s\}$ is a weak dominating set for T. Hence $b_w(T)$ in this case equals 1.

Case 3 In this case delete the edge (s, l). If $\gamma_w(T - \{(s, l)\}) < \gamma_w(T)$, then it will contradict the assumption that the γ_w-set was the smallest wd-set for T. If $\gamma_w(T - \{(s, l)\})$ is greater that $\gamma_w(T)$ then we have done. If $\gamma_w(T - \{(s, l)\}) = \gamma_w(T)$, then the vertex x has a one support vertex s in $T - \{(s, l)\}$, that adjacent to it. then by Case 2, deleting on more edge ($\{m, k\}$) will increase the weak domination number of the resulting graph. So in this case $b_w(T) = 2$.

Case 4 In the last case, either s or l is any weak dominating set of T. By removing edges (k, x) and (x, s), we make the necessary for any γ_w-set for the resulting graph to contain x and so $b_w(T) = 2$ this completes the proof. □

Theorem 3.5 *Let T be a tree. Then $b_w(T) = \Delta(T)$ if and only if $T = K_{1,r}$.*

Proof This follows from Theorem 3.4. □

§4. General Bounds on Strong and Weak Bondage Numbers

Proposition 4.1([2]) *If G is a nonempty graph, then*

$$b(G) \leq min\{deg\ u + deg\ v - 1 : \ u \ and \ v \ are \ adjacent\}.$$

Theorem 4.2 *If $\gamma(G) = \gamma_s(G)$ and $\gamma(G) = \gamma_w(G)$ then,*

(i) $b_s(G) \leq b(G)$;
(ii) $b_w(G) \leq b(G)$.

Proof Let E be a b-set of G. Then $\gamma_s(G) = \gamma(G) < \gamma(G - E) \leq \gamma_s(G - E)$. Thus $b_s(G) \leq b(G)$ and for (ii) proof is same. □

Theorem 4.3 *If G is a nonempty graph and $\gamma(G) = \gamma_s(G)$ then*

$$b_s(G) \leq min\{deg\ u + deg\ v - 1 \mid\ u \ \ and \ \ v \ \ are \ \ adjacent\}.$$

Proof This follows from Proposition 4.1 and Theorem 4.2. □

Theorem 4.4 *For any graph G,*

$$b_s(G) \leq q - p + \gamma_s(G) + 1$$

Proof Let D be a γ_s-set of a graph G. For each vertex $v \in V \setminus D$ choose exactly one edge which is incident to v and to a vertex in D. Let E_0 be the set of all such edges. Then clearly $\gamma_s(G-(E-E_0)) = \gamma_s(G)$ and $|E-E_0| = q-p+\gamma_s(G)$. So for any edge $e \in G-(E-E_0) = E_0$ we see that $\{E - E_0\} \cup \{e\}$ is a strong bondage set of G. Thus

$$b_s(G) \leq q - p + \gamma_s(G) + 1 \quad \square$$

Corollary 4.5 *For any graph G,*

$$b_s(G) \leq q - \Delta(G) + 1$$

Proof In [8], We have known that $\gamma_s(G) \leq p - \Delta(G)$. By applying Theorem 4.4, we get that $b_s(G) \leq q - \Delta(G) + 1$. □

Theorem 4.6 *If G is a nonempty graph with strong domination number $\gamma_s(G) \geq 2$, Then*

$$b_s(G) \leq (\gamma_s(G) - 1)\Delta(G) + 1.$$

Proof We proceed by induction on the strong domination number $\gamma_s(G)$. Let G be a nonempty graph with $\gamma_s(G) = 2$, and assume that $b_s(G) \geq \Delta(G) + 2$, then, if u is a vertex of maximum degree in G, we have $\gamma_s(G-u) = \gamma_s(G)-1 = 1$, and $b_s(G-u) \geq 2$. Since $\gamma_s(G) = 2$ and $\gamma_s(G - u) = 1$, there is a vertex v that is adjacent with every vertex of G but u, that $deg_G v = \Delta(G)$ also, and u is adjacent with every vertex of G except v. Since $b_s(G-u) \geq 2$, the removal from $G-u$ of any one edge incident with v again leaves a graph with strong domination number 1. Thus there is a vertex $w \neq v$ that is adjacent with every vertex of $G-u$. But, since v is the only vertex of G that is not adjacent with u, vertex w must be adjacent in G with u. This however implies that $\gamma_s(G) = 1$, a contradiction. Thus $b_s(G) \leq \Delta(G) + 1$ if $\gamma_s(G) = 2$.

Now, let $(k \geq 2)$ be any integer for which the following statement is true: If H is nonempty graph with $\gamma_s(G) = k$, then $\gamma_s(H) \leq (k-1)\cdot\Delta(H)+1$. Let G be a graph nonempty graph with

$\gamma_s(G) = k+1$, and assume that $b_s(G) > k\cdot\Delta(G)+1$. Then. But then, $b_s(G) \le b_s(G-u)+deg\ u$, and by the inductive hypothesis we have

$$b_s(G) \le [(k-1)\cdot\Delta(G-u)+1] + deg\ u \le (k-1)\cdot\Delta(G)+1+\Delta(G),$$

or

$$b_s(G) \le k\cdot\Delta(G)+1,$$

a contradiction to our assumption that $b_s(G) > k\cdot\Delta(G)+1$. Thus, $b_s(G) \le k\cdot\Delta(G)+1$, and, by the principle of mathematical induction, the proof is complete. □

Theorem 4.7 *If G is a planar graph, then*

$$b_w(G) \le \Delta(G).$$

Proof Suppose G has a vertex u with maximum degree that dose not belong to a weak dominating set. Let E_u denote the set of edges incident with u. And let D be a minimum weak dominating set for $G - E_u$. Then u is in D and $D \setminus u$ is a weak dominating set for G. Hence $\gamma_w(G-E_u) > \gamma_w(G)$ thus $b_w(G) \le |E_u| = deg\ u \le \Delta(G)$. □

§5. Open Problems

We strongly believe the following to be true.

Theorem 5.1 *If G is a nonempty graph of order $(n \ge 2)$ then $b_w(G) \le n-1$.*

Theorem 5.2 *If G is a nonempty graph of order $(n \ge 2)$ then $b_w(G) \le n-\delta(G)$.*

Theorem 5.3 *If G is a nonempty graph of order $(n \ge 2)$ then $b_s(G) \le n-1$.*

Other bounds for the strong and weak bondage of a graph exist. For several classes of graphs, $b_s(G) \le \Delta(G)$ and $b_w(G) \le \Delta(G)$. Let F be the set of edges incident with a vertex of maximum degree. Then it can be shown that $\gamma_s(G-F) \ge \gamma_s(G)$ and similarly $\gamma_w(G-F) \ge \gamma_w(G)$. But it is not necessary that this action would result in an increase in the strong and weak domination numbers. See Fig.2. The calculation for the strong and weak bondage for multipartite graphs remains open. Unions, joins and product of graphs could be investigated for their strong and weak bondage in terms of the constituent graphs. This implies that we need to calculate the strong and weak domination of these graphs. The problem of strong and weak domination is virtually unexplored and so there are several classes of graphs for which the strong and weak domination numbers could be calculated.

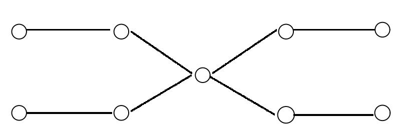

Fig. 2

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