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On the Bicoset of a Bivector Space

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Abstract: The study of bivector spaces was first intiated by Vasantha Kandasamy in [1]. The objective of this paper is to present the concept of bicoset of a bivector space and obtain some of its elementary properties.

Key Words: bigroup, bivector space, bicoset, bisum, direct bisum, inner biproduct space, biprojection.

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§1. Introduction and Preliminaries

The study of bialgebraic structures is a new development in the field of abstract algebra. Some of the bialgebraic structures already developed and studied and now available in several literature include: bigroups, bisemi-groups, biloops, bigroupoids, birings, binear-rings, bisemi-rings, biseminear-rings, bivector spaces and a host of others. Since the concept of bialgebraic structure is pivoted on the union of two non-empty subsets of a given algebraic structure for example a group, the usual problem arising from the union of two substructures of such an algebraic structure which generally do not form any algebraic structure has been resolved. With this new concept, several interesting algebraic properties could be obtained which are not present in the parent algebraic structure. In [1], Vasantha Kandasamy initiated the study of bivector spaces. Further studies on bivector spaces were presented by Vasantha Kandasamy and others in [2], [4] and [5]. In the present work however, we look at the bicoset of a bivector space and obtain some of its elementary properties.

Definition 1.1([2]) *A set* $(G, +, \cdot)$ *with two binary operations* $+$ *and* $\cdot$ *is called a bigroup if there exists two proper subsets* G_1 *and* G_2 *of* G *such that:*

(*i*) $G = G_1 \cup G_2$;

(*ii*) $(G_1, +)$ *is a group;*

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(iii) $(G_2, .)$ *is a group.*

Definition 1.2([2]) *A nonempty subset* H *of a bigroup* $(G, +, \cdot)$ *is called a subbigroup if* H *is itself a bigroup under* $+$ *and* $\cdot$ *defined on* G.

Theorem 1.3([2]) *Let* $(G, +, \cdot)$ *be a bigroup. The nonempty subset* H *of* G *is a subbigroup if and only if there exists two proper subsets* G_1 *and* G_2 *such that:*

(i) $G = G_1 \cup G_2$, *where* $(G_1, +)$ *and* $(G_2, .)$ *are groups;*

(ii) $(H \cap G_1, +)$ *is a subgroup of* $(G_1, +)$;

(iii) $(H \cap G_2, .)$ *is a subgroup of* $(G_2, .)$.

Definition 1.4([2]) *Let* $(G, +, \cdot)$ *be a bigroup where* $G = G_1 \cup G_2$. G *is said to be commutative if both* $(G_1, +)$ *and* $(G_2, \cdot)$ *are commutative.*

Definition 1.5([1]) *Let* $V = V_1 \cup V_2$ *where* V_1 *and* V_2 *are proper subsets of* V. V *is said to be a bivector space over the field* F *if* V_1 *and* V_2 *are vector spaces over the same field F. In this case,* V *is a bigroup.*

Definition 1.6([1]) *Let* $V = V_1 \cup V_2$ *be a bivector space. If* $\dim V_1 = m$ *and* $\dim V_2 = n$, *then* $\dim V = m + n$. *Thus there exists only* $m + n$ *linearly independent elements that can span* V. *In this case,* V *is said to be finite dimensional.*

If one of V_1 *or* V_2 *is infinite dimensional, we call* V *an infinite dimensional bivector space.*

Theorem 1.7([1]) *The bivector spaces of the same dimension over the same field need not be isomorphic in general.*

Theorem 1.8([1]) *Let* $V = V_1 \cup V_2$ *and* $W = W_1 \cup W_2$ *be two bivector spaces of the same dimension over the same field* F. *Then* V *is isomorphic to* W *if and only if* V_1 *is isomorphic to* W_1 *and* V_2 *is isomorphic to* W_2.

Example 1.9 Let $V = V_1 \cup V_2$ and $W = W_1 \cup W_2$ be two bivector spaces over a field $F = \mathcal{R}$. Suppose that $V_1 = F^4$, $V_2 = \left\{ \begin{bmatrix} v_2^1 & v_2^2 \\ 0 & v_2^3 \end{bmatrix} : v_2^i \in F, i = 1, 2, 3 \right\}$, $W_1 = P_3(F)$ (a space of polynomials of degrees ≤ 3 with coefficients in F) and $W_2 = F^3$. Clearly $\dim \mathrm{V} = \dim \mathrm{W} = 7$, $\dim V_1 = \dim W_1 = 4$ and $\dim V_2 = \dim W_2 = 3$. Since $V_1 \cong W_1$ and $V_2 \cong W_2$ in this case, it follows that V and W are isomorphic bivector spaces.

Theorem 1.10 *Let* $V = V_1 \cup V_2$ *be a bivector space over a field* F. *A nonempty subset* $W = W_1 \cup W_2$ *of* V *is a sub-bivector space of* V *if and only if* $W_1 = W \cap V_1$ *and* $W_2 = W \cap V_2$ *are subspaces of* V_1 *and* V_2 *respectively.*

Proof Suppose that $W = W_1 \cup W_2$ is a sub-bivector space of a bivector space $V = V_1 \cup V_2$ over F. It is clear that $W \cap V_1$ and $W \cap V_2$ are subspaces of V_1 and V_2 respectively over F. The required result follows immediately by taking $W_1 = W \cap V_1$ and $W_2 = W \cap V_2$.

Conversely, suppose that $V = V_1 \cup V_2$ is a bivector space over F and and $W = W_1 \cup W_2$ is a nonempty subset of V such that $W_1 = W \cap V_1$ and $W_2 = W \cap V_2$ are subspaces of V_1 and V_2, respectively. We then have to show that W is a bivector space over F. To do this, it suffices to show that $W = (W \cap V_1) \cup (W \cap V_2)$. Obviously, $W \subseteq V_1 \cup W, W \cup V_2 \subseteq V$ and $W \subseteq W \cup V_2$. Now,

$$\begin{aligned}
(W \cap V_1) \cup (W \cap V_2) &= [(W \cap V_1) \cup W] \cap [(W \cap V_1) \cup V_2] \\
&= [(W \cup W) \cap (V_1 \cup W)] \cap [(W \cup V_2) \cap (V_1 \cup V_2)] \\
&= [W \cap (V_1 \cup W)] \cap [(W \cup V_2) \cap V] \\
&= W \cap (W \cup V_2) \\
&= W.
\end{aligned}$$

This shows that $W = (W \cap V_1) \cup (W \cap V_2)$ is a bivector space over F. □

§2. Main Results

Definition 2.1 *Let $V = V_1 \cup V_2$ be a bivector space over a field F and let $W = W_1 \cup W_2$ be a sub-bivector space of V. Let $v_0 \in V$ and $w \in W$ be such that $v_0 = v_0^1 \cup v_0^2$ and $w = w^1 \cup w^2$ where $v_0^i \in V_i, i = 1,2$ and $w^i \in W_i, i = 1,2$. Let P be a set defined by*

$$\begin{aligned}
P &= \{v_0 + W : v_0 \in V\} \\
&= \left\{\left(v_0^1 \cup v_0^2\right) + \left(w^1 \cup w^2\right) : v_0^i \in V_i, i = 1,2\right\} \\
&= \left\{\left(v_0^1 + W_1\right) \cup \left(v_0^2 + W_2\right) : v_0^i \in V_i, i = 1,2\right\} \\
&= \left\{\left(v_0^1 + w^1\right) \cup \left(v_0^2 + w^2\right) : v_0^i \in V_i, w^i \in W_i, i = 1,2\right\}.
\end{aligned}$$

Then P is called a bicoset of V determined by W and v_0 is a fixed bivector in V.

Example 2.2 Let $W = W_1 \cup W_2$ be any sub-bivector space of a bivector space $V = V_1 \cup V_2$ over a field $F = \mathcal{R}$. Let $V_1 = F^3$ and $V_2 = P_2(F)$ (a space of polynomials of degrees ≤ 2 with coefficients in F). Let W_1 and W_2 be defined by

$$\begin{aligned}
W_1 &= \{(a,b,c) : 3a + 2b + c = 0, a,b,c \in F\}, \\
W_2 &= \left\{p(x) : p(x) = a_2x^2 + a_1x + a_0, a_i \in F, i = 0,1,2\right\}.
\end{aligned}$$

If $v = v_1 \cup v_2$ is any bivector in V, then $v_1 = \left(v_1^1, v_1^2, v_1^3\right) \in V_1$, where $v_1^i \in F, i = 1,2,3$ and also $v_2 = b_2x^2 + b_1x + b_0$, where $b_i \in F, i = 0,1,2$. Now, the bicoset of V determined by W is obtained as

$$\left[3\left(a - v_1^1\right) + 2\left(b - v_1^2\right) + \left(c - v_1^3\right)\right] \cup \left[(b_2 - a_2)x^2 + (b_1 - a_1)x^1 + (b_0 - a_0)\right].$$

Proposition 2.3 *Let S be a collection of bicosets of a bivector space $V = V_1 \cup V_2$ over a field F determined by sub-bivector space $W = W_1 \cup W_2$. Then S is not a bivector space over F.*

Proof Let $P = P_1 \cup P_2 = \left(v_1^1 + W_1\right) \cup \left(v_1^2 + W_2\right)$ and $Q = Q_1 \cup Q_2 = (v_2^1 + W_1) \cup (v_2^2 + W_2)$ be arbitrary members of S with $v_i^j \in V_i, i, j = 1, 2$. Clearly, $P_1 = v_1^1 + W_1, P_2 = v_1^2 + W_2, Q_1 = v_2^1 + W_1, Q_2 = v_2^2 + W_2$ are vector spaces over F and $P \cup Q = [P_1 \cup (P_1 \cup Q_1)] \cup [P_1 \cup (P_2 \cup Q_2)]$. Since $[P_1 \cup (P_1 \cup Q_1)]$ and $[P_1 \cup (P_2 \cup Q_2)]$ are obviously not vector spaces over F, it follows that S is not a bivector space over F. □

This is another marked difference between a vector space and a bivector space. We also note that $P \cap Q = [(P_1 \cap Q_1) \cup (P_2 \cap Q_1)] \cup [(P_1 \cap Q_2) \cup (P_2 \cap Q_2)]$ is also not a bivector space over F since it is a union of two bivector spaces and not a union of two vector spaces over F.

Proposition 2.4 *Let $W = W_1 \cup W_2$ be a sub-bivector space of a bivector space $V = V_1 \cup V_2$ and let $P = (v_0^1 + W_1) \cup (v_0^2 + W_2)$ be a bicoset of V determined by W where $v_0 = v_0^1 \cup v_0^2$ is any bivector in V. Then P is a sub-bivector space of V if and only if $v_0 \in W$.*

Proof Suppose that $v_0 = v_0^2 + W_2 \in W = W_1 \cup W_2$. It follows that $v_0^1 \in W_1$ and $v_0^2 \in W_2$ and consequently, $P = (v_0^1 + W_1) \cup (v_0^2 + W_2) = W_1 \cup W_2 = W$. Since W is a sub-bivector space of V , it follows that P is a sub-bivector space of V.

The converse is obvious. □

Proposition 2.5 *Let $W = W_1 \cup W_2$ be a sub-bivector space of a bivector space $V = V_1 \cup V_2$ and let $P = (v_0^1 + W_1) \cup (v_0^2 + W_2)$ and $Q = (v_1^1 + W_1) \cup (v_1^2 + W_2)$ be two bicosets of V determined by W where $v_0 = v_0^1 \cup v_0^2$ and $v_1 = v_1^1 \cup v_1^2$. Then $P = Q$ if and only if $v_0 - v_1 \in W$.*

Proof Suppose that $P = Q$. Then $(v_0^1 + W_1) \cup (v_0^2 + W_2) = (v_1^1 + W_1) \cup (v_1^2 + W_2)$ and this implies that $v_0^1 + W_1 = v_1^1 + W_1$ or $v_0^2 + W_2 = v_1^2 + W_2$ which also implies that $v_0^1 - v_1^1 \in W_1$ or $v_0^2 - v_1^2 \in W_2$ from which we obtain $(v_0^1 - v_1^1) \cup (v_0^2 - v_1^2) \in W_1 \cup W_2$ and thus $(v_0^1 \cup v_0^2) - (v_1^1 \cup v_1^2) \in W_1 \cup W_2$ that is $v_0 - v_1 \in W$.

The converse is obvious and the proof is complete. □

Proposition 2.6 *Let $P = (v_0^1 + W_1) \cup (v_0^2 + W_2)$ be a bicoset of $V = V_1 \cup V_2$ determined by $W = W_1 \cup W_2$ where $v_0 = v_0^1 \cup v_0^2$. If $v_1 = v_1^1 \cup v_1^2$ is any bivector in V such that $v_1 \in P$, then P can be expressed as $P = (v_1^1 + W_1) \cup (v_1^2 + W_2)$.*

Proof This result is obvious. □

Proposition 2.7 *Let $W = W_1 \cup W_2$ and $W' = W_3 \cup W_4$ be two distinct sub-bivector spaces of a bivector space $V = V_1 \cup V_2$ and let $P = (v_1^1 + W_1) \cup (v_1^2 + W_2)$ and $Q = (v_2^1 + W_3) \cup (v_2^2 + W_4)$ be two bicosets of V determined by W and W' respectively. If $v_0 = v_0^1 \cup v_0^2$ is any bivector in V such that $v_0 \in P$ and $v_0 \in Q$, then $P \cup Q$ is also a bicoset of V and $P \cup Q = v_0 + (W \cup W')$.*

Proof Suppose that $v_0 \in P$ and $v_0 \in Q$. It follows from Proposition 2.6 that $P = (v_0^1 + W_1) \cup (v_0^2 + W_2)$ and $Q = (v_0^1 + W_3) \cup (v_0^2 + W_4)$ and therefore

$$\begin{aligned} P \cup Q &= [(v_0^1 + W_1) \cup (v_0^2 + W_2)] \cup [(v_0^1 + W_3) \cup (v_0^2 + W_4)] \\ &= [(v_0^1 \cup v_0^2) + (W_1 \cup W_2)] \cup [(v_0^1 \cup v_0^2) + (W_3 \cup W_4)] \\ &= [v_0 + W] \cup [v_0 + W'] \\ &= v_0 + (W \cup W'). \end{aligned}$$

The required results follow. □

Definition 2.8([2]) *Let $V = V_1 \cup V_2$ be a bivector space over the field F. An inner biproduct on V is a bifunction $<,>=<,>_1 \cup <,>_2$ which assigns to each ordered pair of bivectors $x = x_1 \cup x_2$, $y = y_1 \cup y_2$ in V with $x_i, y_i \in V_i$ $(i = 1, 2)$ a pair of scalars $< x, y >=< x_1, y_1 >_1 \cup < x_2, y_2 >_2$ in F in such a way that $\forall x, y, z = z_1 \cup z_2 \in V$ and all scalars $\alpha = \alpha_1 \cup \alpha_2$ in F, the following conditions hold:*

(i) $< x + y, z >=< x_1 + y_1, z_1 >_1 \cup < x_2 + y_2, z_2 >_2$;

(ii) $< \alpha x, y >= \alpha < x, y >$;

(iii) $< y, x >= \overline{< x, y >}$;

(iv) $< x, x >> 0$ if $x \neq 0 \cup 0$.

$V = V_1 \cup V_2$ together with a specified inner biproduct $<,>=<,>_1 \cup <,>_2$ is called an inner biproduct space over the field F.

If V is a finite dimensional real inner biproduct space, it is called a Euclidean bispace. A complex inner biproduct space is called a unitary bispace.

Definition 2.9 *Let $V = V_1 \cup V_2$ be an inner biproduct space over a field F. If $x = x_1 \cup x_2$ and $y = y_1 \cup y_2$ in V with $x_i, y_i \in V_i$ $(i = 1, 2)$ are such that*

$$< x, y >=< x_1, y_1 >_1 \cup < x_2, y_2 >_2 = 0 \cup 0,$$

we say that x is biorthogonal to y. If $< x, y > \neq 0 \cup 0$ but $< x_1, y_1 >_1 = 0$ or $< x_2, y_2 >_2 = 0$, then we say that x and y are semi biorthogonal.

If $B = B_1 \cup B_2$ is any set in $V = V_1 \cup V_2$ such that all pairs of distinct vectors in B_1 and all pairs of distinct vectors in B_2 are orthogonal, then we say that B is a biorthogonal set.

If $W = W_1 \cup W_2$ is any set in $V = V_1 \cup V_2$ and $\forall v \in V, w \in W$ with $v = v_1 \cup v_2, w = w_1 \cup w_2$ is such that

$$< v, w >=< v_1, w_1 >_1 \cup < v_2, w_2 >_2 = 0 \cup 0,$$

then we call the set

$$W^\perp = W_1^\perp \cup W_2^\perp = \{v \in V :< v_1, w_1 >_1 \cup < v_2, w_2 >_2 = 0 \cup 0, \forall w \in W\}$$

biorthogonal complement of W.

Definition 2.10 *Let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be sub-bivector spaces of a bivector space $V = V_1 \cup V_2$. The bisum of W_1 and W_2 denoted by $W_1 + W_2$ is defined by*

$$\begin{aligned} W_1 + W_2 &= \{(W_1^1 \cup W_1^2) + (W_2^1 \cup W_2^2) : W_i^j \subset V_i, i, j = 1, 2\} \\ &= \{(W_1^1 + W_2^1) \cup (W_1^2 + W_2^2), W_i^j \subset V_i, i, j = 1, 2\}. \end{aligned}$$

Definition 2.11 *Let $V = V_1 \cup V_2$ be a bivector space over a field F and let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be sub-bivector spaces of V. If $V_1 = W_1^1 \oplus W_2^1$ and $V_2 = W_1^2 \oplus W_2^2$, then we call*

$$\begin{aligned} V &= W_1 \oplus W_2 \\ &= (W_1^1 \oplus W_2^1) \cup (W_1^2 \oplus W_2^2) \end{aligned}$$

a direct bisum of W_1 and W_2 and any bivector $v = v_1 \cup v_2$ in V can be expressed uniquely as

$$v = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2),\ w_i^j \in W_i^j,\ i, j = 1, 2.$$

Proposition 2.12 *Let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be sub-bivector spaces of a bivector space $V = V_1 \cup V_2$. Then the bisum of W_1 and W_2 is also a sub-bivector space of V.*

Proof Obviously, $(W_1 + W_2) \cap V_1$ and $(W_1 + W_2) \cap V_2$ are subspaces of V_1 and V_2 respectively. Direct expansion of $[(W_1 + W_2) \cap V_1] \cup [(W_1 + W_2) \cap V_2]$ shows that

$$W_1 + W_2 = [(W_1 + W_2) \cap V_1] \cup [(W_1 + W_2) \cap V_2].$$

Consequently by Theorem 1.10 it follows that $W_1 + W_2$ is a sub-bivector space of V. □

Proposition 2.13 *Let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be sub-bivector spaces of a bivector space $V = V_1 \cup V_2$. Then $V = W_1 \oplus W_2$ if and only if:*

(*i*) $V = W_1 + W_2$;

(*ii*) $W_1 \cap W_2 = \{0\}$.

Proof Suppose that $V = W_1 \oplus W_2$. Then any bivector $v = W_1^1 \cup W_1^2$ in V can be written uniquely as $v = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2),\ w_i^j \in W_i^j,\ i, j = 1, 2$, which is an element of $W_1 + W_2$ and therefore, $V = W_1 + W_2$. Also since $V = W_1 \oplus W_2$, it follows that $V_1 = W_1^1 \oplus W_2^1$ and $V_2 = W_1^2 \oplus W_2^2$. Now, let $v = W_1^1 \cup W_1^2 \in W_1 \cup W_2$. Then $v \in W_1$ and $v \in W_2$ and thus, $v \in V_1$ and $v \in V_2$. If $v \in V_1$, then we can write $v = W_1^1 \cup W_1^2 = W_2^1 \cup W_2^2$ from which we obtain

$$v_1 = w_1^1 + w_2^1 \equiv v_1 + 0, v_1 \in W_1^1, 0 \in W_2^1 \quad \text{and also,}$$

$$v_1 = w_1^1 + w_2^1 \equiv 0 + v_1, 0 \in W_1^1, v_1 \in W_2^1.$$

Since $V_1 = W_1^1 \oplus W_2^1$, it follows that $v_1 = 0$. By similar argument, we obtain $v_2 = 0$ and therefore, $v = 0 \cup 0$. Hence, $W_1 \cap W_2 = \{0\}$.

Conversely, suppose that $V = W_1 + W_2$ and $W_1 \cup W_2 = \{0\}$. Let $v = W_1^1 \cup W_1^2$ be an arbitrary bivector in V. Suppose we can write v in two ways as

$$v = W_1^1 \cup W_1^2 = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2) = (w_{11}^1 + w_{22}^1) \cup (w_{11}^2 + w_{22}^2), w_i^j, w_{ii}^j \in W_i^j, i, j = 1, 2.$$

Then we have $w_1^1 + w_2^1 = w_{11}^1 + w_{22}^1$, $w_1^2 + w_2^2 = w_{11}^2 + w_{22}^2$ from which we obtain $w_1^1 - w_{11}^1 = w_{22}^1 - w_2^1$, $w_1^2 - w_{11}^2 = w_{22}^2 - w_2^2$. But then $w_1^1 - w_{11}^1, w_1^2 - w_{11}^2 \in W_1$ and $w_{22}^1 - w_2^1, w_{22}^2 - w_2^2 \in W_2$ and since $W_1 \cup W_2 = \{0\}$, it follows that $w_1^1 - w_{11}^1 = 0 = w_1^2 - w_{11}^2$ and $w_{22}^1 - w_2^1 = 0 = w_{22}^2 - w_2^2$ from which we obtain $w_1^1 = w_{11}^1, w_1^2 = w_{11}^2$, $w_{22}^1 = w_2^1, w_{22}^2 = w_2^2$. This shows that $v \in V$ can be expressed uniquely as $v = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2), w_i^j \in W_i^j,\ i, j = 1, 2$ and hence $V = W_1 \oplus W_2$ and the proof is complete. □

Proposition 2.14 *Let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be two distinct sub-bivector spaces of a bivector space $V = V_1 \cup V_2$ such that $V = W_1 + W_2$. If $P = (v_1^1 + W_1^1) \cup (v_1^2 + W_1^2)$ and $Q = (v_2^1 + W_2^1) \cup (v_2^2 + W_2^2)$ are two bicosets of V determined by W_1 and W_2 respectively, then $P \cap Q$ is also a bicoset of V.*

Proof Suppose that $V = W_1 + W_2$. Let $v = x_1 \cup x_2$ and $u = y_1 \cup y_2$ be bivectors in V. Clearly, $u - v \in V$ and $u - v = (y_1 - x_1) \cup (y_2 - x_2) = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2), w_i^j \in W_i^j$ from which we obtain $y_1 - x_1 = w_1^1 + w_2^1, y_2 - x_2 = w_1^2 + w_2^2$ which implies that $y_1 - w_1^1 = x_1 + w_2^1, y_2 - w_1^2 = x_2 + w_2^2$ and thus, $(y_1 - w_1^1) \cup (y_2 - w_1^2) = (x_1 + w_2^1) \cup (x_2 + w_2^2) = v_0 = v_0^1 \cup v_0^2$. Since the LHS belongs to P and RHS belongs to Q, it follows that $v_0 \in P \cap Q$ and therefore, $P \cap Q$ is a bicoset of V that is $P \cap Q = v_0 + (W_1 \cap W_2)$. □

Definition 2.15 *Let $V = V_1 \cup V_2$ be a finite dimensional inner biproduct space and $W = W_1 \cup W_2$ a sub-bispace of V. Let $W^{\perp} = W_1^{\perp} \cup W_2^{\perp}$ be a biorthogonal complement of W and $P = (v_1 + W_1) \cup (v_2 + W_2)$ a bicoset of V determined by W, where $v = v_1 \cup v_2$ is a fixed bivector in V. It can be shown that*

$$V = W \oplus W^{\perp} = (W_1 \oplus W_1^{\perp}) \cup (W_2 \oplus W_2^{\perp})$$

and consequently we have

$$W \cup W^{\perp} = (W_1 \cup W_1^{\perp}) \cup (W_2 \cup W_2^{\perp}) = \{0\} \cup \{0\}.$$

Suppose that $x = x_1 \cup x_2$ and $y = y_1 \cup y_2$ are bivectors such that $x_i \in W_i$ and $y_i \in W_i^{\perp}, i = 1, 2$. Suppose also that $v = v_1 \cup v_2 = (x_1 + y_1) \cup (x_2 + y_2)$. Then P can be represented by

$$\begin{aligned} P &= (x_1 + y_1 + W_1) \cup (x_2 + y_2 + W_2) \\ &= (y_1 + W_1) \cup (y_2 + W_2),\ \text{since } x_i \in W_i, i = 1, 2. \end{aligned}$$

This representation is called the biprojection of v on W and it is unique.

To establish the uniqueness, let $z = z_1 \cup z_2$ be any bivector in $W^{\perp}$ and let P have another representation $P = (z_1 + W_1) \cup (z_2 + W_2), z_i \in W_i^{\perp}, i = 1, 2$. Then we have $y_1 + W_1 = z_1 + W_1$,

$y_2 + W_2 = z_2 + W_2$ so that $y_1 - z_1 \in W_1$, $y_2 - z_2 \in W_2$ and thus $y_1 - z_1 \in W_1 \cap W_1^{\perp} = \{0\}$, $y_2 - z_2 \in W_2 \cap W_2^{\perp} = \{0\}$ which implies that $y_1 - z_1 = 0$, $y_2 - z_2 = 0$ from which we obtain $y_1 = z_1$, $y_2 = z_2$ and the uniqueness of P is established.

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