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On the ${}_3\psi_3$ Basic Bilateral Hypergeometric Series Summation Formulas

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Abstract: H.Exton has recorded two ${}_3\psi_3$ basic bilateral hypergeometric series summation formula without proof on page 305 of his book entitled *q- hypergeometric functions and applications*. In this paper, we give a proof of them.

Key Words: Basic hypergeometric series, basic bilateral hypergeometric series, contiguous functions.

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§1. Introduction

We follow the standard notation of q-series [4] and we always assume that $|q| < 1$. The q-shifted factorials $(a;q)_n$ and $(a;q)_\infty$ are defined as

$$(a;q)_n = (a)_n := \begin{cases} 1, & \text{if} \quad n = 0, \\ (1-a)(1-aq)(1-aq^2)...(1-aq^n), & \text{if} \quad n \geqslant 1 \end{cases}$$

and

$$(a;q)_\infty = (a)_\infty := (1-a)(1-aq)(1-aq^2)\cdots.$$

The basic hypergeometric series ${}_{r+1}\varphi_r$ is defined by

$${}_{r+1}\varphi_r\left[\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_{r+1})_n}{(q)_n(b_1)_n(b_2)_n\cdots(b_r)_n} z^n, \qquad |z| < 1.$$

One of the most classical identities in q-series is the q-binomial theorem, due to Cauchy:

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$$ {}_1\varphi_0\left[\begin{matrix} a \\ - \end{matrix}; z\right] = \frac{(az)_\infty}{(z)_\infty}, \qquad |z|<1, \tag{1.1} $$

Another classical q-series identity in q-series is Heine's q-analogue of the Gauss ${}_2F_1$ summation formula:

$$ {}_2\varphi_1\left[\begin{matrix} a, b \\ c \end{matrix}; \frac{c}{ab}\right] = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}, \qquad \left|\frac{c}{ab}\right|<1. \tag{1.2} $$

Heine deduced (1.2) as a particular case of his transformation formula [5]

$$ {}_2\varphi_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty}\ {}_2\varphi_1\left[\begin{matrix} c/b, z \\ az \end{matrix}; b\right], |z|<1, |b|<1. \tag{1.3} $$

Another interesting transformation formula due to Sear's [7] is

$$ {}_3\varphi_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; \frac{de}{abc}\right] = \frac{(e/a)_\infty (de/bc)_\infty}{(e)_\infty (de/abc)_\infty}\ {}_3\varphi_2\left[\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; e/a\right], \tag{1.4} $$

$|de/abc|<1$, $|e/a|<1$. The basic bilateral hypergeometric series ${}_r\psi_r$is defined by

$$ {}_r\psi_r\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_r \end{matrix}; z\right] := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_r)_n} z^n, $$

$\left|\frac{b_1 b_2 \cdots b_r}{a_1 a_2 \cdots a_r}\right| < |z| < 1$. There are many generalizations of q-binomial theorem (1.1) of which, one of the interesting is the following Ramanujan's ${}_1\psi_1$ summation [1] [6]:

$$ {}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix}; z\right] = \frac{(az)_\infty (b/a)_\infty (q/az)_\infty (q)_\infty}{(z)_\infty (q/a)_\infty (b/az)_\infty (b)_\infty}, \quad |b/a|<|z|<1. \tag{1.5} $$

A variety of proofs have been given of (1.5). For more details of (1.5), one may refer [1], [4]. H. Exton [3, p. 305] has given following two ${}_3\psi_3$ basic bilateral series summation formula without proof :

$$ {}_3\psi_3\left[\begin{matrix} a, b, cq \\ d, bq, c \end{matrix}; \frac{1}{a}\right] = \frac{(1-(b/c))(d/b)_\infty (bq/a)_\infty (q)_\infty^2}{(1-(1/c))(q/b)_\infty (q/a)_\infty (bq)_\infty (d)_\infty}, \tag{1.6} $$

$|d|<1$, $|1/a|<1$ and

$$ {}_3\psi_3\left[\begin{matrix} a, b, cq \\ d, bq, c \end{matrix}; \frac{q}{a}\right] = \frac{(1-(c/b))(d/b)_\infty (bq/a)_\infty (q)_\infty^2}{(1-c)(q/b)_\infty (q/a)_\infty (bq)_\infty (d)_\infty}, \tag{1.7} $$

$|d/q| < 1$, $|q/a| < 1$. Exton [3, p. 305] has incorrectly given $(q/c)_\infty$ instead of $(q/b)_\infty$ in the denominator of (1.7). W. Chu [2], deduced (1.6) and (1.7) as a special cases of his integral-summation formula. In this paper, we give a proof of (1.6) and (1.7) on the lines of G. E. Andrews and R. Askey [1] proof of (1.5).

§2. Proof of (1.6) and (1.7)

Lemma 2.1 *We have*

$$\frac{a}{d}(1-d)\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right] + (1-(a/d))\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ dq,e,f \end{matrix};z\right]$$

$$= \frac{(1-d)(1-(e/q))(1-(f/q))}{z(1-(b/q))(1-(c/q))}\ {}_3\psi_3\left[\begin{matrix} a,b/q,c/q \\ d,e/q,f/q \end{matrix};z\right], \tag{2.1}$$

$$\frac{((d/q)-b)}{1-b}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right] - \frac{(d-a)}{(q-a)}\ {}_3\psi_3\left[\begin{matrix} a/q,bq,c \\ d,e,f \end{matrix};z\right]$$

$$= \frac{z((a/q)-b)(1-c)}{(1-e)(1-f)}\ {}_3\psi_3\left[\begin{matrix} a,bq,cq \\ d,eq,fq \end{matrix};z\right], \tag{2.2}$$

$$\frac{b(d-1)(d-a)}{d(d-b)}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right] = (1-(a/d))\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ dq,e,f \end{matrix};z\right]$$

$$+\frac{(d-1)(d-a)(1-(e/q))(1-(f/q))}{z((d/q)-(b/q))(1-(c/q))(q-a)}\ {}_3\psi_3\left[\begin{matrix} a/q,b,c/q \\ d,e/q,f/q \end{matrix};z\right], \tag{2.3}$$

$$\left[f-(a/q)-\frac{(d-a)}{(q-a)}\right]\ {}_3\psi_3\left[\begin{matrix} a/q,bq,c \\ d,e,f \end{matrix};z\right] = \frac{((d/q)-f)(f-(a/q))}{(1-f)}$$

$${}_3\psi_3\left[\begin{matrix} a/q,bq,c \\ d,e,fq \end{matrix};zq\right] - \frac{((d/q)-f)}{(1-f)}\ {}_3\psi_3\left[\begin{matrix} a,bq,c \\ d,e,fq \end{matrix};z\right]. \tag{2.4}$$

Proof of (2.1). It is easy to see that

$$a\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ dq,e,f \end{matrix};zq\right] + \frac{a(1-d)}{d}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right]$$

$$= \frac{a}{d}\sum_{n=-\infty}^{\infty} \frac{(a)_n(b)_n(c)_n}{(dq)_{n-1}(e)_n(f)_n} z^n \left[\frac{dq^n}{(1-dq^n)}+1\right]$$

$$= \frac{a}{d} \sum_{n=-\infty}^{\infty} \frac{(a)_n (b)_n (c)_n}{(dq)_n (e)_n (f)_n} z^n.$$

Hence,

$$a \ {}_3\psi_3 \left[\begin{matrix} a, b, c \\ dq, e, f \end{matrix} ; zq \right] + \frac{a(1-d)}{d} \ {}_3\psi_3 \left[\begin{matrix} a, b, c \\ d, e, f \end{matrix} ; z \right] = \frac{a}{d} \ {}_3\psi_3 \left[\begin{matrix} a, b, c \\ dq, e, f \end{matrix} ; z \right]. \tag{2.5}$$

Also, we have

$${}_3\psi_3 \left[\begin{matrix} a, b, c \\ d, e, f \end{matrix} ; z \right] - a \ {}_3\psi_3 \left[\begin{matrix} a, b, c \\ d, e, f \end{matrix} ; zq \right] = \sum_{n=-\infty}^{\infty} \frac{(a)_{n+1} (b)_n (c)_n}{(d)_n (e)_n (f)_n} z^n$$

$$= \frac{(1-(d/q))(1-(e/q))(1-(f/q))}{z(1-(b/q))(1-(c/q))} \sum_{n=-\infty}^{\infty} \frac{(a)_n (b/q)_n (c/q)_n}{(d/q)_n (e/q)_n (f/q)_n} z^n.$$

Thus,

$${}_3\psi_3 \left[\begin{matrix} a, b, c \\ d, e, f \end{matrix} ; z \right] - a \ {}_3\psi_3 \left[\begin{matrix} a, b, c \\ d, e, f \end{matrix} ; zq \right]$$

$$= \frac{(1-(d/q))(1-(e/q))(1-(f/q))}{z(1-(b/q))(1-(c/q))} \ {}_3\psi_3 \left[\begin{matrix} a, b/q, c/q \\ d/q, e/q, f/q \end{matrix} ; z \right]. \tag{2.6}$$

Changing d to dq in (2.6) and then adding resulting identity with (2.5), we obtain (2.1).

Proof of (2.2). We have

$$\frac{((d/q)-b)}{(1-b)} \sum_{n=-\infty}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n (f)_n} z^n - \frac{(d-a)}{(q-a)} \sum_{n=-\infty}^{\infty} \frac{(q/a)_n (bq)_n (c)_n}{(d)_n (e)_n (f)_n} z^n$$

$$= \sum_{n=-\infty}^{\infty} \frac{(a)_{n-1} (bq)_{n-1} (c)_n}{(d)_n (e)_n (f)_n} z^n \left[((d/q)-b)(1-aq^{n-1}) - ((d/q)-(a/q))(1-bq^n) \right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{(a)_{n-1} (bq)_{n-1} (c)_n}{(d)_n (e)_n (f)_n} z^n ((a/q)-b)(1-dq^{n-1})$$

$$= \frac{z((a/q)-b)(1-c)}{(1-e)(1-f)} \sum_{n=-\infty}^{\infty} \frac{(a)_n (bq)_n (cq)_n}{(d)_n (eq)_n (fq)_n} z^n.$$

This proves (2.2).

Proof of (2.3). Changing b to b/q, c to c/q, e to e/q and f to f/q in (2.2), and multiplying throughout by $\frac{q(1-d)(1-(e/q))(1-(f/q))}{z(1-(c/q))(d-b)}$ and adding the resulting identity with (2.1), we find (2.3).

Proof of (2.4). From [8], we have

$$(1-f)\ {}_3\psi_3\left[\begin{array}{c} a,b,c \\ d,e,f \end{array};z\right]-((d/q)-f)\ {}_3\psi_3\left[\begin{array}{c} a,b,c \\ d,e,fq \end{array};zq\right]$$

$$=\frac{z(1-a)(1-b)(1-c)}{(1-fq)(1-e)}\ {}_3\psi_3\left[\begin{array}{c} aq,bq,cq \\ d,eq,fq^2 \end{array};z\right],$$

and

$$\frac{((d/q)-a)}{(1-a)}\ {}_3\psi_3\left[\begin{array}{c} a,b,c \\ d,e,f \end{array};z\right]-\frac{((d/q)-f)}{(1-f)}\ {}_3\psi_3\left[\begin{array}{c} aq,b,c \\ d,e,fq \end{array};z\right].$$

$$=\frac{z(f-a)(1-b)(1-c)}{(1-f)(1-fq)(1-e)}\ {}_3\psi_3\left[\begin{array}{c} aq,bq,cq \\ d,eq,fq^2 \end{array};z\right]$$

Eliminating ${}_3\psi_3\left[\begin{array}{c} aq,bq,cq \\ d,eq,fq \end{array};z\right]$ between above two identities and then replacing a by a/q and b by bq , we obtain(2.4).

Proof of (1.6). Setting $c=cq,\ e=bq,\ f=c$ and $z=1/a$ in (2.4), we deduce that

$$\left[c-(a/q)-\frac{(d-a)}{(q-a)}\right]\ {}_2\psi_2\left[\begin{array}{c} a/q,cq \\ d,c \end{array};1/a\right]=\frac{((d/q)-c)(c-(a/q))}{(1-c)}\ {}_1\psi_1\left[\begin{array}{c} a/q \\ a \end{array};q/a\right]$$

$$-\frac{((d/q)-c)}{(1-c)}\ {}_1\psi_1\left[\begin{array}{c} a \\ d \end{array};1/a\right].$$

Employing (1.5) in the right side of the above, we obtain

$${}_2\psi_2\left[\begin{array}{c} a/q,cq \\ d,c \end{array};1/a\right]=0. \tag{2.7}$$

Let

$$f(d)=\ {}_3\psi_3\left[\begin{array}{c} a,b,cq \\ d,bq,c \end{array};1/a\right].$$

As a function of d, $f(d)$ is clearly analytic for $|d| < 1$ and $|a| > 1$. Setting $c = cq$, $e = bq$, $f = c$ and $z = 1/a$ in (2.3) and then employing (2.7), we find that

$$f(d) = \frac{(1-(d/b))}{(1-d)} f(dq). \tag{2.8}$$

Iterating (2.8) $n-1$ times , we find that

$$f(d) = \frac{(d/b)_n}{(d)_n} f(dq^n).$$

Since $f(d)$ is analytic for $|d| < 1$, $|a| > 1$, by letting $n \to \infty$, we obtain

$$f(d) = \frac{(d/b)_\infty}{(d)_\infty} f(0).$$

Setting $c = cq$, $d = c$, $e = bq$ in (1.4), we deduce that

$${}_3\varphi_2 \left[\begin{matrix} a, b, cq \\ bq, c \end{matrix} ; 1/a \right]$$

$$= \frac{(bq/a)_\infty (q)_\infty}{(bq)_\infty (1/a)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (c/b)_n (1/q)_n}{(q)_n (c)_n (q)_{n-1}} (bq/a)^n$$

$$= \frac{(bq/a)_\infty (q)_\infty}{(bq)_\infty (1/a)_\infty} \frac{(1-a)(1-(c/b))(1-(1/q))(bq/a)}{(1-q)(1-c)} \sum_{n=0}^{\infty} \frac{(aq)_n (cq/b)_n (1)_n}{(q)_n (cq)_n (q^2)_n} (bq/a)^n$$

$$= \frac{b(1-(c/b))(bq/a)_\infty (q)_\infty}{(1-c)(bq)_\infty (q/a)_\infty}.$$

Thus,

$$f(q) = \frac{(1-(b/c))(bq/a)_\infty (q)_\infty}{(1-(1/c))(bq)_\infty (q/a)_\infty}.$$

Setting $d = q$ in (2.9), and using the above, we find that

$$f(0) = \frac{(1-(b/c))(bq/a)_\infty (q)_\infty^2}{(1-(1/c))(bq)_\infty (q/a)_\infty (q/b)_\infty}.$$

Using this in (2.9), we deduce that

$$f(d) = \frac{(1-(b/c))(bq/a)_\infty (d/b)_\infty (q)_\infty^2}{(1-(1/c))(bq)_\infty (q/a)_\infty (q/b)_\infty (d)_\infty}.$$

This completes the proof of (1.6).

Proof of (1.7). Setting $c = cq$, $e = bq$, $f = c$ and $z = q/a$ in (2.4) and then employing (1.5), we find that

$$ {}_2\psi_2 \left[\begin{matrix} a/q, cq \\ d, c \end{matrix} ; q/a \right] = 0. \tag{2.9} $$

Let

$$ f(d) := {}_3\psi_3 \left[\begin{matrix} a, b, cq \\ d, bq, c \end{matrix} ; q/a \right]. $$

As a function of d, $f(d)$ is clearly analytic for $|d| < 1$, when $|q/a| < 1$. Setting $c = cq$, $e = bq$, $f = c$ and $z = q/a$ in (2.3) and then employing (2.10), we find that

$$ f(d) = \frac{(1 - (d/b))}{(1 - d)} f(dq). $$

Iterating the above $n - 1$ times , we get

$$ f(d) = \frac{(d/b)_n}{(d)_n} f(dq^n). $$

Since $f(d)$ is analytic for $|d| < 1$, $|q/a| < 1$, by letting $n \to \infty$, we obtain

$$ f(d) = \frac{(d/b)_\infty}{(d)_\infty} f(0). $$

Setting $c = bq$, $z = q^2/a$ in (1.3) and employing (1.1), we obtain

$$ {}_2\varphi_1 \left[\begin{matrix} a, b \\ bq \end{matrix} ; q^2/a \right] = \frac{(bq/a)_\infty (q)_\infty}{b(bq)_\infty (q/a)_\infty}. $$

Also by (1.2), we deduce that

$$ \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(q)_n (bq)_n} (q/a)^n = \frac{(bq/a)_\infty (q)_\infty}{(bq)_\infty (q/a)_\infty}. $$

Thus,

$$f(q) = {}_3\varphi_2 \left[\begin{matrix} a, b, cq \\ bq, c \end{matrix} ; q/a \right]$$

$$= \frac{1}{(1-c)} \left[{}_2\varphi_1 \left[\begin{matrix} a, b \\ bq \end{matrix} ; q/a \right] - c \ {}_2\varphi_1 \left[\begin{matrix} a, b \\ bq \end{matrix} ; q^2/a \right] \right]$$

$$= \frac{(1-(c/b))(bq/a)_\infty (q)_\infty}{(1-c)(q/a)_\infty (bq)_\infty}.$$

Now setting in $d = q$ in (2.11) and employing above, we find that

$$f(0) = \frac{(1-(c/b))(bq/a)_\infty (q)_\infty^2}{(1-c)(q/a)_\infty (bq)_\infty (q/b)_\infty}.$$

Using this in (2.11), we deduce (1.7).

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$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)} x + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})} x^2 + \cdots,$$

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