

Minimal Retraction of Space-time and Their Foldings

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Abstract: A *Smarandache multi-spacetime* is such a union spacetime $\bigcup_{i=1}^n S_i$ of spacetimes $S_1, S_2, \dots, S_n$ for an integer $n \geq 1$. In this article, we will be deduced the geodesics of space-time, i.e., a Smarandache multi-spacetime with $n = 1$ by using Lagrangian equations. The deformation retract of space-time onto itself and into a geodesics will be achieved. The concept of retraction and folding of zero dimension space-time will be obtained. The relation between limit of folding and retraction presented.

Key Words: Folding, deformation retract, space-time, Smarandache multi-spacetime.

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§1. Introduction

The folding of a manifold was, firstly introduced by Robertson in [1977] [14]. Since then many authors have studied the folding of manifolds such as in [4,6,12,13]. The deformation retracts of the manifolds defined and discussed in [5,7]. In this paper, we will discuss the folding restricted by a minimal retract and geodesic. We may also mention that folding has many important technical applications, for instance, in the engineering problems of buckling and post-buckling of elastic and elastoplastic shells [1]. More studies and applications are discussed in [4], [8], [9], [10], [13].

§2. Definitions

1. A subset A of a topological space X is called a retract of X , if there exists a continuous map $r : X \rightarrow A$ such that ([2]):

- (i) X is open;
- (ii) $r(a) = a, \forall a \in A$.

2. A subset A of a topological space X is said to be a deformation retract if there exists a retraction $r : X \rightarrow A$, and a homotopy $f : X \times I \rightarrow X$ such that ([2]):

$$f(x, 0) = x, \forall x \in X;$$

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$$\begin{aligned} f(x, 1) &= r(x), \forall x \in X; \\ f(a, t) &= a, \forall a \in A, t \in [0, 1]. \end{aligned}$$

3. Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \rightarrow N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : J \rightarrow M$, the induced path $f \circ \gamma : J \rightarrow N$ is a piecewise geodesic and of the same length as γ ([14]). If f does not preserve the lengths, it is called topological folding.
4. Let M be an m -dimensional manifold. M is said to be minimal m -dimensional manifold if the mean curvature vanishes everywhere, i.e., $H(\sigma.p) = 0$ for all $p \in M$ ([3]).
5. A subset A of a minimal manifold M is a minimal retraction of M , if there exists a continuous map $r : M \rightarrow A$ such that ([12]):

- (i) M is open;
- (ii) $r(M) = A$;
- (iii) $r(a) = a, \forall a \in A$;
- (iv) $r(M)$ is minimal manifold.

§3. Main Results

Using the Neugebaure-Bcklund transformation, the space-time T take the form [11]

$$ds^2 = dt^2 - dp^2 - dz^2 - p^2 d\phi^2 \quad (1)$$

Using the relationship between the cylindrical and spherical coordinates, the metric becomes

$$\begin{aligned} \overline{ds}^2 &= r^2(\sin^2 \theta_2 - \cos^2 \theta_2) \overline{d\theta_2}^2 - r^2 \sin^2 \theta_2 \overline{d\theta_1}^2 + (\cos^2 \theta_2 - \sin^2 \theta_2) \overline{dr}^2 \\ &\quad - r^2 \sin^2 \theta_1 \sin^2 \theta_2 \overline{d\varphi}^2 - 4r \sin \theta_2 \cos \theta_2 d\theta_2 dr. \end{aligned}$$

The coordinates of space-time T are:

$$\left. \begin{aligned} y_1 &= \sqrt{c_1(r, \theta_2) - r^2 \sin^2 \theta_2 \theta_1^2} \\ y_2 &= \sqrt{4r^2 \cos 2\theta_2 + k_1} \\ y_3 &= \sqrt{r^2 \cos 2\theta_2 + c_3(\theta_2)} \\ y_4 &= \sqrt{c_4(r, \theta_1, \theta_2) - r^2 \sin^2 \theta_1 \sin^2 \theta_2 \phi^2} \end{aligned} \right\} \quad (2)$$

where c_1, k_1, c_3, c_4 are the constant of integrations. Applying the transformation

$$\begin{aligned} x_1^2 &= y_1^2 - c_1(r, \theta_2), \\ x_2^2 &= y_2^2 - k_1, \\ x_3^2 &= y_3^2 - c_3(\theta_2), \\ x_4^2 &= y_4^2 - c_4(r, \theta_1, \theta_2) \end{aligned}$$

Then, the coordinates of space-time T becomes:

$$\left. \begin{aligned} x_1 &= ir \sin \theta_2 \theta_1 \\ x_2 &= 2r \sqrt{\cos 2\theta_2} \\ x_3 &= r \sqrt{\cos 2\theta_2} \\ x_4 &= ir \sin \theta_1 \sin \theta_2 \phi. \end{aligned} \right\} \quad (3)$$

Now, we apply Lagrangian equations

$$\frac{d}{ds} \left(\frac{\partial T}{\partial G_i} \right) - \frac{\partial T}{\partial G_i} = 0, i = 1, 2, 3, 4.$$

to find a geodesic which is a subset of the space-time T . Since

$$T = \frac{1}{2} \{ -r^2 \cos 2\theta_2 \theta_2'^2 - r^2 \sin^2 \theta_2 \theta_1'^2 + \cos 2\theta_2 r'^2 - r^2 \sin^2 \theta_1 \sin^2 \theta_2 \phi'^2 - 2r \sin 2\theta_2 \theta_2' r' \}$$

then, the Lagrangian equations for space-time T are:

$$\frac{d}{ds} (r^2 \sin^2 \theta_2 \theta_1') + (r^2 \sin \theta_1 \cos \theta_1 \sin^2 \theta_2 \phi'^2) = 0 \quad (4)$$

$$\begin{aligned} \frac{d}{ds} (r^2 \cos 2\theta_2 \theta_2' + r \sin \theta_2 r') + (r^2 \sin 2\theta_2 \theta_2'^2 + r^2 \sin \theta_2 \cos \theta_2 \theta_1'^2 \\ + \sin 2\theta_2 r'^2 + r^2 \sin^2 \theta_1 \sin \theta_2 \cos \theta_2 \phi'^2 + 2r \cos 2\theta_2 \theta_2' r') = 0 \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{d}{ds} (\cos 2\theta_2 r' - r \sin 2\theta_2 \theta_2') + (r \cos 2\theta_2 \theta_2'^2 + r \sin^2 \theta_2 \theta_1'^2 + \\ r \sin^2 \theta_1 \sin^2 \theta_2 \phi'^2 + \sin 2\theta_2 \theta_2' r') = 0 \end{aligned} \quad (6)$$

$$\frac{d}{ds} (r^2 \sin^2 \theta_1 \sin^2 \theta_2 \phi') = 0. \quad (7)$$

From equation (7) we obtain $r^2 \sin^2 \theta_1 \sin^2 \theta_2 \phi^1 = \text{constant } \mu$. If $\mu = 0$, we obtain the following cases:

(i) If $r = 0$, hence we get the coordinates of space-time T_1 , which are defined as

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0,$$

which is a hypersphere T_1 , $x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0$ on the null cone since the distance between any two different points equal zero, it is a minimal retraction and geodesic.

(ii) If $\sin^2 \theta_1 = 0$, we get

$$x_1 = 0, x_2 = 2r \sqrt{\cos 2\theta_2}, x_3 = r \sqrt{\cos 2\theta_2}, x_4 = 0.$$

Thus, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 5r^2 \cos 2\theta_2$, which is a hypersphere S_1 in space-time T with $x_1 = x_4 = 0$. It is a geodesic and retraction.

(iii) If $\sin^2 \theta_2 = 0$, then $\theta_2 = 0$ we obtain the following geodesic retraction

$$x_1 = 0, x_2 = 2r, x_3 = r, x_4 = 0, \quad x_1^2 + x_2^2 + x_3^2 - x_4^2 = 5r^2,$$

which is the hypersphere $S_2 \subset T$ with $x_1 = x_4 = 0$.

(iv) If $\phi' = 0$ this yields the coordinate of $T_2 \subset T$ given by

$$x_1 = ir \sin \theta_2 \theta_1, x_2 = 2r \sqrt{\cos 2\theta_2}, x_3 = r \sqrt{\cos 2\theta_2}, x_4 = 0.$$

It is worth nothing that $x_4 = 0$ is a hypersurface $T_2 \subset T$. Hence, we can formulate the following theorem.

Theorem 1 *The retractions of space-time is null geodesic, geodesic hyperspher and hypersurface.*

Lemma 1 *In space-time the minimal retraction induces null-geodesic.*

Lemma 2 *A minimal geodesic in space-time is a necessary condition for minimal retraction.*

The deformation retract of the space-time T is defined as

$$\rho : T \times I \rightarrow T$$

where T is the space-time and I is the closed interval $[0,1]$. The retraction of the space-time T is defined as

$$R : T \rightarrow T_1, T_2, S_1 \text{ and } S_2.$$

The deformation retract of space-time T into a geodesic $T_1 \subset T$ is defined by

$$\begin{aligned} \rho(m, t) = & (1-t)\{ir \sin \theta_2 \theta_1, 2r \sqrt{\cos 2\theta_2}, r \sqrt{\cos 2\theta_2}, \\ & ir \sin \theta_1 \sin \theta_2 \phi\} + t\{0, 0, 0, 0\}. \end{aligned}$$

where $\rho(m, 0) = \{ir \sin \theta_2 \theta_1, 2r \sqrt{\cos 2\theta_2}, r \sqrt{\cos 2\theta_2}, ir \sin \theta_1 \sin \theta_2 \phi\}$, $\rho(m, 1) = \{0, 0, 0, 0\}$.

The deformation retract of space-time T into a geodesic $T_2 \subset T$ is defined as

$$\begin{aligned} \rho(m, t) = & (1-t)\{ir \sin \theta_2 \theta_1, 2r \sqrt{\cos 2\theta_2}, r \sqrt{\cos 2\theta_2}, ir \sin \theta_1 \sin \theta_2 \phi\} \\ & + t\{ir \sin \theta_2 \theta_1, 2r \sqrt{\cos 2\theta_2}, r \sqrt{\cos 2\theta_2}, 0\}. \end{aligned}$$

The deformation retract of space-time T into a geodesic $S_1 \subset T$ is defined by

$$\begin{aligned} \rho(m, t) = & (1-t)\{ir \sin \theta_2 \theta_1, 2r \sqrt{\cos 2\theta_2}, r \sqrt{\cos 2\theta_2}, ir \sin \theta_1 \sin \theta_2 \phi\} \\ & + t\{0, 2r \sqrt{\cos 2\theta_2}, r \sqrt{\cos 2\theta_2}, 0\}. \end{aligned}$$

The deformation retract of space-time T into a geodesic $S_2 \subset T$ is defined as

$$\rho(m, t) = (1 - t)\{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir \sin \theta_1 \sin \theta_2 \phi\} + t\{0, 2r, r, 0\}.$$

Now we are going to discuss the folding $\mathfrak{S}$ of the space-time T . Let $\mathfrak{S} : T \rightarrow T$, where

$$\mathfrak{S}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, |x_4|) \quad (8)$$

An isometric folding of the space-time T into itself may be defined as

$$\begin{aligned} \mathfrak{S} : \quad & \{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir \sin \theta_1 \sin \theta_2 \phi\} \\ \rightarrow \quad & \{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir \sin \theta_1 \sin \theta_2 \phi|\}. \end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic T_1 is

$$\begin{aligned} \rho_{\mathfrak{S}} : \quad & \{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir \sin \theta_1 \sin \theta_2 \phi|\} \times I \\ \rightarrow \quad & \{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir \sin \theta_1 \sin \theta_2 \phi|\} \end{aligned}$$

with

$$\rho_{\mathfrak{S}}(m, t) = (1 - t)\{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir \sin \theta_1 \sin \theta_2 \phi|\} + t\{0, 0, 0, 0\}.$$

The deformation retract of the folded space-time T into the folded geodesic T_2 is

$$\begin{aligned} \rho_{\mathfrak{S}}(m, t) &= (1 - t)\{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir \sin \theta_1 \sin \theta_2 \phi|\} \\ &+ t\{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}. \end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic S_1 is

$$\begin{aligned} \rho_{\mathfrak{S}}(m, t) &= (1 - t)\{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir \sin \theta_1 \sin \theta_2 \phi|\} \\ &+ t\{0, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}. \end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic S_2 is

$$\begin{aligned} \rho_{\mathfrak{S}}(m, t) &= (1 - t)\{ir \sin \theta_2 \theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir \sin \theta_1 \sin \theta_2 \phi|\} \\ &+ t\{0, 2r, r, 0\} \end{aligned}$$

Then, the following theorem has been proved.

Theorem 2 *Under the defined folding, the deformation retract of the folded space-time into the folded geodesics is the same as the deformation retract of space-time into the geodesics.*

Now, let the folding be defined as:

$$\mathfrak{S}^*(x_1, x_2, x_3, x_4) = (x, |x_2|, x_3, x_4). \quad (9)$$

The isometric folded space-time $\mathfrak{S}(T)$ is

$$\bar{R} = \{ir \sin \theta_2 \theta_1, \left| 2r \sqrt{\cos 2\theta_2} \right|, r \sqrt{\cos 2\theta_2}, ir \sin \theta_1 \sin \theta_2 \phi\}.$$

Hence, we can formulate the following theorem.

Theorem 3 *The deformation retract of the folded space-time ,i.e., $\rho\mathfrak{S}^*(T)$ is different from the deformation retract of space-time under condition (9).*

Now let $\mathfrak{S}_1 : T^n \rightarrow T^n$,

$\mathfrak{S}_2 : \mathfrak{S}_1(T^n) \rightarrow \mathfrak{S}_1(T^n)$,

$\mathfrak{S}_3 : \mathfrak{S}_2(\mathfrak{S}_1(T^n)) \rightarrow \mathfrak{S}_2(\mathfrak{S}_1(T^n)), \dots$,

$\mathfrak{S}_n : \mathfrak{S}_{n-1}(\mathfrak{S}_{n-2} \dots (\mathfrak{S}_1(T^n)) \dots) \rightarrow \mathfrak{S}_{n-1}(\mathfrak{S}_{n-2} \dots (\mathfrak{S}_1(T^n)) \dots)$,

$\lim_{n \rightarrow \infty} \mathfrak{S}_{n-1}(\mathfrak{S}_{n-2} \dots (\mathfrak{S}_1(T^n)) \dots) = n - 1$ dimensional space-time T^{n-1} .

Let $h_1 : T^{n-1} \rightarrow T^{n-1}$,

$h_2 : h_1(T^{n-1}) \rightarrow h_1(T^{n-1})$,

$h_3 : h_2(h_1(T^{n-1})) \rightarrow h_2(h_1(T^{n-1})), \dots$,

$h_m : h_{m-1}(h_{m-2} \dots (h_1(T^{n-1})) \dots) \rightarrow h_{m-1}(h_{m-2} \dots (h_1(T^{n-1})) \dots)$,

$\lim h_m(h_{m-1}(h_{m-2} \dots (h_1(T^{n-1})) \dots) = n - 2$ dimensional space-time T^{n-2} .

Consequently, $\lim_{s \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \dots k_s(h_m(\mathfrak{S}_n(T^n))) = 0$ -dimensional space-time. Hence, we can formulate the following theorem.

Theorem 4 *The end of the limits of the folding of space-time T^n is a 0-dimensional geodesic, it is a minimal retraction.*

Now let f_i be the foldings and r_i be the retractions. then we have

$$\begin{aligned} T^n &\xrightarrow{f_1^1} T_1^n \xrightarrow{f_2^1} T_2^n \longrightarrow \dots T_{n-1}^n \xrightarrow{\lim f_i^1} T^{n-1}, \\ T^n &\xrightarrow{r_1^1} T_1^n \xrightarrow{r_2^1} T_2^n \longrightarrow \dots T_{n-1}^n \xrightarrow{\lim r_i^1} T^{n-1}, \\ T^n &\xrightarrow{f_1^2} T_1^{n-1} \xrightarrow{f_2^2} T_2^{n-1} \longrightarrow \dots T_{n-1}^{n-1} \xrightarrow{\lim f_i^2} T^{n-2}, \dots, \\ T^{n-1} &\xrightarrow{r_1^1} T_1^{n-1} \xrightarrow{r_2^2} T_2^{n-1} \longrightarrow \dots T_{n-1}^{n-1} \xrightarrow{\lim r_i^2} T^{n-2}, \dots, \\ T^1 &\xrightarrow{f_1^n} T_1^1 \xrightarrow{f_2^n} T_2^1 \longrightarrow \dots T_{n-1}^1 \xrightarrow{\lim f_i^n} T^0, \\ T^1 &\xrightarrow{r_1^n} T_1^1 \xrightarrow{r_2^n} T_2^1 \longrightarrow \dots T_{n-1}^1 \xrightarrow{\lim f_i^n} T^0. \end{aligned}$$

Then the end of the limits of foldings = the limit of retractions = 0-dimensional space-time.

Whence, the following theorem has been proved.

Theorem 5 *In space-time the end of the limits of foldings of T^n into itself coincides with the minimal retraction.*

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