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Genus Distribution for a Graph

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Abstract: In this paper we develop the technique of a distribution decomposition for a graph. A formula is given to determine genus distribution of a cubic graph. Given any connected graph, it is proved that its genus distribution is the sum of those for some cubic graphs by using the technique.

Key Words: Joint tree; genus distribution; embedding distribution; Smarandachely k-drawing.

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§1. Introduction

We consider finite connected graphs. Surfaces are orientable 2-dimensional compact manifolds without boundaries. Embeddings of a graph considered are always assumed to be orientable 2-cell embeddings. Given a graph G and a surface S, a *Smarandachely k-drawing* of G on S is a homeomorphism ϕ: $G \to S$ such that $\phi(G)$ on S has exactly k intersections in $\phi(E(G))$ for an integer k. If $k = 0$, i.e., there are no intersections between in $\phi(E(G))$, or in another words, each connected component of $S - \phi(G)$ is homeomorphic to an open disc, then G has an 2-cell embedding on S. If G can be embedded on surfaces S_r and S_t with genus r and t respectively, then it is shown in [1] that for any k with $r \leqslant k \leqslant t$, G has an embedding on S_k. Naturally, the *genus of a graph* is defined to be the minimum genus of a surface on which the graph can be embedded. Given a graph, *how many distinct embeddings does it have on each surface*? This is the genus distribution problem, first investigated by Gross and Furst [4]. As determining the genus of a graph is NP-complete [15], it appears more difficult and significant to determine the genus distribution of a graph.

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There have been results on genus distribution for some particular types of graphs (see [3], [5], [8], [9], [11]-[17], among others). In [6], Liu discovered the joint trees of a graph which provide a substantial foundation for us to solve the genus distribution of a graph. For a given embedding G_σ of a graph G, one can find the surface, embedding surface or associate surface, which G_σ embeds on by applying the associated joint tree. In fact, genus distribution of G is that of the set of all of its embedding surfaces. This paper first study genus distributions of some sets of surfaces and then investigate the genus distribution of a generic graph by using the surface sorting method developed in [16].

Preliminaries will be briefed in the next section. In Section 3, surfaces Q_j^i will be introduced. We shall investigate the genus distribution of surface sets Q_j^0 and Q_j^1 for $1 \leqslant j \leqslant 24$, and derive the related recursive formulas. In Section 4, a recursion formula of the genus distribution for a cubic graph is given. In the last section, we show that the genus distribution of a general graph can be transformed into genus distribution of some cubic graphs by using a technique we develop in this paper.

§2. Preliminaries

For a graph G, a *rotation at a vertex* v is a cyclic permutation of edges incident with v. A *rotation system* of G is obtained by giving each vertex of G a rotation. Let ρ_v denote the valence of vertex v which is the number of edges incident with v. The number of rotations systems of G is $\prod\limits_{v\in V(G)}(\rho_v-1)!$. Edmonds found that there is a bijection between the rotations systems of a graph and its embeddings [2]. Youngs provided the first proof published [18]. Thus, the number of embeddings of G is $\prod\limits_{v\in V(G)}(\rho_v-1)!$. Let $g_i(G)$ denote the number of embeddings of G with the genus i $(i \geq 0)$. Then, the *genus distribution* of G is the sequence $g_0(G), g_1(G), g_2(G), \cdots$. The *genus polynomial* of G is $f_G(x) = \sum\limits_{i\geq 0} g_i(G)x^i$.

Given a spanning tree T of G, the joint trees of G are obtained by splitting each non-tree edge e into two semi-edges e and e^-. Given a rotation system σ of G, G_σ, $\widetilde{T}_\sigma$ and $\mathcal{P}^\sigma_{\widetilde{T}}$ denote the associated embedding, joint tree and embedding surface which G_σ embedded on respectively. There is a bijection btween embeddings and joint trees of G such that G_σ corresponds to $\widetilde{T}_\sigma$. Given a joint tree $\widetilde{T}$, a *sub-joint tree* $\widetilde{T}_1$ of $\widetilde{T}$ is a graph consisting of T_1 and semi-edges incident with vertices of T_1 where T_1 is a tree and $V(T_1) \subseteq V(T)$. A sub-joint tree $\widetilde{T}_1$ of $\widetilde{T}$ is called *maximal* if there is not a tree T_2 such that $V(T_1) \subset V(T_2) \subseteq V(T)$.

A *linear sequence* $S = abc\cdots z$ is a sequence of letters satisfying with a relation $a \prec b \prec c \prec \cdots \prec z$. Given two linear sequences S_1 and S_2, the *difference sequence* S_1/S_2 is obtained by deleting letters of S_2 in S_1. Since a surface is obtained by identifying a letter with its inverse letter on a special polygon along the direction, a surface is regarded as that polygon such that a and a^- occur only once for each $a \in S$ in this sense.

Let $\mathcal{S}$ be the collection of surfaces. Let $\gamma(S)$ be the genus of a surface S. In order to determine $\gamma(S)$, an equivalence is defined by Op1, Op2 and Op3 on $\mathcal{S}$ as follows:

Op 1. $AB \sim (Ae)(e^-B)$ where $e \notin AB$;

Op 2. $Ae_1e_2Be_2^-e_1^- \sim AeBe^- = Ae^-Be$ where $e \notin AB$;
Op 3. $Aee^-B \sim AB$ where $AB \neq \emptyset$

where AB is a surface.

Thus, S is equivalent to one, and only one of the canonical forms of surfaces $a_0a_0^-$ and $\prod_{k=1}^{i} a_kb_ka_k^-b_k^-$ which are the sphere and orientable surfaces of genus $i(i \geq 1)$.

Lemma 2.1 ([6]) *Let A and B be surfaces. If $a, b \notin B$, and if $A \sim Baba^-b^-$, then $\gamma(A) = \gamma(B) + 1$.*

Lemma 2.2 ([7]) *Let A, B, C, D and E be linear sequences and let $ABCDE$ be a surface. If $a, b \notin ABCDE$, then $AaBbCa^-Db^-E \sim ADCBEaba^-b^-$.*

Lemma 2.3 ([13],[16]) *Let A, B, C and D be linear sequences and let $ABCD$ be a surface. If $a \neq b \neq c \neq a^- \neq b^- \neq c^-$ and if $a, b, c \notin ABCD$, then each of the following holds.*

(i) $aABa^-CD \sim aBAa^-CD \sim aABa^-DC$.
(ii) $AaBa^-bCb^-cDc^- \sim aBa^-AbCb^-cDc^- \sim aBa^-bCb^-AcDc^-$.
(iii) $AaBa^-bCb^-cDc^- \sim BaAa^-bCb^-cDc^- \sim CaAa^-bBb^-cDc^- \sim DaAa^-bBb^-cCc^-$.

For a set of surfaces M, let $g_i(M)$ denote the number of surfaces with the genus i in M. Then, the *genus distribution* of M is the sequence $g_0(M), g_1(M), g_2(M), \cdots$. The *genus polynomial* is $f_M(x) = \sum_{i \geq 0} g_i(M)x^i$.

§3. Genus Distribution for Q_j^1

Let $a, b, c, d, a^-, b^-, c^-, d^-$ be distinct letters and let A_0, B_0, C, D_0 be linear sequences. Then, surface sets Q_j^k are defined as follows for $j = 1, 2, 3, \cdots, 24$:

$$
\begin{array}{lll}
Q_1^k = \{A_kB_kCD_k\} & Q_2^k = \{A_kCD_kaB_ka^-\} & Q_3^k = \{A_kB_kCaD_ka^-\} \\
Q_4^k = \{A_kB_kaCD_ka^-\} & Q_5^k = \{A_kD_kaB_kCa^-\} & Q_6^k = \{A_kD_kCB_k\} \\
Q_7^k = \{B_kCD_kaA_ka^-\} & Q_8^k = \{B_kD_kCaA_ka^-\} & Q_9^k = \{A_kB_kD_kC\} \\
Q_{10}^k = \{A_kD_kCaB_ka^-\} & Q_{11}^k = \{A_kB_kD_kaCa^-\} & Q_{12}^k = \{A_kD_kB_kaCa^-\} \\
Q_{13}^k = \{A_kCB_kD_k\} & Q_{14}^k = \{A_kCB_kaD_ka^-\} & Q_{15}^k = \{A_kCD_kB_k\} \\
Q_{16}^k = \{A_kCaB_kD_ka^-\} & Q_{17}^k = \{A_kD_kB_kC\} & Q_{18}^k = \{CD_kaA_ka^-bB_kb^-\} \\
Q_{19}^k = \{B_kD_kaA_ka^-bCb^-\} & Q_{20}^k = \{B_kCaA_ka^-bD_kb^-\} & Q_{21}^k = \{A_kD_kaB_ka^-bCb^-\} \\
Q_{22}^k = \{A_kCaB_ka^-bD_kb^-\} & Q_{23}^k = \{A_kB_kaCa^-bD_kb^-\} & Q_{24}^k = \{A_kaB_ka^-bCb^-cD_kc^-\}
\end{array}
$$

where $k = 0$ and 1, $A_1 \in \{dA_0, A_0d\}$, $(B_1, D_1) \in \{(B_0d^-, D_0), (B_0, d^-D_0)\}$ and $a, a^-, b, b^-, c, c^-, d, d^- \notin ABCD$. Let $f_{Q_j^0}(x)$ denote the genus polynomial of Q_j^0. If $A_1^0A_0^0D_0B_1B_2^0C_2^0C_1D_1 = \emptyset$, then $f_{Q_j^0}(x) = 1$. Otherwise, suppose that $f_{Q_j^0}(x)$ are given for $1 \leqslant j \leqslant 24$. Then,

Theorem 3.1 *Let $g_{i_j}(n)$ be the number of surfaces with genus i in Q_j^n. Each of the following holds.*

$$
g_{i_j}(1)=\begin{cases}
g_{i_2}(0)+g_{i_3}(0)+g_{i_4}(0)+g_{i_5}(0), \text{ if } j=1\\
g_{i_{21}}(0)+g_{i_{22}}(0)+g_{(i-1)_1}(0)+g_{(i-1)_{15}}(0), \text{ if } j=2\\
g_{i_{22}}(0)+g_{i_{23}}(0)+g_{(i-1)_1}(0)+g_{(i-1)_{17}}(0), \text{ if } j=3\\
g_{i_4}(0)+g_{i_{18}}(0)+g_{(i-1)_6}(0)+g_{(i-1)_9}(0), \text{ if } j=4\\
g_{i_5}(0)+g_{i_{20}}(0)+g_{(i-1)_6}(0)+g_{(i-1)_{13}}(0), \text{ if } j=5\\
2g_{i_6}(0)+2g_{i_8}(0), \text{ if } j=6\\
2g_{(i-1)_{15}}(0)+2g_{(i-1)_{17}}(0), \text{ if } j=7 \text{ and } 16\\
4g_{(i-1)_6}(0), \text{ if } j=8\\
2g_{i_4}(0)+2g_{i_{10}}(0), \text{ if } j=9\\
g_{i_{10}}(0)+g_{i_{18}}(0)+g_{(i-1)_6}(0)+g_{(i-1)_9}(0), \text{ if } j=10\\
2g_{i_{21}}(0)+2g_{i_{23}}(0), \text{ if } j=11\\
2g_{i_{12}}(0)+2g_{i_{19}}(0), \text{ if } j=12\\
2g_{i_5}(0)+2g_{i_{14}}(0), \text{ if } j=13\\
g_{i_{14}}(0)+g_{i_{20}}(0)+g_{(i-1)_6}(0)+g_{(i-1)_{13}}(0), \text{ if } j=14\\
g_{i_7}(0)+g_{i_{12}}(0)+g_{i_{15}}(0)+g_{i_{16}}(0), \text{ if } j=15\\
g_{i_7}(0)+g_{i_{12}}(0)+g_{i_{16}}(0)+g_{i_{17}}(0), \text{ if } j=17\\
2g_{(i-1)_4}(0)+2g_{(i-1)_{10}}(0), \text{ if } j=18\\
4g_{(i-1)_{12}}(0), \text{ if } j=19\\
2g_{(i-1)_5}(0)+2g_{(i-1)_{14}}(0), \text{ if } j=20\\
g_{i_{21}}(0)+g_{i_{24}}(0)+g_{(i-1)_{11}}(0)+g_{(i-1)_{12}}(0), \text{ if } j=21\\
g_{(i-1)_2}(0)+g_{(i-1)_3}(0)+g_{(i-1)_{10}}(0)+g_{(i-1)_{14}}(0), \text{ if } j=22\\
g_{i_{23}}(0)+g_{i_{24}}(0)+g_{(i-1)_{11}}(0)+g_{(i-1)_{12}}(0), \text{ if } j=23\\
2g_{(i-1)_{21}}(0)+2g_{(i-1)_{23}}(0), \text{ if } j=24
\end{cases}
$$

Proof We shall prove the equation for $g_{i_6}(1)$, and the proofs for others are similar. Let

$U_1=\{A_0dd^-D_0CB_0\}$ $\qquad U_2=\{dA_0D_0CB_0d^-\}$

$U_3=\{A_0dD_0CB_0d^-\}$ $\qquad U_4=\{dA_0d^-D_0CB_0\}$.

By the definition of Q_6^1, we have $Q_6^1=\{U_1,U_2,U_3,U_4\}$. By the definition of g_i,

$$g_{i_6}(1)=g_i(U_1)+g_i(U_2)+g_i(U_3)+g_i(U_4).$$

By Op3,

$$A_0dd^-D_0CB_0\sim A_0D_0CB_0, \text{ and } dA_0D_0CB_0d^-=A_0D_0CB_0d^-d\sim A_0D_0CB_0.$$

It follows that

$$g_i(U_1)=g_i(U_2)=g_{i_6}(0). \tag{8}$$

By Lemma 2.3 (i) and Op2, we have

$$A_0dD_0CB_0d^-=D_0CB_0d^-A_0d\sim B_0D_0Cd^-A_0d\sim B_0D_0CaA_0a^-$$

and

$$dA_0d^-D_0CB_0 = B_0D_0CdA_0d^- \sim B_0D_0CaA_0a^-.$$

So

$$g_i(U_3) = g_i(U_4) = g_{i_8}(0). \tag{9}$$

Combining (1) and (2), we have

$$g_{i_6}(1) = 2g_{i_6}(0) + 2g_{i_8}(0).$$

§4. Embedding Surfaces of a Cubic Graph

Given a cubic graph G with n non-tree edges y_l ($1 \leqslant l \leqslant n$), suppose that T is a spanning tree such that T contains the longest path of G and that $\widetilde{T}$ is an associated joint tree. Let X_l, Y_l, Z_l and F_l be linear sequences for $1 \leq l \leq n$ such that $X_l \cup Y_l = y_l$, $Z_l \cup F_l = y_l^-$, $X_l \neq Y_l$ and $Z_l \neq F_l$.

RECORD RULE: Choose a vertex u incident with two semi-edges as a starting vertex and travel $\widetilde{T}$ along with tree edges of $\widetilde{T}$. In order to write down surfaces, we shall consider three cases below.

Case 1: If v is incident with two semi-edges y_s and y_t. Suppose that the linear sequence is R when one arrives v. Then, write down $RX_sy_tY_s$ going away from v.

Case 2: If v is incident with one semi-edge y_s. Suppose that R_1 is the linear sequence when one arrives v in the first time. Then the sequence is R_1X_s when one leaves v in the first time. Suppose that R_2 is the linear sequence when one arrives v in the second time. Then the sequence is R_2Y_s when one leaves v in the second time.

Case 3: If v is not incident with any semi-edge. Suppose that R_1, R_2 and R_3 are, respectively, the linear sequences when one leaves v in the first time, the second time and the third time. Then, the sequences are $(R_2/R_1)R_1(R_3/R_2)$ and R_3 when one leaves v in the third time.

Here, $1 \leq s, t \leq n$ and $s \neq t$. If v is incident with a semi-edge y_s^-, then replace X_s with Z_s and replace Y_s with F_s.

Lemma 4.1 *There is a bijection between embedding surfaces of a cubic graph and surfaces obtained by the record rule.*

Proof Let T be a spanning tree such that $\widetilde{T}$ is a joint tree of G above. Suppose that σ_v is a rotation of v and that R_1, R_2 and R_3 are given above.

$$\sigma_v = \begin{cases} (y_s, y_t, e_r), \text{ if } X_s = y_s \text{ or } F_s = y_s^- \\ \qquad\qquad \text{and } v \text{ is incident with } y_s, y_t \text{ and } e_r; \\ (y_t, y_s, e_r), \text{ if } Y_s = y_s \text{ or } Z_s = y_s^- \\ \qquad\qquad \text{and } v \text{ is incident with } y_s, y_t \text{ and } e_r; \\ (y_s, e_1, e_2), \text{ if } X_s = y_s \text{ or } F_s = y_s^- \\ \qquad\qquad \text{and } v \text{ is incident with } y_s, e_p \text{ and } e_q; \\ (e_1, y_s, e_2), \text{ if } Y_s = y_s \text{ or } Z_s = y_s^- \\ \qquad\qquad \text{and } v \text{ is incident with } y_s, e_p \text{ and } e_q; \\ (e_1, e_2, e_3), \text{ if the linear sequence is } R_3 \\ \qquad\qquad \text{and } v \text{ is incident with } e_p, e_q \text{ and } e_r; \\ (e_2, e_1, e_3), \text{ if the linear sequence is } (R_2/R_1)R_1(R_3/R_2) \\ \qquad\qquad \text{and } v \text{ is incident with } e_p, e_q \text{ and } e_r \end{cases}$$

where e_p, e_q and e_r are tree-edges for $1 \leqslant p, q, r \leqslant 2n-3$ and $e_p \neq e_q \neq e_r$ for $p \neq q \neq r$. Hence the conclusion holds. □

By the definitions for X_l, Y_l, Z_l and F_l, we have the following observation:

Observation 4.2 A surface set $H^{(0)}$ of G has properties below.
(1) Either $X_l,\ Y_l \in H^{(0)}$ or $X_l,\ Y_l \notin H^{(0)}$;
(2) Either $Z_l,\ F_l \in H^{(0)}$ or $Z_l,\ F_l \notin H^{(0)}$;
(3) If for some l with $1 \leqslant l \leqslant n$, $X_l, Y_l, Z_l, F_l \in H^{(0)}$, then $H^{(0)}$ has one of the following forms $X_lA^{(0)}Y_lB^{(0)}Z_lC^{(0)}F_l\ D^{(0)}$, $Y_lA^{(0)}X_lB^{(0)}Z_lC^{(0)}F_lD^{(0)}$, $X_lA^{(0)}Y_lB^{(0)}F_lC^{(0)}Z_lD^{(0)}$ or $Y_lA^{(0)}X_lB^{(0)}F_lC^{(0)}Z_lD^{(0)}$. These forms are regarded to have no difference through this paper.

If either $X_l \in H^{(0)}, Z_l \notin H^{(0)}$ or $X_l \notin H^{(0)}, Z_l \in H^{(0)}$, then replace X_l, Y_l, Z_l and F_l according to the definition of X_l, Y_l, Z_l and F_l.

RECURSION RULE: Given a surface set $H^{(0)} = \{X_lA^{(0)}Y_lB^{(0)}Z_lC^{(0)}F_lD^{(0)}\}$ where $A^{(0)}, B^{(0)}$, $C^{(0)}$ and $D^{(0)}$ are linear sequences.

Step 1. Let $A_0 = A^{(0)}$, $B_0 = B^{(0)}$, $C = C^{(0)}$ and $D_0 = D^{(0)}$. Q_j^1 is obtained for $2 \leqslant j \leqslant 5$. Then $H_j^{(1)}$ is obtained by replacing a, a^- and Q_j^1 with a_1, a_1^- and $H_j^{(1)}$ respectively.

Step 2. Given a surface set $H^{(k)}_{j_1,j_2,j_3,\cdots,j_k}$ for a positive integer k and $2 \leqslant j_1, j_2, j_3, \cdots, j_k \leqslant 5$, without loss of generality, suppose that $H^{(k)}_{j_1,j_2,j_3,\cdots,j_k} = \{X_sA^{(k)}Y_sB^{(k)}Z_sC^{(k)}F_sD^{(k)}\}$ where $A^{(k)}, B^{(k)}, C^{(k)}$ and $D^{(k)}$ are linear sequences for certain s ($1 \leqslant s \leqslant n$). Let $A_0 = A^{(k)}$, $B_0 = B^{(k)}$, $C = C^{(k)}$ and $D_0 = D^{(k)}$. Q_j^1 is obtained for $2 \leqslant j \leqslant 5$. Then $H^{(k+1)}_{j_1,j_2,j_3,\cdots,j_k,j}$ is obtained by replacing a, a^- and Q_j^1 with a_{k+1}, a_{k+1}^- and $H^{(k+1)}_{j_1,j_2,j_3,\cdots,j_k,j}$ respectively.

Some surface sets $H^{(m)}_{j_1,j_2,j_3,\cdots,j_m}$ which contain a_l, a_l^-, y_l, y_l^- can be obtained by using step 2 for a positive integer m, $2 \leqslant j_1, j_2, j_3, \cdots, j_m \leqslant 5$ and $1 \leqslant l \leqslant n$. It is easy to compute $f_{H^{(m)}_{j_1,j_2,j_3,\cdots,j_m}}(x)$.

By Theorem 3.7,

$$\begin{aligned} g_i(H^{(r)}_{j_1,j_2,j_3,\cdots,j_r}) &= g_i(H^{(r+1)}_{j_1,j_2,j_3,\cdots,j_r,2}) + g_i(H^{(r+1)}_{j_1,j_2,j_3,\cdots,j_r,3}) \\ &+ g_i(H^{(r+1)}_{j_1,j_2,j_3,\cdots,j_r,4}) + g_i(H^{(r+1)}_{j_1,j_2,j_3,\cdots,j_r,5}), \qquad (1) \\ &\qquad \text{if } 0 \le r \le m-1, 2 \le j_1, j_2, j_3, \cdots, j_r \le 5. \end{aligned}$$

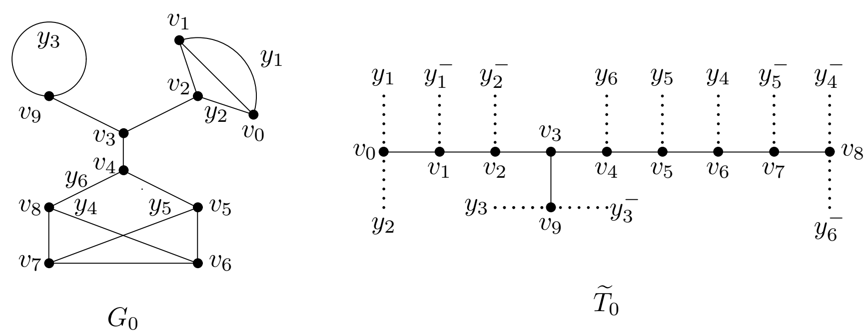

Fig.1: G_0 and $\widetilde{T}_0$

Example 4.3 The graph G_0 is given in Fig.1. A joint tree $\widetilde{T}_0$ is obtained by splitting non-tree edges y_l $(1 \leqslant l \leqslant 6)$. Travel $\widetilde{T}_0$ by regarded v_0 as a starting point. By using record rule we obtain surface sets

$$\{X_1y_2Y_1Z_1Z_2Z_3y_3F_3Y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5X_6F_2F_1\}$$

and

$$\{X_1y_2Y_1Z_1Z_2Y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5X_6F_2F_1Z_3y_3F_3\}.$$

By replacing Z_2, F_2, Z_3, F_3, X_6 and Y_6 according the definition 16 surface sets U_r $(1 \leqslant r \leqslant 16)$ are listed below.

$$U_1 = \{X_1y_2Y_1Z_1y_2^-y_3^-y_3y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5F_1\}$$

$$U_2 = \{X_1y_2Y_1Z_1y_2^-y_3^-y_3Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6F_1\}$$

$$U_3 = \{X_1y_2Y_1Z_1y_2^-y_3y_3^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5F_1\}$$

$$U_4 = \{X_1y_2Y_1Z_1y_2^-y_3y_3^-Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6F_1\}$$

$$U_5 = \{X_1y_2Y_1Z_1y_3^-y_3y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_2^-F_1\}$$

$$U_6 = \{X_1y_2Y_1Z_1y_3^-y_3Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6y_2^-F_1\}$$

$$U_7 = \{X_1y_2Y_1Z_1y_3y_3^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_2^-F_1\}$$

$$U_8 = \{X_1y_2Y_1Z_1y_3y_3^-Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6y_2^-F_1\}$$

$$U_9 = \{X_1y_2Y_1Z_1y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5F_1y_3^-y_3\}$$

$$U_{10} = \{X_1y_2Y_1Z_1y_2^-Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6F_1y_3^-y_3\}$$

$$U_{11} = \{X_1y_2Y_1Z_1y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5F_1y_3y_3^-\}$$

$$U_{12} = \{X_1y_2Y_1Z_1y_2^-Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6F_1y_3y_3^-\}$$

$$U_{13} = \{X_1y_2Y_1Z_1y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_2^-F_1y_3^-y_3\}$$

$$U_{14} = \{X_1y_2Y_1Z_1Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6y_2^-F_1y_3^-y_3\}$$

$$U_{15} = \{X_1y_2Y_1Z_1y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_2^-F_1y_3y_3^-\}$$

$$U_{16} = \{X_1y_2Y_1Z_1Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6y_2^-F_1y_3y_3^-\}.$$

The genus distribution of U_r can be obtained by using the recursion rule. Since the method is similar, we shall calculate the genus distribution of U_1 and leave the calculation of genus distribution for others to readers.

U_1 is reduced to $\{X_1y_2Y_1Z_1y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5F_1\}$ by Op2. Let $H^{(0)} = S_1$, $A_0 = y_2$, $C_0 = y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5$ and $B_0 = D_0 = \emptyset$. Then $H_2^{(1)} = H_3^{(1)} = \{y_2y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5\}$ and $H_4^{(1)} = H_5^{(1)} = \{y_2a_1y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5a_1^-\}$.

$H_2^{(1)}$ is reduced to $\{y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5\}$ by Op2. Let $A_0 = X_5y_6Y_5$, $B_0 = Z_5$, $C_0 = y_6^-$ and $D_0 = F_5$. Then $H_{2,2}^{(2)} = \{X_5y_6Y_5y_6^-F_5a_2Z_5a_2^-\}$, $H_{2,3}^{(2)} = \{X_5y_6Y_5Z_5y_6^-a_2F_5a_2^-\}$, $H_{2,4}^{(2)} = \{X_5y_6Y_5Z_5a_2y_6^-F_5a_2^-\}$ and $H_{2,5}^{(2)} = \{X_5y_6Y_5F_5a_2Z_5y_6^-a_2^-\}$. $H_{4,2}^{(2)} = \{X_5a_1^-y_2a_1y_2^-y_6Y_5 y_6^-F_5a_2Z_5a_2^-\}$, $H_{4,3}^{(2)} = \{X_5a_1^-y_2a_1y_2^-y_6Y_5Z_5y_6^-a_2F_5a_2^-\}$, $H_{4,4}^{(2)} = \{X_5a_1^-y_2a_1y_2^-y_6Y_5Z_5a_2y_6^-F_5 a_2^-\}$ and $H_{4,5}^{(2)} = \{X_5a_1^-y_2a_1y_2^-y_6Y_5F_5a_2Z_5y_6^-a_2^-\}$ by letting $A_0 = X_5a_1^-y_2a_1y_2^-y_6Y_5$, $B_0 = Z_5$, $C_0 = y_6^-$ and $D_0 = F_5$.

Similarly, $H_{2,2,2}^{(3)} = \{y_6a_2a_2^-a_3y_6^-a_3^-\}$, $H_{2,2,3}^{(3)} = \{y_6y_6^-a_2a_3a_2^-a_3^-\}$, $H_{2,2,4}^{(3)} = \{y_6y_6^-a_3a_2a_2^-a_3^-\}$ and $H_{2,2,5}^{(3)} = \{y_6a_2^-a_3y_6^-a_2a_3^-\}$. $H_{2,3,2}^{(3)} = \{y_6y_6^-a_2a_2^-\}$, $H_{2,3,3}^{(3)} = \{y_6y_6^-a_2a_3a_2^-a_3^-\}$, $H_{2,3,4}^{(3)} = \{y_6a_3y_6^-a_2a_2^-a_3^-\}$ and $H_{2,3,5}^{(3)} = \{y_6a_2^-a_3y_6^-a_2a_3^-\}$. $H_{2,4,2}^{(3)} = \{y_6a_2y_6^-a_2^-\}$, $H_{2,4,3}^{(3)} = \{y_6a_2y_6^-a_3a_2^- a_3^-\}$, $H_{2,4,4}^{(3)} = \{y_6a_3a_2y_6^-a_2^-a_3^-\}$ and $H_{2,4,5}^{(3)} = \{y_6a_2^-a_3a_2y_6^-a_3^-\}$. $H_{2,5,2}^{(3)} = \{y_6a_2y_6^-a_2^-\}$, $H_{2,5,3}^{(3)} = \{y_6a_2a_3y_6^-a_2^-a_3^-\}$, $H_{2,5,4}^{(3)} = \{y_6a_3a_2y_6^-a_2^-a_3^-\}$ and $H_{2,5,5}^{(3)} = \{y_6y_6^-a_2^-a_3a_2a_3^-\}$. $H_{4,2,2}^{(3)} = \{a_1^-y_2a_1 y_2^-y_6a_2a_2^-a_3y_6^-a_3^-\}$, $H_{4,2,3}^{(3)} = \{a_1^-y_2a_1y_2^-y_6y_6^-a_2a_3a_2^-a_3^-\}$, $H_{4,2,4}^{(3)} = \{a_1^-y_2a_1y_2^-y_6y_6^- a_3a_2a_2^-a_3^-\}$ and $H_{4,2,5}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_2^-a_3y_6^-a_2a_3^-\}$. $H_{4,3,2}^{(3)} = \{a_1^-y_2a_1y_2^-y_6y_6^-a_2a_2^-\}$, $H_{4,3,3}^{(3)} = \{a_1^-y_2a_1y_2^- y_6y_6^-a_2a_3a_2^-a_3^-\}$, $H_{4,3,4}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_3y_6^-a_2a_2^-a_3^-\}$ and $H_{4,3,5}^{(3)} = \{a_1^-y_2a_1y_2^-y_6 a_2^-a_3y_6^-a_2a_3^-\}$. $H_{4,4,2}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_2y_6^-a_2^-\}$, $H_{4,4,3}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_2y_6^-a_3a_2^-a_3^-\}$, $H_{4,4,4}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_3a_2 y_6^-a_2^-a_3^-\}$ and $H_{4,4,5}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_2^-a_3a_2y_6^-a_3^-\}$. $H_{4,5,2}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_2 y_6^-a_2^-\}$, $H_{4,5,3}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_2a_3y_6^-a_2^-a_3^-\}$, $H_{4,5,4}^{(3)} = \{a_1^-y_2a_1y_2^-y_6a_3a_2y_6^-a_2^-a_3^-\}$ and $H_{4,5,5}^{(3)} = \{a_1^-y_2a_1y_2^-y_6 y_6^-a_2^-a_3a_2a_3^-\}$.

By using (1),

$$f_{U_1}(x) = 4 + 32x + 28x^2.$$

Thus,

$$f_{G_0}(x) = 64 + 512x + 448x^2.$$

§5. Genus Distribution for a Graph

Theorem 5.1 *Given a graph, the genus distribution of G is determined by using the genus distribution of some cubic graphs.*

Proof Given a finite graph G_0, suppose that u is adjacent to $k+1$ distinct vertices v_0, v_1, v_2, $\cdots$, v_k of G_0 with $k \geq 3$. Actually, the supposition always holds by subdividing some edges of G.

A *distribution decomposition* of a graph is defined below: add a vertex u_s of valence 3 such that u_s is adjacent to u, v_0 and v_s for each s with $1 \leq s \leq k$ and then obtain a graph G_s by deleting the edges uv_0 and uv_s.

Choose the spanning trees T_s of G_s such that uv_s, uu_s and u_sv_s are tree edges for $0 \leq s \leq k$. Consider a joint tree $\widetilde{T}_0$ of G. Let $\widetilde{T}_s^*$ be the maximal joint tree of $\widetilde{T}_0$ such that $v_s \in V(T_s^*)$ and $v_t \notin V(T_s^*)$ for $t \neq s$ and $0 \leqslant s, t \leqslant k$.

Let v_s be the starting vertex of $\widetilde{T}_s^*$ for $0 \leqslant s \leqslant k$. Suppose that $\mathcal{A}_s$ is the set of all sequences by travelling $\widetilde{T}_s^*$ and that Q_s is the embedding surface set of G_s. Then

$$Q_0 = \{A_0A_{r_1}A_{r_2}A_{r_3}\cdots A_{r_k}|A_{r_p} \in \mathcal{A}_{r_p},\ 1 \leqslant r_p \leqslant k, r_p \neq r_q \text{ for } p \neq q\}$$

and for $1 \leqslant s \leqslant k$

$$Q_s = \{A_0A_sA_{r_1}A_{r_2}A_{r_3}\cdots A_{r_{k-1}}, A_0A_{r_1}A_{r_2}A_{r_3}\cdots A_{r_{k-1}}A_s|A_{r_p} \in \mathcal{A}_{r_p},$$
$$1 \leqslant r_p \leqslant k, r_p \neq s, 1 \leqslant p, q \leqslant k-1, \text{ and } r_p \neq r_q \text{ for } p \neq q\}.$$

Let $f_{Q_s}(x)$ denote the genus distribution of Q_s. It is obvious that

$$f_{Q_0}(x) = \frac{1}{2}\sum_{s=1}^{k} f_{Q_s}(x).$$

Thus,

$$f_{G_0}(x) = \frac{1}{2}\sum_{s=1}^{k} f_{G_s}(x).$$

Since G_0 has finite vertices, the genus distribution of G_0 can be transformed into those of some cubic graphs in homeomorphism by using the distribution decomposition. □

Next we give a simple application of Theorem 5.1.

Example 5.2 The graph W_4 is shown in Fig.2. In order to calculate its genus distribution, we use the distribution decomposition and then we obtain three graph G_s for $1 \leqslant s \leqslant 3$ (Fig.2). It is obvious that G_2 are isomorphic to Möbius ladder ML_3 and G_s are isomorphic to Ringel ladder RL_2 for $s = 1$ and 3. Since (see [8], [15])

$$f_{ML_3}(x) = 40x + 24x^2$$

and since (see [9], [15])

$$f_{RL_2}(x) = 2 + 38x + 24x^2,$$

$$\begin{aligned} f_{W_4}(x) &= \frac{1}{2}\sum_{s=1}^{3} f_{G_s}(x) \\ &= \frac{1}{2}[40x + 24x^2 + 2(2 + 38x + 24x^2)] \\ &= 2 + 58x + 36x^2. \end{aligned}$$

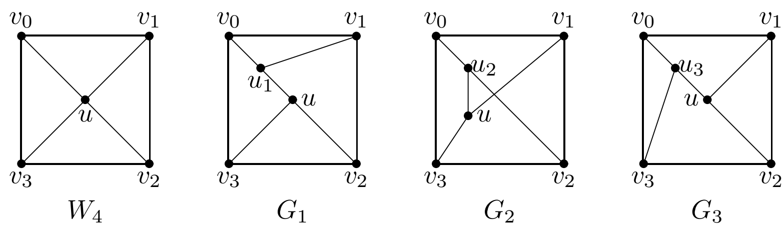

Fig.2: W_4 and G_s

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