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Efficient Domination in Bi-Cayley Graphs

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Abstract: A Cayley graph is constructed out of a group Γ and its generating set X and it is denoted by $\mathbb{C}(\Gamma, X)$. A *Smarandachely n-Cayley graph* is defined to be $G = ZC(\Gamma, X)$, where $V(G) = \Gamma \times \mathbb{Z}_n$ and $E(G) = \{((x,0),(y,1))_a, ((x,1),(y,2))_a, \cdots, ((x,n-2),(y,n-1))_a :$ $x, y \in \Gamma, a \in X$ such that $y = x * a\}$. Particularly, a Smarandachely 2-Cayley graph is called as a *Bi-Cayley graph*, denoted by $B\mathbb{C}(\Gamma, X)$. Necessary and sufficient conditions for the existence of an efficient dominating set and an efficient open dominating set in Bi-Cayley graphs are determined.

Key Words: Cayley graphs, Smarandachely n-Cayley graph, Bi-Cayley graphs, efficient domination, efficient open domination, covering of a graph.

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§1. Introduction

The terminology and notation in this paper follows that found in [3]. The fact that Cayley graphs are excellent models for interconnection networks, investigated in connection with parallel processing and distributed computation. The concept of domination for Cayley graphs has been studied by various authors and one can refer to [2, 4, 6]. I.J. Dejter, O. Serra [2], J.Huang, J-M. Xu [4] obtained some results on efficient dominating sets for Cayley graphs. The existence of independent perfect dominating sets in Cayley graphs was studied by J.Lee [6]. Tamizh Chelvam and Rani [8-10], obtained the domination, independent domination, total domination and connected domination numbers for some Cayley graphs constructed on $\mathbb{Z}_n$ for some generating set of $\mathbb{Z}_n$.

Let $(\Gamma, *)$ be a group with e as the identity and X be a symmetric generating set(if $a \in X$, then $a^{-1} \in X$) with $e \notin X$. The Cayley graph $G = \mathbb{C}(\Gamma, X)$, where $V(G) = \Gamma$ and $E(G) = \{(x, y)_a/x, y \in V(G), a \in X$ such that $y = x*a\}$. Since X is a generating set for Γ, $\mathbb{C}(\Gamma, X)$ is a connected and regular graph of degree $|X|$. The Bi-Cayley graph is defined as $G = B\mathbb{C}(\Gamma, X)$, where $V(G) = \Gamma \times \{0, 1\}$ and $E(G) = \{((x,0),(y,1))_a/x, y \in \Gamma, a \in X$ such that $y = x * a\}$. Now the operation $+$ is defined by $(x,0)+(y,1) = (x*y,1)$ and $(x,0)+(y,0) = (x*y,0)$.

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The *Smarandachely n-Cayley graph* is defined to be $G = ZC(\Gamma, X)$, where $V(G) = \Gamma \times \mathbb{Z}_n$ and $E(G) = \{((x,0),(y,1))_a, ((x,1),(y,2))_a, \cdots, ((x,n-2),(y,n-1))_a : x, y \in \Gamma, a \in X$ such that $y = x * a\}$. When $n = 2$, the Smarandachely n-Cayley graphs are called as *Bi-Cayley graphs*. By the definition of Bi-Cayley graph, it is a regular graph of degree $|X|$.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V - S$ is adjacent to an element u of S. The domination number $\gamma(G)$ is the minimum cardinality among all the dominating sets in G [3] and a corresponding dominating set is called a γ-set. A dominating set S is called an efficient dominating set if for every vertex $v \in V, |N[v] \cap S| = 1$. Note that if S is an efficient dominating set then $\{N[v] : v \in S\}$ is a partition of $V(G)$ and if G has an efficient dominating set, then all efficient dominating sets in G have the same cardinality namely $\gamma(G)$. A set $S \subseteq V$ is called a total dominating set if every vertex $v \in V$ is adjacent to an element $u(\neq v)$ of S. The total domination number $\gamma_t(G)$ of G equals the minimum cardinality among all the total dominating sets in G [3] and a corresponding total dominating set is called a γ_t-set. A dominating set S is called an efficient open dominating set if for every vertex $v \in V$, $|N(v) \cap S| = 1$.

A graph $\tilde{G}$ is called covering of G with projection $f : \tilde{G} \to G$ if there is a surjection $f : V(\tilde{G}) \to V(G)$ such that $f|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in f^{-1}(v)$. Also the projection $f : \tilde{G} \to G$ is said to be an *n-fold covering* if f is $n-$to-one.

In this paper, we prove that the Bi-Cayley graph obtained from Cayley graph for an Abelian group $(\Gamma, *)$ has an efficient dominating set if and only if it is a covering of the graph $\overline{K_n \times K_2}$. It is also proved that the Bi-Cayley graph obtained from Cayley graph for an Abelian group $(\Gamma, *)$ has an efficient open dominating set if and only if it is a covering of the graph $K_{n,n}$.

Theorem 1.1([4]) *Let G be a $k-$regular graph. Then $\gamma(G) \geq \frac{|V(G)|}{k+1}$, with the inequality if and only if G has an efficient dominating set.*

Theorem 1.2([6]) *Let $p : \tilde{G} \to G$ be a covering and let S be a perfect dominating set of G. The $p^{-1}(S)$ is a perfect dominating set of $\tilde{G}$. Moreover, if S is independent, then $p^{-1}(S)$ is independent.*

Theorem 1.3([3]) *If G has an efficient open dominating set S, then $|S| = \gamma_t(G)$ and all efficient open dominating sets have the same cardinality.*

§2. Efficient Domination and Bi-Cayley Graphs

In this section, we find the necessary and sufficient condition for the existence of an efficient dominating set in $B\mathbb{C}(\Gamma, X)$. Since $B\mathbb{C}(\Gamma, X)$ is regular bi-partite graph, in $B\mathbb{C}(\Gamma, X)$ every efficient dominating set S is of the form $S = A \cup B$ where $A \subseteq (\Gamma \times 0) \cap S$ and $B \subseteq (\Gamma \times 1) \cap S$ with $|A| = |B| = \frac{|S|}{2}$.

Through out this section, the vertex set of $V(\overline{K_n \times K_2})$ is taken to be $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ such that $\langle\{a_1, a_2, \cdots, a_n\}\rangle$, $\langle\{b_1, b_2, \cdots, b_n\}\rangle$ are null graphs and $(a_i, b_j) \in E(\overline{K_n \times K_2})$ if and only if $i \neq j$.

Lemma 2.1 *Let $S_1, S_2, \cdots, S_n$ be n efficient dominating sets of $B\mathbb{C}(\Gamma, X)$ which are mutually pairwise disjoint. Then the induced subgraph $\tilde{G} =< S_1 \cup S_2 \cup \cdots \cup S_n >$ is a $m-$fold covering graph of the graph $G = \overline{K_n \times K_2}$, where $m = \frac{|S_i|}{2}$ for each $i = 1, 2, \cdots, n$.*

Proof Note that in a graph all the efficient dominating sets have the same cardinality. Since S_1 is efficient, $S_1 = A_1 \cup B_1$ where $A_1 \subseteq (\Gamma \times 0) \cap S_1$ and $B_1 \subseteq (\Gamma \times 1) \cap S_1$ with $|A_1| = |B_1| = \frac{|S_1|}{2}$. Define $A_i = N(B_1) \cap S_i$ and $B_i = N(A_1) \cap S_i$ for $2 \leq i \leq n$. Note that $A_i \subset \Gamma \times 0$ and $B_i \subset \Gamma \times 1$ for $1 \leq i \leq n$ and $\tilde{G} =< A_1 \cup B_1 \cup A_2 \cup B_2 \cup \cdots \cup A_n \cup B_n >$.

Let $V(G) = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$. Define $f : \tilde{G} \to G$ by $f(s) = a_i \ if \ s \in A_i$ and $f(s) = b_i \ if \ s \in B_i$ for $1 \leq i \leq n$. Let $v \in V(G)$. Suppose $v = a_i$. Then $N(v) = \{b_1, b_2, \ldots, b_{i-1}, b_{i+1}, b_{i+2}, \cdots, b_n\}$ and $f^{-1}(v) = A_i$. Let $\tilde{v} \in f^{-1}(v)$. Since S_i's are efficient, $N(\tilde{v}) = \{\beta_1, \beta_2, \cdots, \beta_{i-1}, \beta_{i+1}, \cdots, \beta_n\}$ where $\beta_j \in B_j$ for $1 \leq j \leq i-1$ and $i+1 \leq j \leq n$. By the definition of f, we have $f(\beta_j) = b_j$. Thus $f : N(\tilde{v}) \to N(v)$ is a bijection when $v = a_i$. Similarly one can prove that $f : N(\tilde{v}) \to N(v)$ is a bijection when $v = b_i$. Since $\frac{|S_i|}{2} = |A_i| = |B_i| = m$ for all $1 \leq i \leq n$, f is an $m-$fold covering of the graph $\overline{K_n \times K_2}$. □

Theorem 2.2 *Let $G = B\mathbb{C}(\Gamma, X)$ and n be a positive integer. Then G is a covering graph of $\overline{K_n \times K_2}$ if and only if G has a vertex partition of n efficient dominating sets.*

Proof Suppose G is a covering of $\overline{K_n \times K_2}$. Since $\{a_i, b_i\}$ is an efficient dominating set in $\overline{K_n \times K_2}$, by Theorem 1.2, we have $f^{-1}(\{a_i, b_i\})$ is an efficient dominating set in G for $1 \leq i \leq n$. Since f is a function, $f^{-1}(\{a_i, b_i\}) \cap f^{-1}(\{a_j, b_j\}) = \emptyset$ for $i \neq j$. Hence $\{f^{-1}(\{a_i, b_i\}) : 1 \leq i \leq n\}$ is a vertex partition of efficient dominating sets in G. The other part follows from Lemma 2.1. □

Lemma 2.3 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let S be an efficient dominating set for the Bi-Cayley graph $G = B\mathbb{C}(\Gamma, X)$. Then we have the following:*

(a) For each $1 \leq i \leq n, S + (x_i, 0)$ is an efficient dominating set.

(b) $\{S, S + (x_1, 0), S + (x_2, 0), \cdots, S + (x_n, 0)\}$ is a vertex partition in $B\mathbb{C}(\Gamma, X)$.

Proof (a) Let $(v, 0) \in V(G)$. If $(x_i^{-1} * v, 0) \in S$, then $(v, 0) \in S + (x_i, 0)$. Suppose $(x_i^{-1} * v, 0) \notin S$. Since S is efficient, there exists unique $(s, 1) \in S$ such that $s = (x_i^{-1} * v) * x$ for some $x \in X$. That is $x_i * s = v * x$. Hence the vertex $(v, 0)$ is dominated by $(x_i * s, 1) \in S + (x_i, 0)$. Thus in all the cases we have $(v, 0) \in N[S + (x_i, 0)]$. Similarly when $(v, 1) \in V(G)$, one can prove that $(v, 1) \in N[S + (x_i, 0)]$. Thus $S + (x_i, 0)$ is a dominating set for $1 \leq i \leq n$. Since S is efficient and $|S + (x_i, 0)| = |S|$, by Theorem 1.1, we have $S + (x_i, 0)$ is an efficient dominating set for $1 \leq i \leq n$.

(b) Since S is a dominating set, for every $(u, 0) \in V(G)$, we have $(u, 0) \in S$ or $(u, 0)$ is adjacent to some vertex $(s, 1) \in S$ and so $u = s * x_i$ for some $x_i \in X$. Similar thing is holds for $(u, 1) \in V(G)$. This means that $V(G) = S \cup (S + (x_1, 0)) \cup (S + (x_2, 0)) \cup \cdots \cup (S + (x_n, 0))$. Since G is $|X|-$regular and S is an efficient dominating set, $|S| = \frac{2|\Gamma|}{|X|+1}$. That is $2|\Gamma| = (|X| + 1)|S|$. Since $|S| = |S + (x_1, 0)| = |S + (x_2, 0)| = \cdots = |S + (x_n, 0)|$, one can conclude that $\{S, S+(x_1,0), S+(x_2,0), \cdots, S+(x_n,0)\}$ is a vertex partition of G. □

From Lemmas 2.1, 2.3 one can have the following:

Corollary 2.4 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let S be an efficient dominating set in $B\mathbb{C}(\Gamma, X)$. If $(x_i,0)+S = S+(x_i,0)$ for each $1 \le i \le n$, then there exist a covering $f : B\mathbb{C}(\Gamma, X) \to \overline{K_{n+1} \times K_2}$ such that $S, S+(x_1,0), S+(x_2,0), \cdots, S+(x_n,0)$ are the fibers of $\{a_i, b_i\}$ under the map f.*

Now we define the following: For $S \subset V(B\mathbb{C}(\Gamma, X))$, define $S^0 = S \cup \{(e,0)\}$.

Theorem 2.5 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let M be a normal subset of Γ and $S = (M \times 0) \cup (M \times 1)$. Then the following are equivalent.*

(a) S is an efficient dominating set in $B\mathbb{C}(\Gamma, X)$.

(b) There exists a covering $f : B\mathbb{C}(\Gamma, X) \to \overline{K_{n+1} \times K_2}$ such that $f^{-1}(\{a_i, b_i\}) = S$ for some $1 \le i \le n$.

(c) $|S| = \frac{2|\Gamma|}{|X|+1}$ and $S \cap [S + (((X \times 0)^0 + (X \times 0)^0) - \{(e,0)\})] = \emptyset$.

Proof $(a) \Rightarrow (b)$: Since M is a normal subset, we have $(x_i,0)+S = S+(x_i,0)$ for $1 \le i \le n$ and so the proof follows from Corollary 2.4.

$(b) \Rightarrow (a)$: Since $\{a_i, b_i\}$ is an efficient dominating set in $\overline{K_n \times K_2}$, the proof follows from Theorem 1.2.

$(a) \Rightarrow (c)$: Since S is an efficient dominating set and G is $|X|-$regular, the fact $|S| = \frac{2|\Gamma|}{|X|+1}$ follows from Theorem 1.1. Suppose $S \cap [S + (((X \times 0)^0 + (X \times 0)^0) - \{(e,0)\})] \neq \emptyset$. Then there exist $(s,0)$(or $(s,1)$) $\in S$ such that $(s,0) = (s_1,0)+(x,0)+(x_1,0)$ with $x, x_1 \in X, x \neq x_1^{-1}$ and $(s_1,0)$ (or $(s_1,1)$) $\in S$. Since $x \neq x_1^{-1}$, we have $s \neq s_1$. Thus $s * x^{-1} = s_1 * x_1$ and so $(s_1 * x_1, 1)$ is adjacent to two vertices $(s,0), (s_1,0) \in S$, a contradiction to S is efficient.

$(c) \Rightarrow (a)$: Let $x_i, x_j \in X$ with $x_i \neq x_j$. Suppose $(S+(x_i,0)) \cap (S+(x_j,0)) \neq \emptyset$. Let $a \in (S+(x_i,0)) \cap (S+(x_j,0))$. Then $a = (s_1,0)+(x_i,0) = (s_2,0)+(x_j,0)$ or $(s_1,1)+(x_i,0) = (s_2,1)+(x_j,0)$. Hence $s_1 * x_i = s_2 * x_j$ and so $s_1 = s_2 * x_j * x_i^{-1}$. Since $x_i \neq x_j$, we have $x_i^{-1} * x_j \neq e$. Thus $(s_1,0) \in S \cap [S + (((X \times 0)^0 + (X \times 0)^0) - \{(e,0)\})]$, a contradiction. Suppose $S \cap (S+(x,0)) \neq \emptyset$ for some $x \in X$. Then $(s,0) = (s_1,0)+(x,0)$ or $(s,1) = (s_1,1)+(x,0)$. Thus $(s,0) = (s_1,0)+(x,0)+(e,0)$ or $(s,1) = (s_1,1)+(x,0)+(e,0)$. Since $x \neq e$, $(s,0) \in S \cap [S + (((X \times 0)^0 + (X \times 0)^0) - \{(e,0)\})]$, a contradiction. Thus $S \cup \{S+(x_i,0) : 1 \le i \le n\}$ is a collection of pairwise disjoint sets. Now $N[S] = \bigcup_{s \in S} N[s] = \bigcup_{s \in S} [s + (X \times 0)^0] = \bigcup_{(x,0) \in (X \times 0)^0} [S + (x,0)]$. Since $|S+(x,0)| = |S|$, $|N[S]| = |S||(X \times 0)^0| = |S|(|X|+1) = |S|(\frac{2|\Gamma|}{|S|}) = 2|\Gamma|$. Thus S is a dominating set. Since $|S| = \frac{2|\Gamma|}{|X|+1}$, by Theorem 1.1, S is an efficient dominating set. □

§3. Efficient Open Domination and Bi-Cayley Graphs

In this section, we find the necessary and sufficient condition for the existence of an efficient open dominating set in $B\mathbb{C}(\Gamma, X)$. Note that if S is an efficient open dominating set of a graph

G, then $\{N(v) : v \in S\}$ is a partition of $V(G)$ and if G has an efficient open dominating set, then all efficient open dominating sets in G have the same cardinality namely $\gamma_t(G)$.

Through out this section, the vertex set of $K_{n,n}$ is taken as $\{c_1, c_2, \cdots, c_n, d_1, d_2, \cdots, d_n\}$ where no two c_i's are adjacent and no two d_i's are adjacent.

Remark 3.1 If S is an efficient open dominating set in $G = B\mathbb{C}(\Gamma, X)$, then $|S|$ is even and we can write $S = C \cup D$ where $|C| = |D| = \frac{|S|}{2}$ and every edge of $\langle C \cup D\rangle$ has one end in C and another end in D. Note that if G is a $k-$regular graph, then $\gamma_t(G) \geq \frac{|V(G)|}{k}$ and equality holds if and only if G has an efficient open dominating set.

Lemma 3.1 *Let $S_1, S_2, \cdots, S_n$ be n mutually pairwise disjoint efficient open dominating sets of $B\mathbb{C}(\Gamma, X)$. Then the induced subgraph $\tilde{G} = \langle S_1 \cup S_2 \cup \cdots \cup S_n\rangle$ is a $m-$fold covering graph of $G = K_{n,n}$, where $m = \frac{|S_i|}{2}$ for each $i = 1, 2, \cdots, n$.*

Proof Since S_i is efficient open for each $1 \leq i \leq n$, we have $S_i = C_i \cup D_i$ where $C_i \subseteq (\Gamma \times 0) \cap S_i$ and $D_i \subseteq (\Gamma \times 1) \cap S_i$ with $|C_i| = |D_i| = \frac{|S_i|}{2}$ and every edge in the induced subgraph $\langle C_i \cup D_i\rangle$ has one end in C_i and other in D_i. Note that $\tilde{G} = \langle C_1 \cup D_1 \cup C_2 \cup D_2 \cup \cdots \cup C_n \cup D_n\rangle$. Let $V(G) = \{c_1, c_2, \cdots, c_n, d_1, d_2, \cdots, d_n\}$.

Define $f : \tilde{G} \to G$ by $f(s) = c_i$ if $s \in C_i$ and $f(s) = d_i$ if $s \in D_i$ for $1 \leq i \leq n$. Let $v \in V(G)$. Suppose $v = c_i$. Then $N(v) = \{d_1, d_2, \ldots, d_n\}$ and $f^{-1}(v) = C_i$. Let $\tilde{v} \in f^{-1}(v)$. Since S_i's are efficient open, $N(\tilde{v}) = \{\beta_1, \beta_2, \cdots, \beta_n\}$ where $\beta_j \in D_j$ for $1 \leq j \leq n$. By the definition of f, we have $f(\beta_j) = d_j$. Thus $f : N(\tilde{v}) \to N(v)$ is a bijection when $v = c_i$. Similarly one can prove that $f : N(\tilde{v}) \to N(v)$ is a bijection when $v = d_i$. Since $\frac{|S_i|}{2} = |C_i| = |D_i| = m$ for all $1 \leq i \leq n$, f is an $m-$fold covering of the graph $K_{n,n}$. □

Remark 3.3 Let $f : \tilde{G} \to G$ be a covering and S be an efficient open dominating set of G. By the definition of an efficient open domination, S is perfect and so by Theorem 1.2, $f^{-1}(S)$ is perfect. That is $|N(\tilde{v}) \cap f^{-1}(S)| = 1$ for all $\tilde{v} \in \tilde{G} - f^{-1}(S)$. Let $\tilde{v} \in f^{-1}(S)$. Then $f(\tilde{v}) = v \in S$. Since S is an efficient open dominating set, there exist unique $w \in S$ such that v and w are adjacent. Since $f|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection, $\tilde{w} = f^{-1}(w)$ is the only vertex adjacent to $\tilde{v}$ in $f^{-1}(S)$. That is $|N(\tilde{v}) \cap f^{-1}(S)| = 1$ for all $\tilde{v} \in f^{-1}(S)$. Hence inverse image of an efficient open dominating set under a covering function is an efficient open dominating set.

Theorem 3.4 *Let $G = B\mathbb{C}(\Gamma, X)$ and n be a positive integer. Then G is a covering of $K_{n,n}$ if and only if G has a vertex partition of efficient open dominating sets.*

Proof Suppose G is a covering graph of $K_{n,n}$. Since the pair $\{c_i, d_i\}$ is an efficient open dominating set in $K_{n,n}$, by Remark 3.3, $f^{-1}(\{c_i, d_i\})$ is an efficient open dominating set in G for $1 \leq i \leq n$. Since f is a function, $\{f^{-1}(\{c_i, d_i\}) : 1 \leq i \leq n\}$ is a partition of efficient open dominating sets in G. The other part follows from Lemma 3.2. □

Lemma 3.5 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let S be an efficient open dominating set for the Bi-Cayley graph $G = B\mathbb{C}(\Gamma, X)$. Then we have the following:*

(a) For each $1 \le i \le n, S + (x_i, 0)$ is an efficient open dominating set.

(b) $\{S + (x_1, 0), S + (x_2, 0), \cdots, S + (x_n, 0)\}$ is a vertex partition of $B\mathbb{C}(\Gamma, X)$.

Proof (a) Let $(v, 0) \in V(G)$. Consider the vertex $(x_i^{-1} * v, 0) \in V(G)$. Since S is an open dominating set, there exists $(s, 1) \in S$ such that $s = (x_i^{-1} * v) * x$ for some $x \in X$. That is $x_i * s = v * x$. Hence the vertex $(v, 0)$ is dominated by $(x_i * s, 1) \in S + (x_i, 0)$ and so $(v, 0) \in N(S+(x_i, 0))$. Similarly when $(v, 1) \in V(G)$, one can prove that $(v, 1) \in N(S+(x_i, 0))$. Thus $S+(x_i, 0)$ is an open dominating set for $1 \le i \le n$. Since S is an efficient open dominating set and $|S| = |S + (x_i, 0)|$, by Remark 3.1, $S + (x_i, 0)$ is an efficient open dominating set for $1 \le i \le n$.

(b) Since S is an open dominating set, for every $(u, 0) \in V(G)$ there exists $(s, 1) \in S$ such that $u = s * x_i$ for some $x_i \in X$. Similar thing is holds for $(u, 1) \in V(G)$. This means that $V(G) = (S + (x_1, 0)) \cup (S + (x_2, 0)) \cup \cdots \cup (S + (x_n, 0))$. Since G is $|X|-$regular and S is an efficient open dominating set, $|S| = \frac{2|\Gamma|}{|X|}$. That is $2|\Gamma| = |X|\ |S|$. Since $|S| = |S + (x_1, 0)| = |S+(x_2, 0)| = \cdots = |S+(x_n, 0)|$, one can conclude that $\{S, S+(x_1, 0), S+(x_2, 0), \cdots, S+(x_n, 0)\}$ is a vertex partition of G. □

From the proof of Lemma 3.2 and by Lemma 3.5, the following corollary follows:

Corollary 3.6 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let S be an efficient dominating set in $B\mathbb{C}(\Gamma, X)$. If $(x_i, 0) + S = S + (x_i, 0)$ for each $1 \le i \le n$, then there exists a covering $f : B\mathbb{C}(\Gamma, X) \to K_{n,n}$ such that $S + (x_1, 0), S + (x_2, 0), \cdots, S + (x_n, 0)$ are the fibers of $\{c_i, d_i\}$ under the map f.*

Theorem 3.7 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ, M be a normal subset of Γ and $S = (M \times 0) \cup (M \times 1)$. Then the following are equivalent.*

(a) S is an efficient open dominating set in $B\mathbb{C}(\Gamma, X)$.

(b) There exists a covering $f : B\mathbb{C}(\Gamma, X) \to K_{n,n}$ such that $f^{-1}(\{c_i, d_i\}) = S$ for some $1 \le i \le n$.

(c) $|S| = \frac{2|\Gamma|}{|X|}$ and $S \cap [S + (((X \times 0) + (X \times 0)) - \{(e, 0)\})] = \emptyset$.

Proof $(a) \Rightarrow (b)$: Proof follows from Corollary 3.6.

$(b) \Rightarrow (a)$: Since $\{c_i, d_i\}$ is an efficient open dominating set in $K_{n,n}$, the proof follows from Remark 3.3.

$(a) \Rightarrow (c)$: Since S is an efficient open and G is $|X|-$regular, the fact $|S| = \frac{2|\Gamma|}{|X|}$ follows from Remark 3.1. Suppose $S \cap [S + (((X \times 0) + (X \times 0)) - \{(e, 0)\})] \ne \emptyset$. Then there exist $(s, 0)$(or $(s, 1)$) $\in S$ such that $(s, 0) = (s_1, 0) + (x, 0) + (x_1, 0)$ with $x, x_1 \in X, x \ne x_1^{-1}$ and $(s_1, 0)$ (or $(s_1, 1)$) $\in S$. Since $x \ne x_1^{-1}$, we have $s \ne s_1$. Since $s = s_1 * x * x_1$, we have $s * x^{-1} = s_1 * x_1$ and so $(s_1 + x_1, 1)$ is adjacent with two vertices $(s, 0), (s_1, 0) \in S$, a contradiction to S is efficient open.

$(c) \Rightarrow (a)$: Let $x_i, x_j \in X$ with $x_i \ne x_j$. Suppose $(S + (x_i, 0)) \cap (S + (x_j, 0)) \ne \emptyset$. Let $a \in (S+(x_i, 0)) \cap (S+(x_j, 0))$. Then $a = (s_1, 0)+(x_i, 0) = (s_2, 0)+(x_j, 0)$ or $a = (s_1, 1)+(x_i, 0) = (s_2, 1) + (x_j, 0)$. Since $x_i * x_j^{-1} \ne e$, $(s_1, 0) \in S \cap [S + (((X \times 0) + (X \times 0)) - \{(e, 0)\})]$, a contradiction. Thus $\{S + (x_i, 0) : 1 \le i \le n\}$ is a collection of pairwise disjoint sets. Now $N(S) = \bigcup_{s\in S} N(s) = \bigcup_{s\in S} [s + (X \times 0)] = \bigcup_{(x,0)\in(X\times 0)} [S + (x, 0)]$. Since $|S + (x_i, 0)| = |S|$, $|N(S)| = |S||(X \times 0)| = |S||X| = |S|(\frac{2|\Gamma|}{|S|}) = 2|\Gamma|$. Thus S is an open dominating set. Since $|S| = \frac{2|\Gamma|}{|X|}$, one can conclude that S is an efficient open dominating set. □

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