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Directionally n-signed graphs-II

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Abstract: Let $G = (V, E)$ be a graph. Let V be a vector space of dimensional n. A *Smarandachely labeling* on a graph G is labeling an edge $uv \in E(G)$ by an vector $v \in V$ on (u, v) and $-v$ on (v, u). Then turn the conception *directional labeling* as a special case to Smarandachely labeling. By *directional labeling (or d-labeling)* of an edge $x = uv$ of G by an ordered n-tuple $(a_1, a_2, ..., a_n)$, we mean a labeling of the edge x such that we consider the label on uv as $(a_1, a_2, ..., a_n)$ in the direction from u to v, and the label on x as $(a_n, a_{n-1}, ..., a_1)$ in the direction from v to u. Here, we study graphs, called *(n, d)-sigraphs*, in which every edge is d-labeled by an n-tuple $(a_1, a_2, ..., a_n)$, where $a_k \in \{+, -\}$, for $1 \leq k \leq n$. In this paper, we obtain another characterization of i-balanced (n, d)-sigraphs, introduced the notion of path balance and generalized the notion of local balance in sigraphs to (n, d)-sigraphs. Further, we obtain characterization of path i-balanced (n, d)-sigraphs.

Key Words: Smarandachely labeling, sigraphs, directional labeling, complementation, balance.

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§1. Introduction

For graph theory terminology and notation in this paper we follow the book [1]. All graphs considered here are finite and simple.

Let V be a vector space of dimensional n. A *Smarandachely labeling* on a graph G is labeling an edge $uv \in E(G)$ by an vector $v \in V$ on (u, v) and $-v$ on (v, u). Then turn the

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conception *directional labeling* as a special case to Smarandachely labeling.

There are two ways of labeling the edges of a graph by an ordered n-tuple $(a_1, a_2, ..., a_n)$ (See [7]).

1. *Undirected labeling* or *labeling*. This is a labeling of each edge uv of G by an ordered n-tuple $(a_1, a_2, ..., a_n)$ such that we consider the label on uv as $(a_1, a_2, ..., a_n)$ irrespective of the direction from u to v or v to u.

2. *Directional labeling* or *d-labeling*. This is a labeling of each edge uv of G by an ordered n-tuple $(a_1, a_2, ..., a_n)$ such that we consider the label on uv as $(a_1, a_2, ..., a_n)$ in the direction from u to v, and $(a_n, a_{n-1}, ..., a_1)$ in the direction from v to u.

Note that the d-labeling of edges of G by ordered n-tuples is equivalent to labeling the symmetric digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, where uv is a symmetric arc in $\overrightarrow{G}$ if, and only if, uv is an edge in G, so that if $(a_1, a_2, ..., a_n)$ is the d-label on uv in G, then the labels on the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are $(a_1, a_2, ..., a_n)$ and $(a_n, a_{n-1}, ..., a_1)$ respectively.

Let H_n be the *n-fold sign group*, $H_n = \{+, -\}^n = \{(a_1, a_2, ..., a_n) : a_1, a_2, ..., a_n \in \{+, -\}\}$ with co-ordinate-wise multiplication. Thus, writing $a = (a_1, a_2, ..., a_n)$ and $t = (t_1, t_2, ..., t_n)$ then $at := (a_1t_1, a_2t_2, ..., a_nt_n)$. For any $t \in H_n$, the *action* of t on H_n is $a^t = at$, the co-ordinate-wise product.

Let $n \geq 1$ be a positive integer. An *n-sigraph (n-sidigraph)* is a graph $G = (V, E)$ in which each edge (arc) is labeled by an ordered n-tuple of signs, i.e., an element of H_n. A *sigraph* $G = (V, E)$ is a graph in which each edge is labeled by $+$ or $-$. Thus a 1-sigraph is a sigraph. Sigraphs are well studied in literature (See for example [2]-[4], [8]-[9]). In this paper, we study graphs in which each edge is labeled by an ordered n-tuple $a = (a_1, a_2, ..., a_n)$ of signs (i.e, an element of H_n) in one direction but in the other direction its label is the reverse: $a^r = (a_n, a_{n-1}, ..., a_1)$, called *directionally labeled n-signed graphs* (or *(n, d)-sigraphs*).

Note that an n-sigraph $G = (V, E)$ can be considered as a symmetric digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, where both $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are arcs if, and only if, uv is an edge in G. Further, if an edge uv in G is labeled by the n-tuple $(a_1, a_2, ..., a_n)$, then in $\overrightarrow{G}$ both the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are labeled by the n-tuple $(a_1, a_2, ..., a_n)$.

In [5,6], we have initiated a study of $(3, d)$ and $(4, d)$-Sigraphs. Also, we discuss some applications of $(3, d)$ and $(4, d)$-Sigraphs in real life situations.

In [7], we introduce the notion of complementation and generalize the notion of balance in sigraphs to the directionally n-signed graphs. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge. Also we gave some motivation to study (n, d)-sigraphs in connection with relations among human beings in society.

In this paper, we introduce the notion of path balance and we generalize the notion of local balance in sigraphs (a graph whose edges have signs) to the more general context of graphs with multiple signs on their edges.

In [7], we define complementation and isomorphism for (n, d)-sigraphs as follows: For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^t = at$. The *reversal* of $a = (a_1, a_2, ..., a_n)$ is: $a^r = (a_n, a_{n-1}, ..., a_1)$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of T is $T^t = \{a^t : a \in T\}$.

For any $t \in H_n$, the *t-complement* of an (n,d)-sigraph $G = (V, E)$, written G^t, is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^t. The *reversal* G^r is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^r.

Let $G = (V, E)$ and $G' = (V', E')$ be two (n,d)-sigraphs. Then G is said to be *isomorphic* to G' and we write $G \cong G'$, if there exists a bijection $\phi : V \to V'$ such that if uv is an edge in G which is d-labeled by $a = (a_1, a_2, ..., a_n)$, then $\phi(u)\phi(v)$ is an edge in G' which is d-labeled by a, and conversely.

For each $t \in H_n$, an (n,d)-sigraph $G = (V, E)$ is *t-self complementary*, if $G \cong G^t$. Further, G is *self reverse*, if $G \cong G^r$.

Proposition 1(E. Sampathkumar et al. [7]) *For all $t \in H_n$, an (n,d)-sigraph $G = (V, E)$ is t-self complementary if, and only if, G^a is t-self complementary, for any $a \in H_n$.*

Let $v_1, v_2, ..., v_m$ be a cycle C in G and $(a_{k1}, a_{k2}, ..., a_{kn})$ be the n-tuple on the edge $v_k v_{k+1}, 1 \le k \le m-1$, and $(a_{m1}, a_{m2}, ..., a_{mn})$ be the n-tuple on the edge $v_m v_1$.

For any cycle C in G, let $\mathcal{P}(\overrightarrow{C})$ denotes the product of the n-tuples on C given by $(a_{11}, a_{12}, ..., a_{1n})(a_{21}, a_{22}, ..., a_{2n})...(a_{m1}, a_{m2}, ..., a_{mn})$ and

$$\mathcal{P}(\overleftarrow{C}) = (a_{mn}, a_{m(n-1)}, ..., a_{m1})(a_{(m-1)n}, a_{(m-1)(n-1)}, ..., a_{(m-1)1})...(a_{1n}, a_{1(n-1)}, ..., a_{11}).$$

Similarly, for any path P in G, $\mathcal{P}(\overrightarrow{P})$ denotes the product of the n-tuples on P given by

$$(a_{11}, a_{12}, ..., a_{1n})(a_{21}, a_{22}, ..., a_{2n})...(a_{m-1,1}, a_{m-1,2}, ..., a_{m-1,n})$$

and

$$\mathcal{P}(\overleftarrow{P}) = (a_{(m-1)n}, a_{(m-1)(n-1)}, ..., a_{(m-1)1})...(a_{1n}, a_{1(n-1)}, ..., a_{11}).$$

An n-tuple $(a_1, a_2, ..., a_n)$ is *identity n-tuple*, if each $a_k = +$, for $1 \le k \le n$, otherwise it is a *non-identity n-tuple*. Further an n-tuple $a = (a_1, a_2, ..., a_n)$ is *symmetric*, if $a^r = a$, otherwise it is a *non-symmetric n-tuple*. In (n,d)-sigraph $G = (V, E)$ an edge labeled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Note that the above products $\mathcal{P}(\overrightarrow{C})$ $(\mathcal{P}(\overrightarrow{P}))$ as well as $\mathcal{P}(\overleftarrow{C})$ $(\mathcal{P}(\overleftarrow{P}))$ are n-tuples. In general, these two products need not be equal. However, the following holds.

Proposition 2 *For any cycle C (path P) of an (n,d)-sigraph $G = (V, E)$, $\mathcal{P}(\overleftarrow{C}) = \mathcal{P}(\overrightarrow{C})^r$ $(\mathcal{P}(\overleftarrow{P}) = \mathcal{P}(\overrightarrow{P})^r)$.*

Proof By the definition, we have

$\mathcal{P}(\overleftarrow{C})^r$
$= ((a_{mn}, a_{m(n-1)}, ..., a_{m1})(a_{(m-1)n}, a_{(m-1)(n-1)}, ..., a_{(m-1)1})...(a_{1n}, a_{1(n-1)}, ..., a_{11}))^r$
$= ((a_{m1}, a_{m2}, ...a_{mn})^r(a_{(m-1)1}, a_{(m-1)2}, ...a_{(m-1)n})^r...(a_{11}, a_{12}, ...a_{1n})^r)^r$
$= ((a_{m1}, a_{m2}, ...a_{mn})(a_{(m-1)1}, a_{(m-1)2}, ...a_{(m-1)n})...(a_{11}, a_{12}, ...a_{1n}))$
$= \mathcal{P}(\overrightarrow{C})$.

Similarly, we can prove $\mathcal{P}(\overleftarrow{P}) = \mathcal{P}(\overrightarrow{P})^r$. □

Corollary 2.1 *For any cycle C (path P), $\mathcal{P}(\overleftarrow{C}) = \mathcal{P}(\overrightarrow{C})$ ($\mathcal{P}(\overleftarrow{P}) = \mathcal{P}(\overrightarrow{P})$) if, and only if, $\mathcal{P}(\overrightarrow{C})$ ($\mathcal{P}(\overrightarrow{P})$) is a symmetric n-tuple. Furthermore, $\mathcal{P}(\overrightarrow{C})$ ($\mathcal{P}(\overrightarrow{P})$) is the identity n-tuple if, and only if, $\mathcal{P}(\overleftarrow{C})$ ($\mathcal{P}(\overleftarrow{P})$) is.*

For any subset Y of $\overrightarrow{E}(G) = \{(u, v) : \textit{uv is an edge in } G\}$, the set of all arcs in G, the product of the set Y is the product of the n-tuples of its arcs and it is denoted by $\mathcal{P}(Y)$. If Y_1 and Y_2 are disjoint sets, the product of the union of Y_1 and Y_2 is the product of the n-tuples of the two sets:

$$\mathcal{P}(Y_1 \cup Y_2) = \mathcal{P}(Y_1).\mathcal{P}(Y_2).$$

The following Proposition gives a similar result about the symmetric difference of two sets of arcs.

Proposition 3 *If Y_1 and Y_2 are two subsets of $\overrightarrow{E}(G)$ of an (n, d)-sigraph $G = (V, E)$, then $\mathcal{P}(Y_1 \oplus Y_2) = \mathcal{P}(Y_1).\mathcal{P}(Y_2)$.*

Proof We know that

$$Y_1 = (Y_1 - Y_2) \cup (Y_1 \cap Y_2) \text{ and } Y_2 = (Y_2 - Y_1) \cup (Y_1 \cap Y_2).$$

Since each of these is a union of disjoin sets, we have

$$\mathcal{P}(Y_1) = \mathcal{P}(Y_1 - Y_2).\mathcal{P}(Y_1 \cap Y_2) \text{ and } \mathcal{P}(Y_2) = \mathcal{P}(Y_2 - Y_1).\mathcal{P}(Y_1 \cap Y_2).$$

Multiplying these equations we get that

$$\mathcal{P}(Y_1).\mathcal{P}(Y_2) = \mathcal{P}(Y_1 - Y_2).\mathcal{P}(Y_2 - Y_1).\mathcal{P}(Y_1 \cap Y_2).\mathcal{P}(Y_1 \cap Y_2).$$

Since $\mathcal{P}(Y_1 \cap Y_2).\mathcal{P}(Y_1 \cap Y_2)$ is always identity n-tuple, and since $Y_1 - Y_2$ and $Y_2 - Y_1$ are disjoint,

$$\mathcal{P}(Y_1).\mathcal{P}(Y_2) = \mathcal{P}[(Y_1 - Y_2) \cup (Y_2 - Y_1)].$$

Thus, $\mathcal{P}(Y_1).\mathcal{P}(Y_2) = \mathcal{P}(Y_1 \oplus Y_2)$. □

Corollary 3.1 *Two sets of edges Y_1 and Y_2 have the same n-tuple if, and only if, their symmetric difference $Y_1 \oplus Y_2$ is identity.*

§2. Balance in an (n, d)-sigraph

In [7], we defined two notions of balance in an (n, d)-sigraph $G = (V, E)$ as follows.

Definition. *Let $G = (V, E)$ be an (n, d)-sigraph. Then,*

(i) G is identity balanced *(or* i-balanced*), if $P(\overrightarrow{C})$ on each cycle of G is the identity n-tuple, and*

(ii) G is balanced*, if every cycle contains an even number of non-identity edges.*

Note An i-balanced (n, d)-sigraph need not be balanced and conversely. For example, consider the $(4, d)$-sigraphs in Fig.1. In Fig.1(a) G is an i-balanced but not balanced, and in Fig.1(b) G is balanced but not i-balanced.

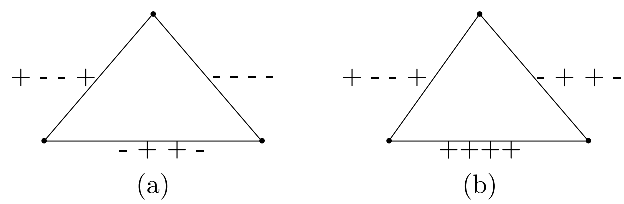

Fig.1

2.1 Criteria for Balance

An (n, d)-sigraph $G = (V, E)$ is i-balanced if each non-identity n-tuple appears an even number of times in $P(\overrightarrow{C})$ on any cycle of G.

However, the converse is not true. For example see Fig.2(a). In Fig.2(b), the number of non-identity 4-tuples is even and hence it is balanced. But it is not i-balanced, since the 4-tuple $(++--)$ (as well as $(--++)$) does not appear an even number of times in $P(\overrightarrow{C})$ of 4-tuples.

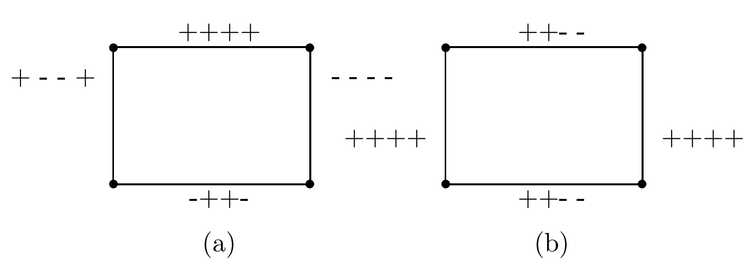

Fig.2

In [7], we obtained following characterizations of balanced and i-balanced (n, d)-sigraphs.

Proposition 4(E. Sampathkumar et al. [7]) *An (n, d)-sigraph $G = (V, E)$ is balanced if, and only if, there exists a partition $V_1 \cup V_2$ of V such that each identity edge joins two vertices in*

V_1 or V_2, and each non-identity edge joins a vertex of V_1 and a vertex of V_2.

As earlier we defined, let $P(C)$ denote the product of the n-tuples in $P(\overrightarrow{C})$ on any cycle C in an (n,d)-sigraph $G=(V,E)$.

Proposition 5(E. Sampathkumar et al. [7]) *An (n,d)-sigraph $G=(V,E)$ is i-balanced if, and only if, for each k, $1 \leq k \leq n$, the number of n-tuples in $P(C)$ whose k^{th} co-ordinate is $-$ is even.*

In H_n, let S_1 denote the set of non-identity symmetric n-tuples and S_2 denote the set of non-symmetric n-tuples. The product of all n-tuples in each $S_k, 1 \leq k \leq 2$ is the identity n-tuple.

Proposition 6(E. Sampathkumar et al. [7]) *An (n,d)-sigraph $G=(V,E)$ is i-balanced, if both of the following hold:*

(i) In $P(C)$, each n-tuple in S_1 occurs an even number of times, or each n-tuple in S_1 occurs odd number of times (the same parity, or equal mod 2).

(ii) In $P(C)$, each n-tuple in S_2 occurs an even number of times, or each n-tuple in S_2 occurs an odd number of times.

In this paper, we obtained another characterization of i-balanced (n,d)-sigraphs as follows:

Proposition 7 *An (n,d)-sigraph $G=(V,E)$ is i-balanced if, and only if, any two vertices u and v have the property that for any two edge distinct $u-v$ paths $\overrightarrow{P_1}=(u=u_0,u_1,...,u_m=v$ and $\overrightarrow{P_2}=(u=v_0,v_1,...,v_n=v)$ in G, $\mathcal{P}(\overrightarrow{P_1})=(\mathcal{P}(\overrightarrow{P_2}))^r$ and $\mathcal{P}(\overrightarrow{P_2})=(\mathcal{P}(\overrightarrow{P_1}))^r$.*

Proof Suppose that G is i-balanced. The paths $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$ may be combined to form is either a cycle or union of cycles. That is, $P_1 \cup P_2=(u=u_0,u_1,...,u_m=v=v_n,v_{n-1},...,v_0=u)$. Since $\mathcal{P}(\overrightarrow{P_1}\cup\overleftarrow{P_2})$=identity n-tuple e.

$$\mathcal{P}(\overrightarrow{P_1}).\ \mathcal{P}(\overleftarrow{P_2})=e, \quad \mathcal{P}(\overrightarrow{P_1})=\mathcal{P}(\overleftarrow{P_2})=(\mathcal{P}(\overrightarrow{P_2}))^r.$$

The converse is obvious. □

Corollary 7.1 *In an i-balanced (n,d)-sigraph G if two vertices are joined by at least 3 paths then the product of n tuples on any paths joining them must be symmetric.*

A graph $G=(V,E)$ is said to be k-connected for some positive integer k, if between any two vertices there exists at least k disjoint paths joining them.

Corollary 7.2 *If the underlying graph of an i-balanced (n,d)-sigraph is 3-connected, then all the edges in G must be labeled by a symmetric n-tuple.*

Corollary 7.3 *A complete (n,d)-sigraph on $p \geq 4$ is i-balanced then all the edges must be labeled by symmetric n-tuple.*

2.2 Complete (n, d)-sigraphs

An (n, d)-sigraph is *complete*, if its underlying graph is complete.

Proposition 8 *The four triangles constructed on four vertices $\{a, b, c, d\}$ can be directed so that given any pair of vertices say (a, b) the product of the edges of these 4 directed triangles is the product of the n-tuples on the arcs $\overrightarrow{ab}$ and $\overrightarrow{ba}$*

Proof The four triangles constructed on these vertices are (abc), (adb), (cad), (bcd). Consider the 4 directed triangles $(\overrightarrow{abc})$, $(\overrightarrow{adb})$, $(\overrightarrow{cad})$, $(\overrightarrow{bcd})$ for the pair ab. Then

$$\begin{aligned}
\mathcal{P} &= \mathcal{P}(\overrightarrow{abc}).\mathcal{P}(\overrightarrow{adb}).\mathcal{P}(\overrightarrow{acd}).\mathcal{P}(\overrightarrow{bcd}) \\
&= [\mathcal{P}(\overrightarrow{ab}).\mathcal{P}(\overrightarrow{ca}).\mathcal{P}(\overrightarrow{bc})].\ [\mathcal{P}(\overrightarrow{ad}).\mathcal{P}(\overrightarrow{db}).\mathcal{P}(\overrightarrow{ba})] \\
&\quad [\mathcal{P}(\overrightarrow{ca}).\mathcal{P}(\overrightarrow{ad}).\mathcal{P}(\overrightarrow{cd})][\mathcal{P}(\overrightarrow{bc}).\mathcal{P}(\overrightarrow{db}).\mathcal{P}(\overrightarrow{cd})] \\
&= [\mathcal{P}(\overrightarrow{ab}).\mathcal{P}(\overrightarrow{ba})].\ [\mathcal{P}(\overrightarrow{ca}).\mathcal{P}(\overrightarrow{ca})].\ [\mathcal{P}(\overrightarrow{bc}).\mathcal{P}(\overrightarrow{bc})] \\
&\quad [\mathcal{P}(\overrightarrow{ad}).\mathcal{P}(\overrightarrow{ad})].\ [\mathcal{P}(\overrightarrow{db}).\mathcal{P}(\overrightarrow{db})].\ [\mathcal{P}(\overrightarrow{cd}).\mathcal{P}(\overrightarrow{cd})] \\
&= \mathcal{P}(\overrightarrow{ab})\mathcal{P}(\overrightarrow{ba})
\end{aligned}$$

□

Corollary 8.1 *The product of the n-tuples of the four triangles constructed on four vertices $\{a, b, c, d\}$ is identity if at least one edge is labeled by a symmetric n-tuple.*

The *i-balance base* with axis a of a complete (n, d)-sigraph $G = (V, E)$ consists list of the product of the n-tuples on the triangles containing a.

Proposition 9 *If the i-balance base with axis a and n-tuple of an edge adjacent to a is known, the product of the n-tuples on all the triangles of G can be deduced from it.*

Proof Given a base with axis a and the n-tuple of the arc $\overrightarrow{ab}$ be $(a_1, a_2, \cdots, a_n)$.. Consider a triangle (bcd) whose n-tuple is not given by the base. Let $\mathcal{P}' = \mathcal{P}(\overrightarrow{abc}).\mathcal{P}(\overrightarrow{adb}).\mathcal{P}(\overrightarrow{acd})$. Hence, $\mathcal{P}'$ is known from the base with axis a. Let $\mathcal{P}$ be defined as in Proposition-8; we then have $\mathcal{P} = \mathcal{P}'.\ \mathcal{P}(\overrightarrow{bcd})$. By Proposition-8, $\mathcal{P} = \mathcal{P}(\overrightarrow{ab}).\mathcal{P}(\overrightarrow{ba})$. Thus, $\mathcal{P}(\overrightarrow{bcd}) = \mathcal{P}'.\mathcal{P}(\overrightarrow{ab}).\mathcal{P}(\overrightarrow{ba})$. □

Remark 10 In the statement of above Proposition, it is not necessary to know the n-tuple of an edge incident at a. But it is sufficient that an edge incident at a is a symmetric n-tuple.

Proposition 11 *A complete (n, d)-sigraph $G = (V, E)$ is i-balanced if, and only if, all the triangles of a base are identity.*

Proof If all the triangles of a base are identity, all the triangles of the (n, d)-sigraph are identity. Indeed, for any triangle (bed) not appearing in the base with axis a, we have

$$\mathcal{P}(\overrightarrow{bcd}) = \mathcal{P}(\overrightarrow{abc}).\ \mathcal{P}(\overrightarrow{abd}).\ \mathcal{P}(\overrightarrow{acd})\text{=identity.}$$

Conversely, if the (n, d)-sigraph is i-balanced, all these triangles are identity and particular those of a base. □

Corollary 11.1 *All the triangles of a complete (n,d)-sigraph $G = (V,E)$ are i-unbalanced if, and only if, all the triangles of a base are non-identity.*

Proposition 12 *The number of i-balanced complete (n,d)-sigraphs of m vertices is p^{m-1}, where $p = 2^{\lceil n/2 \rceil}$.*

Proof In a graph G of m vertices, there are $(m-1)$ edges containing a; each of these edges has $p = 2^{\lceil n/2 \rceil}$ possibilities,since each edge must be labelled by an symmetric n-tuple, by Corollary-7.3. Hence in all, p^{m-1} possibilities, where $p = 2^{\lceil n/2 \rceil}$. Starting from each of these possibilities, a base with axis a can be constructed, of which all the triangles are identity. □

§3. Path Balance in an (n,d)-sigraph

Definition *Let $G = (V,E)$ be an (n,d)-sigraph. Then G is*

1. Path i-balanced, if any two vertices u and v satisfy the property that for any $u-v$ paths P_1 and P_2 from u to v, $\mathcal{P}(\overrightarrow{P}_1) = \mathcal{P}(\overrightarrow{P}_2)$.

2. Path balanced *if any two vertices u and v satisfy the property that for any $u-v$ paths P_1 and P_2 from u to v have same number of non identity n-tuples.*

Clearly, the notion of path balance and balance coincides. That is an (n,d)-sigraph is balanced if, and only if, G is path balanced.

If an (n,d) sigraph G is i-balanced then G need not be path i-balanced and conversely.

The following result gives a characterization path i-balanced (n,d)-sigraphs.

Theorem 13 *An (n,d)-sigraph is path i-balanced if, and only if, any two vertices u and v satisfy the property that for any two vertex disjoint $u-v$ paths P_1 and P_2 from u to v, $\mathcal{P}(\overrightarrow{P}_1) = \mathcal{P}(\overrightarrow{P}_2)$.*

Proof Necessary: Suppose that G is path i-balanced. Then clearly for any two vertex disjoint paths P_1 and P_2 from one vertex to another, $\mathcal{P}(\overrightarrow{P}_1) = \mathcal{P}(\overrightarrow{P}_2)$.

Sufficiency: Suppose that for any two vertex disjoint paths P_1 and P_2 from one vertex to another, $\mathcal{P}(\overrightarrow{P}_1) = \mathcal{P}(\overrightarrow{P}_2)$ and that G is not path i-balanced. Let $S = \{(u,v) :$ there exists paths P and Q from u to v with $\mathcal{P}(\overrightarrow{P}) \neq \mathcal{P}(\overrightarrow{Q})\}$. Let $(u,v) \in S$ such that there exists paths P_1 and P_2 such that P_1 has length $d(u,v)$. Then by the hypothesis, the paths P_1 and P_2 must have a common point say w Let P_3 and P_4 be the subpaths from u to w and P_5 and P_6 be the subpaths from w to v. Now either $(u,w) \in S$ or $(w,v) \in S$. This gives a contradiction to the choice of u and v. This completes the proof. □

§4. Local Balance in an (n,d)-Signed Graph

The notion of local balance in signed graph was introduced by F. Harary [3]. A signed graph $S = (G,\sigma)$ is locally at a vertex v, or S is *balanced at* v, if all cycles containing v are balanced. A cut point in a connected graph G is a vertex whose removal results in a disconnected graph.

The following result due to Harary [3] gives interdependence of local balance and cut vertex of a signed graph.

Theorem 14(F. Harary [3]) *If a connected signed graph $S = (G, \sigma)$ is balanced at a vertex u. Let v be a vertex on a cycle C passing through u which is not a cut point, then S is balanced at v.*

We now extend the notion of local balance in signed graph to (n, d)-signed graphs.

Definition *Let $G = (V, E)$ be a (n, d)-sigraph. Then for any vertices $v \in V(G)$, G is* locally i-balanced at v *(*locally balanced at v*) if all cycles in G containing v is i-balanced (balanced).*

Analogous to the theorem we have the following for an (n, d) sigraph.

Theorem 15 *If a connected (n, d)-signed graph $G = (V, E)$ is locally i-balanced (locally balanced) at a vertex u and v be a vertex on a cycle C passing through u which is not a cut point, then S is locally i-balanced(locally balanced) at v.*

Proof Suppose that G is i-balanced at u and v be a vertex on a cycle C passing through u which is not a cut point. Assume that G is not i-balanced at v. Then there exists a cycle C_1 in G which is not i-balanced. Since G is balanced at u, the cycle C is i-balanced.

With out loss of generality we may assume that $u \notin C$ for if u is in C, then $\mathcal{P}(C)$ is identity, since G is i-balanced at u. Let $e = uw$ be an edge in C. Since v is not a cut point there exists a cycle C_0 containing e and v. Then C_0 consists of two paths P_1 and P_2 joining u and v.

Let v_1 be the first vertex in P_1 and v_2 be a vertex in P_2 such that $v_1 \neq v_2 \in C$, such points do exist since v is not a cut point and $v \in C$. Since $u, v \in C_0$. Let P_3 be the path on C_0 from v_1 and v_2, P_4 be a path in C containing v and P_5 is the path from v_1 to v_2. Then $P_5 \cup P_4$ and $P_3 \cup P_5$ are cycles containing u and hence are i-balanced, since they contain u. That is $\mathcal{P}(P_3) = (\mathcal{P}(P_5))^r$ so that $C = P_3 \cup P_5$ is i-balanced. This completes the proof. By using the same arguments we can prove the result for local balance. □

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