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Study on Hausdorff Supra Fuzzy Bitopological Space — Approach of Quasi-Coincidence

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Abstract: In this paper, we have defined some notions of Hausdorff (T_2) separation on supra fuzzy bitopological spaces in the sense of quasi-coincidence. We have found the relations among the notions. We have shown that these notions satisfy good extension, hereditary, productive and projective properties. We have also shown that Hausdorff supra fuzzy bitopological space is preserved under one-one, onto fuzzy pairwise open and continuous mappings. Finally, we have discussed initial and final supra fuzzy T_2 bitopological spaces.

Key Words: Supra fuzzy bitopological space, quasi-coincidence, supra fuzzy Hausdorff bitopological space, good extension, mappings, initial and final fuzzy bitopology.

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§1. Introduction

The first concept of fuzzy sets proposed by Zadeh [24] in 1965. By using this concept Chang [2] defined fuzzy topological spaces in 1968. The concept of bitopological spaces was introduced by J.C. Kelly [5]. A set equipped with two topologies is called a bitopological spaces. The supra topological spaces have been introduced by A.S. Mashhour at [10] in 1983. In topological space, the arbitrary union condition is enough to have a supra topological space. Here every fuzzy topological space is a supra fuzzy bitopological space but the converse is not always true. Separation axioms [4,11,12, 14] are important parts in fuzzy topological spaces. Also many researchers have contributed various types of separation axioms [6,13,15] on fuzzy bitopological space which is introduced by Kandil and El-Shafee [6] in 1991. Among those axioms, T_2- type separation on fuzzy bitopological space is one and it has been introduced earlier by Abu Sufiya et al. [22], Nouh [20], Amin et al. [1] and others. The purpose of this paper is to further contribute to the development of supra fuzzy Hausdorff bitopological spaces especially on supra fuzzy bitopological spaces. In this paper, we define Hausdorff supra fuzzy bitopological space and showed that the definitions satisfy good extension property, hereditary property, order preserving, productive and projective properties hold on the new concepts, initial and final supra fuzzy bitopologies are discussed also.

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§2. Basic Notions and Preliminary Results

For the purpose of the main results, we need to introduce some definitions and notions.

Definition 2.1([24]) *For a set X, a function $u : X \to [0,1]$ is called a fuzzy set in X. For every $x \in X$, $u(x)$ represents the grade of membership of x in the fuzzy set u. Some authors say that u is a fuzzy subset of X.*

Definition 2.2([17]) *A fuzzy set $u \in X$ is called fuzzy singleton if and only if $u(x) = r, 0 < r \leq 1$ for a certain $x \in X$ an $u(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in x is denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$ then we say that $x_r \in u$ if and only if $r \leq u(x)$.*

Definition 2.3([2]) *A fuzzy singleton x_r is said to be quasi-coincidence with u denoted by $x_r qu$ if and only if $u(x) + r > 1$. If x_r is not quasi-coincidence with u we write $x_r \bar{q} u$ and defined as $u(x) + r \leq 1$.*

Definition 2.4([2]) *Let X and Y be two sets and $f : X \to Y$ be a function. For a fuzzy subset v of Y, the inverse image of v under f is the fuzzy subset $f^{-1}(v) = v_0 f$ in X and is defined by*

$$f^{-1}(v)(x) = v(f(x)), \; for \, x \in X.$$

Definition 2.5([2]) *Let X be a non empty set and t be the collection of fuzzy sets in I^X. Then t is called a fuzzy topology on X if it satisfies the following conditions:*

(i) $1, 0 \in t$;
(ii) If $u_i \in t$ for each $i \in \Lambda$, then $\cup_{i \in \Lambda} u_i \in t$;
(iii) If $u_1, u_2 \in t$ then $u_1 \cap u_2 \in t$.

If t is a fuzzy topology on X, then the pair (X, t) is called a fuzzy topological space (fst, in short) and members of t are called t-open (or simply open) fuzzy sets. If u is open fuzzy set, then the fuzzy sets of the form 1-u are called t-closed (or simply closed) fuzzy sets.

Definition 2.6([10]) *Let X be a nonempty set. A subfamily t^* of I^X is said to be a supra fuzzy topology on X if and only if*

(i) $1, 0 \in t^*$;
(ii) If $u_i \in t^*$ for each $i \in \Lambda$, then $\cup_{i \in \Lambda} u_i \in t^*$,

Then the pair (X, t^*) is called a supra fuzzy topological spaces. The elements of t^* are called supra open sets in (X, t^*) and complement of supra open set is called supra closed set.

Definition 2.7([3]) *Let (X, t^*) and (Y, s^*) be two topological space. Let t^* and s^* are associated supra topological with t and s respectively and $f : (X, t^*) \to (Y, s^*)$ be a function. Then the function f is a supra fuzzy continuous if the inverse image of each i.e., if for any $v \in s^*, f^{-1}(v) \in t^*$. The function f is called supra fuzzy homeomorphic if and only if f is supra bijective and both f and f^{-1} are supra fuzzy continuous.*

Definition 2.8([3]) *The function $f : (X, t^*) \to (Y, s^*)$ is called supra fuzzy open if and only if for each supra open fuzzy set u in (X, t^*), f(u) is supra fuzzy set in (Y, s^*).*

Definition 2.9([3]) *The function $f : (X, t^*) \to (Y, s^*)$ is called supra fuzzy closed if and only if for each supra closed fuzzy set u in (X, t^*), f(u) is supra closed fuzzy set in (Y, s^*).*

Definition 2.10([8]) *Suppose $\{X_i, i \in \Lambda\}$, be any collection of sets and X denoted the Cartesian product of these sets, i.e., $X = \prod_{i\in\Lambda} X_i$. Here X consists of all points $p =< a_i, i \in \Lambda >$, where $a_i \in X_i$. For each $j_0 \in \Lambda$, we define the projection $\pi_{j_0} : X \longrightarrow X_{j_0}$ by $\pi_{j_0}(< a_i : i \in \Lambda >) = a_{j_0}$. These projection are used to defined the product supra topology.*

Definition 2.11([21]) *Let (X, t^*) be a topological space and t^* be associated supra topology with T. Then a function $f : X \longrightarrow I$ is lower semi continuous if and only if $\{x \in X : f(x) > \alpha\}$ is open for all $\alpha \in I$.*

Definition 2.12([3]) *Let (X, t^*) be a supra fuzzy topological space and t^* be associated supra topology with t. Then the lower semi continuous topology on X associated with t^* is $\omega(t^*) = \{\mu : X \to [0, 1], \mu\, is\, supra\, lsc\}$.*

Definition 2.13([17]) *The function $f : (X, t) \to (Y, s)$ is called fuzzy continuous if and only if for every $v \in s, f^{-1}(v) \in t$, the function f is called fuzzy homeomorphic if and only if f is bijective and both f and f^{-1} are fuzzy continuous.*

Definition 2.14([18]) *The function $f : (X, t) \to (Y, s)$ is called fuzzy open if and only if for every open fuzzy set u in $(X, t), f(u)$ is open fuzzy set in (Y, s).*

Definition 2.15([23]) *Let $\{(X_i, s_i, t_i) : i \in \Lambda\}$ is a family of fuzzy bitopological spaces. Then the space $(\prod X_i, \prod s_i, \prod t_i)$ is called the product fuzzy bitopological space of the family $\{(X_i, s_i, t_i) : i \in \Lambda\}$, where $\prod s_i$ and $\prod t_i$ denote the usual product fuzzy topologies of the families $\{\prod s_i : i \in \Lambda\}$ and $\{\prod t_i : i \in \Lambda\}$ of the fuzzy topologies respectively on X.*

A fuzzy bitopological property P is called productive if the product of fuzzy bitopological spaces of a family of fuzzy bitopological space, each having property P, has property P.

A fuzzy bitopological property P is called projective if for a family of fuzzy bitopological space $\{(X_i, s_i, t_i) : i \in \Lambda\}$, the product fuzzy bitopological space $(\prod X_i, \prod s_i, \prod t_i)$ has property P implies that each coordinate space has property P.

Definition 2.16([18]) *Let (X, T) be an ordinary topological space. The set of all lower semi continuous functions from (X, T) into the closed unit interval I equipped with the usual topology constitutive a fuzzy topology associated with (X, T) and is denoted by $(X, \omega(T))$.*

Definition 2.17([8]) *The initial fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i, t_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : X \to (X_i, t_i)\}_{i\in\Lambda}$ is the smallest fuzzy topology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in t_i\}_{i\in\Lambda}$.*

Definition 2.18([8]) *The final fuzzy topology on a set X for the family of fuzzy topological spaces*

$\{(X_i, t_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : (X_i, t_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy topology on X making each f_i fuzzy continuous.

Definition 2.19([19]) *A bijective mapping from an fts (X, t) to an fts (Y, s) preserves the value of a fuzzy singleton (fuzzy point).*

Note 2.1 The preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

§3. Definition and Properties of Supra Fuzzy T_2 Bi-Topological Spaces

We define our notions in Supra fuzzy T_2 bitopological spaces and show relations among the notions.

Definition 3.1 *A supra fuzzy bitopological space (X, s^*, t^*) is called*

(a) $SFPT_2(i)$ if and only if for any pair x_m, $y_n \in S(X)$ for distinct x and y, there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m q\mu, y_n q\eta$, and $\mu \cap \eta = 0$;

(b) $SFPT_2(ii)$ if and only if for any pair x_m, $y_n \in S(X)$ for distinct x and y, there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m q\mu$, $y_n q\eta$ and $\mu\bar{q}\eta$;

(c) $SFPT_2(iii)$ if and only if any pair x_m, $y_n \in S(X)$ for distinct x and y, there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m \in \mu, y_n \in \eta$ and $\mu\bar{q}\eta$;

(d) $SFPT_2(iv)$ if and only if any pair $x, y \in X$ for distinct x and y, there exist $\mu, \eta \in s^* \cup t^*$ such that $\mu(x) = 1, \eta(y) = 1$ and $\mu \cap \eta = 0$.

Lemma 3.1 *For a supra fuzzy bitopological space (X, s^*, t^*) the following implications are true:*

$$SFPT_2(i) \Rightarrow SFPT_2(ii),\ SFPT_2(iv) \Rightarrow SFPT_2(i),\ SFPT_2(iv) \Rightarrow SFPT_2(ii)$$

but in general, the converse is not true.

Proof For $SFPT_2(i) \Rightarrow SFPT_2(ii)$, let (X, s^*, t^*) be a supra fuzzy bitopological space and (X, s^*, t^*) is $SFPT_2(i)$. We have to prove that (X, s^*, t^*) is $SFPT_2(ii)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Since (X, s^*, t^*) is $SFPT_2(i)$ fuzzy bitopological space, we have, there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m q\mu, y_n q\eta$, and $\mu \cap \eta = 0$.

To prove (X, s^*, t^*) is $SFPT_2(ii)$, it is only needed to prove that $\mu\bar{q}\eta$. Now, $\mu \cap \eta = 0 \Rightarrow (\mu \cap \eta)(x) = 0 \Rightarrow \min(\mu(x), \eta(x)) = 0 \Rightarrow \mu(x) = 0\, or\, \eta(x) = 0 \Rightarrow \mu(x) + \eta(x) \leq 1 \Rightarrow \mu\bar{q}\eta$. It follows that there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m q\mu, y_n q\eta$, and $\mu \cap \eta = 0$. Hence it is clear that (X, s^*, t^*) is $SFPT_2(ii)$.

To show (X, s^*, t^*) is $SFPT_2(ii) \not\Longrightarrow (X, s^*, t^*)$ is $SFPT_2(i)$, we give a counterexample following.

Counterexample. Let $X = \{x, y\}$ and $\mu, \eta \in I^X$ be given by $\mu(x) = 1 - \varepsilon$, $\mu(y) = 1 - \frac{\varepsilon}{3}$ and $\eta(y) = 1 - \varepsilon$, $\eta(x) = \frac{\varepsilon}{3}$, where $\varepsilon = \frac{m}{3}$ for $m \in (0, 1]$. Consider the supra fuzzy topologies s^* and t^* on X generated by $\{0, \mu, \eta, 1\}$. Then, $\mu(x) = 1 - \frac{m}{3} \Rightarrow \mu(x) + \frac{m}{3} = 1 \Rightarrow \mu(x) + m > 1 \Rightarrow x_m q\mu$ also, $\eta(y) = 1 - \frac{m}{3} \Rightarrow \eta(y) + \frac{m}{3} = 1\ \eta(y) + m > 1 \Rightarrow y_m q\eta$ and, $\mu(x) + \eta(x) = 1 - \varepsilon + \frac{\varepsilon}{3}$

$\Rightarrow \mu(x)+\eta(x) = 1-\frac{\varepsilon}{3} \leq 1 \Rightarrow \mu(x)+\eta(x) \leq 1 \Rightarrow \mu\bar{q}\eta$. Hence, (X, s^*, t^*) is $SFPT_2(ii)$. But $\min(\mu(x), \eta(x)) \neq 0 \Rightarrow \mu \cap \eta \neq 0$ Thus, (X, s^*, t^*) is not $SFPT_2(i)$.

For $SFPT_2(iv) \Rightarrow SFPT_2(i)$, let (X, s^*, t^*) be a supra fuzzy bitopological space and (X, s^*, t^*) is $SFPT_2(i)$. We have to prove that (X, s^*, t^*) is $SFPT_2(i)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Since (X, s^*, t^*) is $SFPT_2(iv)$ fuzzy bitopological space, we have, there exist $\mu, \eta \in s^* \cup t^*$ such that $\mu(x) = 1, \eta(y) = 1$, and $\mu \cap \eta = 0$.

To prove (X, s^*, t^*) is $SFPT_2(i)$, it is only needed to prove that $x_m q\mu, y_n q\eta$. Now, $\mu(x) = 1 \Rightarrow \mu(x)+m > 1$, for any $m \in (0,1] \Rightarrow x_m q\mu$. Similarly, we can prove that $y_n q\eta$. It follows that there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m q\mu, y_n q\eta$ and $\mu \cap \eta = 0$. Hence it is clear that (X, s^*, t^*) is $SFPT_2(i)$.

To show (X, s^*, t^*) is $SFPT_2(i) \not\Longrightarrow (X, s^*, t^*)$ is $SFPT_2(iv)$, we give a counterexample following.

Counterexample. Let $X = \{x, y\}$ and $\mu, \eta \in I^X$ be given by $\mu(x) = 1-\varepsilon$, $\mu(y) = 0$ and $\eta(y) = 1-\varepsilon$, $\eta(x) = 0$, where $\varepsilon = \frac{m}{3}$ for $m \in (0,1]$. Consider the supra fuzzy topologies s^* and t^* on X generated by $\{0, \mu, \eta, 1\}$. Then, $\mu(x) = 1-\frac{m}{3} \Rightarrow \mu(x)+\frac{m}{3} = 1 \Rightarrow \mu(x)+m > 1 \Rightarrow x_m q\mu$. Similarly, we can prove that $y_n q\eta$. Also, $\min(\mu(x), \eta(x)) = 0 \Rightarrow \mu \cap \eta = 0$. Hence, (X, s^*, t^*) is $SFPT_2(i)$. But $\mu(x) \neq 1, \eta(y) \neq 1$ Thus, (X, s^*, t^*) is not $SFPT_2(iv)$.

For $SFPT_2(iv) \Rightarrow SFPT_2(ii)$, let (X, s^*, t^*) be a supra fuzzy bitopological space and (X, s^*, t^*) is $SFPT_2(iv)$. We have to prove that (X, s^*, t^*) is $SFPT_2(ii)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Since (X, s^*, t^*) is $SFPT_2(iv)$ fuzzy bitopological space, we have, there exist $\mu, \eta \in s^* \cup t^*$ such that $\mu(x) = 1, \eta(y) = 1$, and $\mu \cap \eta = 0$.

To prove (X, s^*, t^*) is $SFPT_2(ii)$, it is only needed to prove that $x_m q\mu, y_n q\eta$ and $\mu\bar{q}\eta$. Now, $\mu(x) = 1 \Rightarrow \mu(x)+m > 1$, for any $m \in (0,1] \Rightarrow x_m q\mu$. Similarly, we can prove that $y_n q\eta$. Now, $\mu \cap \eta = 0 \Rightarrow (\mu \cap \eta)(x) = 0 \Rightarrow \min(\mu(x), \eta(x)) = 0 \Rightarrow \mu(x) = 0 \, or \, \eta(x) = 0 \Rightarrow \mu(x)+\eta(x) \leq 1 \Rightarrow \mu\bar{q}\eta$. It follows that there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m q\mu, y_n q\eta$ and $\mu\bar{q}\eta$. Hence it is clear that (X, s^*, t^*) is $SFPT_2(ii)$.

To show (X, s^*, t^*) is $SFPT_2(ii) \not\Longrightarrow (X, s^*, t^*)$ is $SFPT_2(iv)$, we give a counterexample following.

Counterexample. Let $X = \{x, y\}$ and $\mu, \eta \in I^X$ be given by $\mu(x) = 1-\varepsilon$, $\mu(y) = \frac{\varepsilon}{3}$ and $\eta(y) = 1-\varepsilon$, $\eta(x) = \frac{\varepsilon}{3}$, where $\varepsilon = \frac{m}{3}$ for $m \in (0,1]$. Consider the supra fuzzy topologies s^* and t^* on X generated by $\{0, \mu, \eta, 1\}$. Then, $\mu(x) = 1-\frac{m}{3} \Rightarrow \mu(x)+\frac{m}{3} = 1 \Rightarrow \mu(x)+m > 1 \Rightarrow x_m q\mu$. Similarly, we can prove that $y_n q\eta$. Also, $\mu(x)+\eta(x) = 1-\varepsilon+\frac{\varepsilon}{3} \Rightarrow \mu(x)+\eta(x) = 1-\frac{\varepsilon}{3} \leq 1 \Rightarrow \mu(x)+\eta(x) \leq 1 \Rightarrow \mu\bar{q}\eta$. Hence, (X, s^*, t^*) is $SFPT_2(ii)$. But $\mu(x) \neq 1, \eta(y) \neq 1, min(\mu(x), \eta(x)) \neq 0 \Rightarrow \mu \cap \eta \neq 0$. Thus, (X, s^*, t^*) is not $SFPT_2(iv)$. These complete the proof. □

§4. **Good Extensions**

In this section, we shall show that our notions satisfy good extension property.

Theorem 4.1 *Let (X, S^*, T^*) be a supra bitopological space. Consider the following statements:*

(1) (X, S^*, T^*) be a T_2 supra bitopological space;
(2) $(X, \omega(S^*), \omega(T^*))$ be a $SFPT_2(i)$ bitopological space;
(3) $(X, \omega(S^*), \omega(T^*))$ be a $SFPT_2(ii)$ bitopological space;
(4) $(X, \omega(S^*), \omega(T^*))$ be a $SFPT_2(iii)$ bitopological space;
(5) $(X, \omega(S^*), \omega(T^*))$ be a $SFPT_2(iv)$ bitopological space.

The following implications are true:

$$(1) \Longleftrightarrow (2),\ (1) \Longleftrightarrow (3),\ (1) \Longleftrightarrow (4),\ (1) \Longleftrightarrow (5).$$

Proof For $(1) \Longrightarrow (2)$, let (X, S^*, T^*) be a supra bitopological space and (X, S^*, T^*) is T_2. We have to prove that $(X, \omega(S^*), \omega(T^*))$ is $SFPT_2(i)$. Let x_m, y_n be fuzzy singletons in X with distinct x, y. Since (X, S^*, T^*) is T_2 supra bitopological space we have, there exist $U, V \in S^* \cup T^*$ such that $x \in U$, $y \in V$ and $U \cap V = 0$. From the definition of lower semi continuous we have $1_U, 1_V \in \omega(S^*) \cup \omega(T^*)$ and $1_U(x) = 1$, $1_V(y) = 1$. Then $1_U(x) + m > 1 \Rightarrow x_m q 1_U$. Similarly, $\Rightarrow y_n \bar{q} 1_V$.

Also, $1_U \cap 1_V = 0$. If $1_U \cap 1_V \neq 0$, then there exists $z \in X$ such that $(1_U \cap 1_V)(z) \neq 0 \Rightarrow 1_U(z) \neq 0, 1_V(z) \neq 0 \Rightarrow U(z) = 1, V(z) = 1 \Rightarrow z \in U, z \in V \Rightarrow z \in U \cap V \Rightarrow U \cap V \neq \phi$, a contradiction. So, $1_U \cap 1_V = 0$. Hence, $(X, \omega(S^*), \omega(T^*))$ is $SFPT_2(i)$. Thus $(1) \Longrightarrow (2)$ holds.

For $(2) \Longrightarrow (1)$, let $(X, \omega(S^*), \omega(T^*))$ is $SFPT_2(i)$. We have to prove that (X, S^*, T^*) is T_2. Let x, y be distinct points in X. Since $(X, \omega(S^*), \omega(T^*))$ is $SFPT_2(i)$, we have, for any fuzzy singletons x_m, y_n in X, $\exists \mu, \eta \in \omega(S^*) \cup \omega(T^*)$ such that $x_m q \mu, y_n q \eta$ and $\mu \cap \eta = 0$. Now $x_m q \mu \Rightarrow \mu(x) + m > 1 \Rightarrow \mu(x) > 1 - m = \alpha \Rightarrow x \in \mu^{-1}(\alpha, 1]$ Similarly, $y \in \eta^{-1}(\alpha, 1]$. Also, $\mu^{-1}(\alpha, 1], \eta^{-1}(\alpha, 1) \in S^* \cup T^*$. Now, $\mu \cap \eta = 0 \Rightarrow \mu \cap \eta(z) = 0 \Rightarrow \min(\mu(z), \eta(z)) = 0$.

We claim that $\mu^{-1}(\alpha, 1] \cap \eta^{-1}(\alpha, 1] = \phi$. For, if $z \in \mu^{-1}(\alpha, 1] \cap \eta^{-1}(\alpha, 1]$, then $z \in \mu^{-1}(\alpha, 1]\, and\, z \in \eta^{-1}(\alpha, 1] \Rightarrow \mu(z) > \alpha\, and\, \eta(z) > \alpha \Rightarrow \min(\mu(z), \eta(z)) > \alpha$, a contradiction. Then $\mu^{-1}(\alpha, 1] \cap \eta^{-1}(\alpha, 1] = \phi$.

It follows that there exist $\mu^{-1}(\alpha, 1], \eta^{-1}(\alpha, 1] \in S^* \cup T^*$ such that $x \in \mu^{-1}(\alpha, 1], y \in \eta^{-1}(\alpha, 1]$ and $\mu^{-1}(\alpha, 1] \cap \eta^{-1}(\alpha, 1] = \phi$. Thus $(2) \Longrightarrow (1)$ holds.

Similarly, we can prove the other results. □

§5. Hereditary, Productivity and Projectivity in Supra Fuzzy T_2 Bitopological Spaces

In this section, we shall show that our notions satisfy hereditary, productive and projective properties.

Theorem 5.1 *Let (X, s^*, t^*) be a supra fuzzy bitopological space, $A \subseteq X$, $s^*_A = \{\mu/A : \mu \in s^* \cup t^*\}$, $t^*_A = \{\eta/A : \eta \in s^* \cup t^*\}$, then (X, s^*, t^*) is $SFPT_2(j) \Longrightarrow (A, s^*_A, t^*_A)$ is $SFPT_2(j)$; where $j = i, ii, iii, iv$.*

Proof Let (X, s^*, t^*) be a supra fuzzy bitopological space and (X, s^*, t^*) is $SFPT_2(i)$. We have to prove that (A, s^*_A, t^*_A) is $SFPT_2(i)$. Let x_m, y_n be fuzzy singletons in A for distinct

x and y. Since $A \subseteq X$, these fuzzy singletons are also fuzzy singletons in X. Also since (X, s^*, t^*) is $SFPT_2(i)$ supra fuzzy bitopological space we have, there exist $\mu, \eta \in s^* \cup t^*$ such that $x_m q\mu$, $y_n q\eta$ and $\mu \cap \eta = 0$. For $A \subseteq X$, we have $\mu/A, \eta/A \in s^*_A \cup t^*_A$. Now, $x_m q\mu \Rightarrow \mu(x) + m > 1$, $x \in X \Rightarrow (\mu/A)(x) + m > 1$, $x \in A \subseteq X \Rightarrow x_m q(\mu/A)$. and $y_n q\eta \Rightarrow \eta(y) + n > 1$, $y \in X \Rightarrow (\eta/A)(y) + n > 1$, $y \in A \subseteq X \Rightarrow y_n q(\eta/A)$.

Also,

$$\begin{aligned} \mu \cap \eta = 0 \quad &\Rightarrow \quad (\mu \cap \eta)(x) = 0,\ x \in X \Rightarrow \min(\mu(x), \eta(x)) = 0,\ x \in X \\ &\Rightarrow \quad \min(\mu/A(x), \eta/A(x)) = 0,\ x \in A \subseteq X \\ &\Rightarrow \quad ((\mu/A) \cap (\eta/A)(x)) = 0 \Rightarrow (\mu/A) \cap (\eta/A) = 0. \end{aligned}$$

It follows that there exist $\mu/A, \eta/A \in s^*_A \cup t^*_A$ such that $x_m q(\mu/A)$, $y_n q(\eta/A)$ and $(\mu/A) \cap (\eta/A) = 0$. Hence, (A, s^*_A, t^*_A) is $SFPT_2(i)$. The proof of others is of similar manner. □

Theorem 5.2 *Let (X_i, s^*_i, t^*_i), $i \in \Lambda$ be a supra fuzzy bitopological spaces and $(X = \prod_{i\in\Lambda} X_i, s^*, t^*)$ be the corresponding product bitopological space, then for all $i \in \Lambda$, (X_i, s^*_i, t^*_i) is $SFPT_2(j)$ if and only if (X, s^*, t^*) is $SFPT_2(j)$; where $j = i, ii, iii, iv$.*

Proof Let for all $i \in \Lambda$, (X_i, s^*_i, t^*_i) is $SFPT_2(iii)$ space. We have to prove that (X, s^*, t^*) is $SFPT_2(iii)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Then $(x_i)_m$, $(y_i)_n$ are fuzzy singletons for distinct x_i and y_i for some $i \in \Lambda$. Since (X_i, s^*_i, t^*_i) is $SFPT_2(iii)$, there exist $\mu_i, \eta_i \in s^*_i \cup t^*_i$ such that $(x_i)_m \in \mu_i$, $(y_i)_n \in \eta_i$ and $\mu_i \bar{q} \eta_i$. Now, $(x_i)_m \in \mu_i$, $(y_i)_n \in \eta_i$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$. Now, $(x_i)_m \in \mu_i \Rightarrow \mu(x_i) \geq m \Rightarrow \mu_i(\pi_i(x)) \geq m \Rightarrow (\mu_i \circ \pi_i)(x) \geq m \Rightarrow x_m \in (\mu_i \circ \pi_i)$. Similarly, we can prove that $(y_i)_n \in \mu_i$. Now, $\Rightarrow \mu_i(x_i) + \eta_i(x_i) \leq 1 \Rightarrow \mu_i(\pi_i(x)) + \eta_i(\pi_i(x)) \leq 1 \Rightarrow (\mu_i \circ \pi_i)(x) + (\eta_i \circ \pi_i)(x) \leq 1 \Rightarrow (\mu_i \circ \pi_i) \bar{q} (\eta_i \circ \pi_i)$. It follows that there exist $(\mu_i \circ \pi_i), (\eta_i \circ \pi_i) \in s^*_i \cup t^*_i$ such that $x_m \in (\mu_i \circ \pi_i)$, $y_n \in (\eta_i \circ \pi_i)$ and $(\mu_i \circ \pi_i) \bar{q} (\eta_i \circ \pi_i)$. Hence, (X, s^*, t^*) is $SFPT_2(iii)$.

Conversely, Let (X, s^*, t^*) be a supra fuzzy bitopological space and (X, s^*, t^*) is $SFPT_2(iii)$. We have to prove that (X_i, s^*_i, t^*_i), $i \in \Lambda$ is $SFPT_2(iii)$. Let a_i be a fixed element in X_i. Let

$$A_i = \{x \in X = \prod_{i\in\Lambda} X_i : x_j = a_j \, for \, some \, i \neq j\}$$

.

Then A_i is a subset of X, and hence $(A_i, s^*_{A_i}, t^*_{A_i})$ is subspace of (X, s^*, t^*). Since (X, s^*, t^*) is $SFPT_2(iii)$, so $(A_i, s^*_{A_i}, t^*_{A_i})$ is $SFPT_2(iii)$. Now we have A_i is homeomorphic image of X_i. Hence it is clear that for all $i \in \Lambda$, (X_i, s^*_i, t^*_i) is $SFPT_2(iii)$ space. Similarly, other results can be proved. □

§6. Mappings in Supra Fuzzy T_2 Bitopological Spaces

In this section, we shall show that our notions are preserved under one-one, onto, fuzzy open and fuzzy continuous mappings.

Theorem 6.1 *Let (X, s_1^*, t_1^*) and (Y, s_2^*, t_2^*) be two supra fuzzy bitopological spaces and $f : X \to Y$ be a one-one, onto and fuzzy open map then, (X, s_1^*, t_1^*) is $SFPT_2(j) \Longrightarrow (Y, s_2^*, t_2^*)$ is $SFPT_2(j)$; where $j = i, ii, iii, iv$.*

Proof Let (X, s_1^*, t_1^*) be a supra fuzzy bitopological space and (X, s_1^*, t_1^*) is $SFPT_2(i)$. We have to prove that (Y, s_2^*, t_2^*) is $SFPT_2(i)$. Let x'_m, y'_n be fuzzy singletons in Y for distinct x' and y'. Since f is onto then there exist $x, y \in X$ with $f(x) = x'$, $f(y) = y'$ and x_m, y_n are fuzzy singletons in X with $x \neq y$ as f in one-one. Again since (X, s_1^*, t_1^*) is $SFPT_2(i)$ space, there exist $\mu, \eta \in s_1^* \cup t_1^*$ such that $x_m q \mu$, $y_n q \eta$ and $\mu \cap \eta$. Now, $x_m q \mu \Rightarrow \mu(x) + m > 1$ and, $y_n q \eta \Rightarrow \mu(y) + n > 1$.

Now, $f(\mu)(x') = \{\sup \mu(x) : f(x) = x'\} \Rightarrow f(\mu)(x') = \mu(x)$, for some x and $f(\eta)(y') = \{\sup \eta(y) : f(y) = y'\} \Rightarrow f(\eta)(y') = \eta(y)$ for some y. Also since, f is open map then $f(\mu), f(\eta) \in s_2^* \cup t_2^*$ as $\mu, \eta \in s_1^* \cup t_1^*$.

Again, $\Rightarrow \mu(x)+m > 1 \Rightarrow f(\mu)(x')+m > 1 \Rightarrow x'_m q f(\mu), \Rightarrow \eta(x)+n > 1 \Rightarrow f(\eta)(x')+n > 1 \Rightarrow y'_n q f(\eta)$ and $\mu \cap \eta = 0$. Here, $f(\mu \cap \eta)(x') = \{\sup(\mu \cap \eta)(x) : f(x) = x'\}$, $f(\mu \cap \eta)(x') = 0$ and $f(\mu \cap \eta)(y') = \{\sup(\mu \cap \eta)(y) : f(y) = y'\}$. Therefore, $f(\mu \cap \eta) = 0 \Rightarrow f(\mu) \cap f(\eta) = 0$. It follows that there exist $f(\mu), f(\eta) \in s_2^* \cup t_2^*$ such that $x'_m q f(\mu)$, $y'_n q f(\eta)$ and $f(\mu) \cap f(\eta) = 0$. Hence it is clear that (Y, s_2^*, t_2^*) is $SFT_2(i)$ space. Similarly, we can prove the remaining. □

Theorem 6.2 *Let (X, s_1^*, t_1^*) and (Y, s_2^*, t_2^*) be two supra fuzzy bitopological spaces and $f : X \to Y$ be a one-one and fuzzy continuous map then, (Y, s_2^*, t_2^*) is $SFPT_2(j) \Longrightarrow (X, t^*)$ is $SFT_2(j)$; where $j = i, ii, iii, iv$.*

Proof Let (Y, s_2^*, t_2^*) be a supra fuzzy topological space and (Y, s_2^*, t_2^*) is $SFPT_2(j)$. We have to prove that (X, s_1^*, t_1^*) is $SFPT_2(j)$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Then $(f(x))_m$, $(f(y))_n$ are fuzzy singletons in Y with $f(x) \neq f(y)$ as f is one-one. Again since, (Y, s_2^*, t_2^*) is $SFPT_2(j)$ space, there exist $\mu, \eta \in s_2^* \cup t_2^*$ such that $(f(x))_m \in \mu$, $(f(y))_n \in \eta$ and $\mu \bar{q} \eta$. Now, $(f(x))_m \in \mu \Rightarrow \mu(f(x)) \geq m \Rightarrow f^{-1}(\mu(x)) + m > 1 \Rightarrow (f^{-1}(\mu))(x) + m > 1 \Rightarrow x_m \in f^{-1}(\mu)$ and, $(f(x))_n \in \eta \Rightarrow \eta(f(x)) \geq n \Rightarrow f^{-1}(\eta(x)) \geq n \Rightarrow (f^{-1}(\eta))(x) \geq n \Rightarrow y_n \in f^{-1}(\eta)$. Also, $\mu \bar{q} \eta \Rightarrow u_1(f(x)) + u_2(f(x)) \leq 1 \Rightarrow (f^{-1}(\eta))(x) + (f^{-1}(\eta))(x) \leq 1 \Rightarrow (f^{-1}(\mu) \bar{q} (f^{-1}(\eta))$. Now, since, f is continuous map and $\mu, \eta \in s_2^* \cup t_2^*$ then $f^{-1}(\mu), f^{-1}(\eta) \in s_1^* \cup t_1^*$. It follows that there exist $f^{-1}(\mu), f^{-1}(\eta) \in s_1^* \cup t_1^*$ such that $x_m \in f^{-1}(\mu)$, $y_n \in f^{-1}(\eta)$ and $f^{-1}(\mu) \bar{q} f^{-1}(\eta)$. Hence, (X, s_1^*, t_1^*) is $SFPT_2(iii)$ space. The proofs of others are of similar procedure. □

§7. Initial and Final Supra Fuzzy T_2 Bitopological Spaces

We discuss the initial and final fuzzy bitopologies in this section.

Definition 7.1([16]) *The initial fuzzy bitopology on a set X for the family of fuzzy bitopological spaces $\{(X_i, s_i, t_i)\}_{i \in \Lambda}$ and the family of functions $\{f_i : X \to (X_i, s_i \cup t_i)\}_{i \in \Lambda}$ is the smallest fuzzy bitopology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in s_i \cup t_i\}_{i \in \Lambda}$.*

Definition 7.2([16]) *The final fuzzy bitopology on a set X for the family of fuzzy bitopological spaces $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ and the family of functions $\{f_i : (X_i, s_i \cup t_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy bitopology on X making each f_i fuzzy continuous.*

Theorem 7.1 *If $\{(X_i, s_i^*, t_i^*)\}_{i\in\Lambda}$ is a family of $SFPT_2(j)$ and $\{f_i : X \to (X_i, s_i^* \cup t_i^*)\}_{i\in\Lambda}$, a family of one-one and fuzzy continuous functions, then the initial supra fuzzy bitopology on X for the family $\{f_i\}_{i\in\Lambda}$ is $SFPT_2(j)$ for $j = i, ii, iii, iv$.*

Proof We shall prove the theorem for $j = i, ii$ and the remaining is similar. Let s^*, t^* be the initial supra fuzzy topologies on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Then $f_i(x)$, $f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as f_i is one-one. Since (X_i, s_i^*, t_i^*) is $SFPT_2(i)$, then for every two distinct fuzzy singletons $(f_i(x))_m$, $(f_i(y))_n$ in X_i, there exist fuzzy sets $\mu_i\, \eta_i \in s_i^* \cup t_i^*$ such that $(f_i(x))_m q\mu_i$, $(f_i(y))_n q\eta_i \, and \, \mu_i \cap \eta_i = 0$. Now,

$$(f_i(x))_m q\mu_i \quad \text{and} \quad (f_i(y))_n q\eta_i,$$

i.e.,

$$\mu_i(f_i(x)) + m > 1 \quad \text{and} \quad \eta_i(f_i(y)) + n > 1.$$

That is

$$f_i^{-1}(\mu_i)(x) + m > 1 \quad \text{and} \quad f_i^{-1}(\eta_i)(y) + n > 1.$$

Also,

$$\mu_i \cap \eta_i \Rightarrow \mu_i(f_i(x)) + \eta_i(f_i(x)) \leq 1 \Rightarrow f_i^{-1}(\mu_i)(x) + f_i^{-1}(\eta_i)(x) \leq 1.$$

This is true for every $i \in \Lambda$. So,

$$\inf f_i^{-1}(\mu_i)(x) + m > 1, \quad \inf f_i^{-1}(\eta_i)(y) + n > 1 \quad \text{and} \quad \inf f_i^{-1}(\mu_i)(x) + \inf f_i^{-1}(\eta_i)(x) \leq 1.$$

Let $\mu = \inf f_i^{-1}(\mu_i)$ and $\eta = \inf f_i^{-1}(\eta_i)$. Then $\mu, \eta \in s^* \cup t^*$ as f_i is fuzzy continuous. So,

$$\mu(x) + m > 1, \quad \eta(y) + n > 1 \quad \text{and} \quad \mu(x) + \eta(x) \leq 1.$$

Hence, $x_m q\mu, y_n q\eta$ and $\mu \cap \eta = 0$. Therefore, (X, s^*, t^*) is $SFPT_2(i)$.

Again, Since (X_i, s_i^*, t_i^*) is $SFPT_2(ii)$, then for every two distinct fuzzy singletons $(f_i(x))_m$, $(f_i(y))_n$ in X_i, there exist fuzzy sets $\mu_i, \eta_i \in s_i^* \cup t_i^*$ such that

$$(f_i(x))_m q\mu_i,\, (f_i(y))_n q\eta_i \quad \text{and} \quad \mu_i \bar{q} \eta_i.$$

Now,

$$(f_i(x))_m q\mu_i \quad \text{and} \quad (f_i(y))_n q\eta_i,$$

i.e.,

$$\mu_i(f_i(x)) + m > 1 \quad \text{and} \quad \eta_i(f_i(y)) + n > 1.$$

That is

$$f_i^{-1}(\mu_i)(x) + m > 1 \quad \text{and} \quad f_i^{-1}(\eta_i)(y) + n > 1.$$

Also,

$$\mu_i \bar{q} \eta_i \Rightarrow \mu_i(f_i(x)) + \eta_i(f_i(x)) \leq 1 \Rightarrow f_i^{-1}(\mu_i)(x) + f_i^{-1}(\eta_i)(x) \leq 1.$$

This is true for every $i \in \Lambda$. So,

$$\inf f_i^{-1}(\mu_i)(x) + m > 1, \quad \inf f_i^{-1}(\eta_i)(y) + n > 1 \quad \text{and} \quad \inf f_i^{-1}(\mu_i)(x) + \inf f_i^{-1}(\eta_i)(x) \leq 1.$$

Let $\mu = \inf f_i^{-1}(\mu_i)$ and $\eta = \inf f_i^{-1}(\eta_i)$. Then $\mu, \eta \in s^* \cup t^*$ as f_i is fuzzy continuous. So,

$$\mu(x) + m > 1, \quad \eta(y) + n > 1 \quad \text{and} \quad \mu(x) + \eta(x) \leq 1.$$

Hence, $x_m q \mu, y_n q \eta$ and $\mu \bar{q} \eta$. Therefore, (X, s^*, t^*) is $SFPT_2(ii)$. □

Theorem 7.2 *If $\{(X_i, s_i^*, t_i^*)\}_{i \in \Lambda}$ is a family of $SFPT_2(j)$ and $\{f_i : X \to (X_i, s_i^* \cup t_i^*)\}_{i \in \Lambda}$, a family of fuzzy open and bijective functions, then the final supra fuzzy bitopology on X for the family $\{f_i\}_{i \in \Lambda}$ is $SFPT_2(j)$ for $j = i, ii, iii, iv$.*

Proof We shall prove the theorem for $j = ii$ and the remaining is similar. Let s^*, t^* be the final supra fuzzy topologies on X for the family $\{f_i\}_{i \in \Lambda}$. Let x_m, y_n be fuzzy singletons in X for distinct x and y. Then $f_i^{-1}(x)$, $f_i^{-1}(y) \in X_i$ and $f_i^{-1}(x) \neq f_i^{-1}(y)$ as f_i is bijective. Since (X_i, s_i^*, t_i^*) is $SFPT_2(ii)$, then for every two distinct fuzzy singletons $(f_i^{-1}(x))_m$, $(f_i^{-1}(y))_n$ in X_i, there exist fuzzy sets $\mu_i, \eta_i \in s_i^* \cup t_i^*$ such that $(f_i^{-1}(x))_m q \mu_i$, $(f_i^{-1}(y))_n q \eta_i \, and \, \mu_i \bar{q} \eta_i$.

Now,

$$(f_i^{-1}(x))_m q \mu_i \quad \text{and} \quad (f_i^{-1}(y))_n q \eta_i,$$

i.e.,,

$$\mu_i(f_i^{-1}(x)) + m > 1 \quad \text{and} \quad \eta_i(f_i^{-1}(y)) + n > 1.$$

That is,

$$f_i(\mu_i)(x) + m > 1 \quad \text{and} \quad f_i(\eta_i)(y) + n > 1.$$

Also,

$$\mu_i \bar{q} \eta_i \Rightarrow \mu_i(f_i^{-1}(x)) + \eta_i(f_i^{-1}(x)) \leq 1 \Rightarrow (f_i(\mu_i(x) + f_i(\eta_i)(x) \leq 1.$$

This is true for every $i \in \Lambda$. So,

$$\inf f_i(\mu_i)(x) + m > 1, \quad \inf f_i(\eta_i)(y) + n > 1 \quad \text{and} \quad \inf f_i(\mu_i)(x) + \inf f_i(\eta_i)(x) \leq 1.$$

Let $\mu = \inf f_i(\mu_i)$ and $\eta = \inf f_i(\eta_i)$. Then $\mu, \eta \in s^* \cup t^*$ as f_i is fuzzy open. So,

$$\mu(x) + m > 1, \eta(y) + n > 1 \quad \text{and} \quad \mu(x) + \eta(x) \leq 1.$$

Hence, $x_m q \mu, y_n q \eta$ and $\mu \bar{q} \eta$. Therefore, (X, s^*, t^*) is $SPFT_2(ii)$. □

§8. Conclusion

The main result of this paper is introducing some new concepts of supra fuzzy T_2 bitopological spaces. We discuss some features of these concepts and present the hereditary, productive and projective properties. Also, we have observed that these notions are preserved under one-one, onto, supra fuzzy open and supra fuzzy continuous mappings. We think that this research work will contribute to the development of the field of modern mathematics. Initial and final bitopologies introduced in $SFPT_2$ spaces are interesting.

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