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Generalised Sasakian-Space-Form in Submanifolds

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Abstract: In this paper, we obtain necessary and sufficient condition for an invariant submanifold of generalised sasakian space form with semi-symmetric metric connections to be totally geodesic.

Key Words: Invariant submanifolds, generalised sasakian space form, totally geodesic, semi-symmetric metric connection.

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§1. Introduction

In differential geometry, Invariant submanifolds (I.S.M.) of a contact manifold have been a major area of research for long time since the concept was borrowed from complex geometry. A submanifold of a contact manifold is said to be totally geodesic if every geodesic in that submanifold is also geodesic in the ambient manifold. The generalised Sasakian space forms (G.S.S.F.) have been investigated by numerous researchers like Alegre and Carriazo [1], [2], [3]. Thereafter, (G.S.S.F.) have been study by many authors [4], [9], [10], [14], [16], [19]. The conception of a semi-symmetric metric connection(S.S.M.C.) on a Riemannian manifold is introduced by H. A. Hayden [15] and studied by various authors [17], [18], [33] and [34]. Submanifolds of a Riemannian manifold with S.S.M.C. was studied by Z. Nakao [22] and I.S.M. which was established by B. Y. Chen [11], [12] and [13].

In this paper, we procure essential and competent condition for an I.S.M. of G.S.S.F with S.S.M.C.to be totally geodesic. We have considered many geometrical conditions by using

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different curvature tensors such as concircular, Weyl and Conformal curvature tensor on I.S.M. of G.S.S.F. with S.S.M.C.

An almost contact metric manifold $\overline{M}$ is called G.S.S.F if

$$\begin{aligned}\overline{R}(X,Y)Z =& f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ &- g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\end{aligned} \tag{1.1}$$

for all vector fields X, Y, Z on $\overline{M}$, where $\overline{R}$ is the curvature tensor of $\overline{M}$ of dimension $(2n+1)$. It is indicated as

$$\overline{M}^{2n+1}(f_1, f_2, f_3), \quad f_1 = \frac{c+3}{4}, \quad f_2 = f_3 = \frac{c-1}{4}.$$

For readers who are unfamiliar with terminology, notations, recent overviews and introductions, we suggest the auhthors to refer the papers [5, 6, 7, 8, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

§2. Preliminaries

Let $(\overline{M})$ be a $(2n+1)$ dimensional manifold equipped with almost contact metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\eta(\xi) = 1,\ \eta(X) = g(X,\xi),\ \phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) + g(X, \phi Y) = 0,\ \eta(\phi X) = 0, \tag{2.3}$$

for all vector fields X, Y.

In a G.S.S.F $\overline{M}^{2n+1}(f1_1, f_2, f_3)$, the following hold:

$$(\overline{\nabla}_X \phi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.4}$$

$$\overline{\nabla}_X \xi = -(f_1 - f_3)\phi X, \tag{2.5}$$

$$\overline{S}(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y) \tag{2.6}$$

for all X, Y, Z on $\overline{M}^{2n+1}$ and $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$ and $\overline{S}$ is the Ricci tensor and $\overline{r}$ is the scalar curvature of $\overline{M}$.

Let M be a submanifold immersed in a $(2n+1)$ dimensional contact metric manifold $\overline{M}$ induced with metric g. TM is the tangent bundle of the manifold M and $T^{\perp}M$ is the set of vector fields normal to M.

Gauss and Weingarten formula are given by,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X N = \nabla^{\perp}_X N - A_N X, \tag{2.7}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in the normal bundle. The second fundamental form h and A_N are related by

$$g(A_N X, Y) = g(h(X,Y), N) \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$, $N \in T^{\perp}M$.

If $h = 0$, then the submanifold is said to be totally geodesic , which implies that the geodesics in M are also geodesics in $\overline{M}$. Also, we indicate $Q(E,T)$ a $(0, k+2)$-type tensor field interpret as follows

$$\begin{aligned} Q(E,T)(X_1, X_2, \ldots, X_k; X, Y) &= -T((X \wedge_E Y)X_1, X_2, \ldots, X_k) \\ - T(X_1, (X \wedge_E Y)X_2, \cdots, X_k) &- \cdots - T(X_1, X_2, \cdots, X_{k-1}, (X \wedge_E Y)X_k), \end{aligned} \tag{2.9}$$

where $(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y$.

A submanifold is said to be pseudo-parallel if

$$\overline{R}(X,Y) \cdot h = fQ(g,h). \tag{2.10}$$

In an (I.S.M.) of a (G.S.S.F.) N is identically zero. We have

$$h(X,\xi) = 0 \tag{2.11}$$

for any vector field X tangent to M. In a $(2n+1)$ dimensional Riemannian manifold, The concircular curvature tensor $\overline{C}$, Weyl curvature tensor $\overline{W}$ and Conformal curvature tensor $\overline{V}$ are given by,

$$\overline{C}(X,Y)Z = \overline{R}(X,Y)Z - \left(\frac{\overline{r}}{2n(2n+1)}\right)[g(Y,Z)X - g(X,Z)Y], \tag{2.12}$$

$$\overline{W}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{2n}[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y], \tag{2.13}$$

$$\begin{aligned} \overline{V}(X,Y)Z =& \overline{R}(X,Y)Z - \frac{1}{2n-1}[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y + g(Y,Z)\overline{Q}X \\ &- g(X,Z)\overline{Q}Y] + \frac{\overline{r}}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.14}$$

A semi-symmetric connection $\tilde{\overline{\nabla}}$ is called S.S.M.C. if it satisfies $\tilde{\overline{\nabla}} g = 0$.

The connection among the S.S.M.C. $\tilde{\overline{\nabla}}$ and the Riemannian connection $\overline{\nabla}$ of a G.S.S.F. $\overline{M}^{2n+1}(f_1, f_2, f_3)$ is given by

$$\tilde{\overline{\nabla}}_X Y = \overline{\nabla}_X Y + \eta(Y)X - g(X,Y)\xi. \tag{2.15}$$

If $\overline{R}$ and $\tilde{\overline{R}}$ are the Riemannian Curvature tensor of G.S.S.F. $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to Levi-civita connection and S.S.M.C. , then

$$\tilde{\overline{R}}(X,Y)Z = \overline{R}(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y + g(Y,Z)JX + g(X,Z)JY, \tag{2.16}$$

where α is a $(0,2)$ tensor field given by,

$$\alpha(X,Y) = (\tilde{\overline{\nabla}}_X \eta)Y + \frac{1}{2}g(X,Y), \tag{2.17}$$

$$g(JX,Y) = g(\tilde{\overline{\nabla}}_X \xi, Y) + \frac{1}{2}g(X,Y) = \alpha(X,Y), \tag{2.18}$$

$$\tilde{\overline{S}}(X,Y) = \overline{S}(X,Y) - (2n-1)\alpha(X,Y) - cg(X,Y), \tag{2.19}$$

where $c = trace(\alpha)$, $\tilde{\overline{S}}$, $\tilde{\overline{r}}$ and $\overline{S}$,$\overline{r}$ are the Ricci tensor and scalar curvature with respect to S.S.M.C. $\tilde{\overline{\nabla}}$ and $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to Levi-civita connection respectively.

§3. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{C}(X,Y) \cdot h = fQ(g,h)$

Theorem 3.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric metric connection. Then M satisfies $\overline{C}(X,Y) \cdot h = fQ(g,h)$ iff. M is totally geodesic provided*

$$f \neq \left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \left(\frac{r}{2n(2n+1)}\right)\right]. \tag{3.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric metric connection satisfying

$$\overline{C}(X,Y) \cdot h = fQ(g,h), \tag{3.2}$$

Notice that (3.2) can be written as,

$$R^{\perp}(X,Y)h(U,V) - h(\overline{C}(X,Y)U,V) - h(U,\overline{C}(X,Y)V) = -f[h((X \wedge_g Y),V) + h(U,(X \wedge_g Y)V)]. \tag{3.3}$$

Using (3.3) and also putting $X = V = \xi$, we get,

$$R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{C}(\xi,Y)U,\xi) - h(U,\overline{C}(\xi,Y)\xi) = -f[g(Y,U)h(\xi,\xi) - g(\xi,U)h(Y,\xi) + g(Y,\xi)h(U,\xi) - g(\xi,\xi)h(U,Y)]. \tag{3.4}$$

Applying (2.11) in (3.4), we acquire,

$$-h(\overline{C}(\xi,Y)U,\xi) - h(U,\overline{C}(\xi,Y)\xi) = f[h(U,Y)]. \tag{3.5}$$

By virtue of (2.12), (2.11), (2.15), (2.16), (2.17) (2.18) and (2.19), we obtain

$$h(\overline{C}(\xi, Y)U, \xi) = 0 \tag{3.6}$$

and

$$-h(U, \overline{C}(\xi, Y)\xi) = \Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \Big(\frac{r}{2n(2n+1)}\Big)\Big] h(U, Y). \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.5) we get

$$\Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \Big(\frac{r}{2n(2n+1)}\Big)\Big] h(U, Y) = f[h(U, Y)]. \tag{3.8}$$

That is, $h(U, Y) = 0$ implies M is totally geodesic provided,

$$f \neq \Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \Big(\frac{r}{2n(2n+1)}\Big)\Big]. \tag{3.9}$$

Conversely, If M is totally geodesic, then we obtain M fulfilling $\overline{C}(X, Y) \cdot h = fQ(g, h)$. This completes the proof. □

§4. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{W}(X, Y) \cdot h = fQ(g, h)$

Theorem 4.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{W}(X, Y) \cdot h = fQ(g, h)$ iff M is totally geodesic, provided,*

$$f \neq \Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{1}{2n}\Big(2n(f_1 - f_3) - c - n + \frac{1}{2}\Big)\Big]. \tag{4.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{W}(X, Y) \cdot h = fQ(g, h), \tag{4.2}$$

Notice that (3.11) which follows as,

$$R^{\perp}(X, Y)h(U, V) - h(\overline{W}(X, Y)U, V) - h(U, \overline{W}(X, Y)V = -f[h((X \wedge_g Y), V) + h(U, (X \wedge_g Y)V)]. \tag{4.3}$$

Taking $X = V = \xi$ and using (2.9) we obtain,

$$\begin{aligned} &R^{\perp}(\xi, Y)h(U, \xi) - h(\overline{W}(\xi, Y)U, \xi) - h(U, \overline{W}(\xi, Y)\xi) \\ &= -f[g(Y, U)h(\xi, \xi) - g(\xi, U)h(Y, \xi) + g(Y, \xi)h(U, \xi) - g(\xi, \xi)h(U, Y)]. \end{aligned} \tag{4.4}$$

Putting (2.11) in (4.4) we get,

$$-h(\overline{W}(\xi, Y)U, \xi) - h(U, \overline{W}(\xi, Y)\xi) = f[h(U, Y)]. \tag{4.5}$$

By virtue of (2.13),(2.11), (2.15), (2.16), (2.17), (2.18) and (2.19), we get

$$h(\overline{W}(\xi, Y)U, \xi) = 0 \tag{4.6}$$

and

$$\begin{aligned} -h(U, \overline{W}(\xi, Y)\xi) = &\Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} \\ &- \frac{1}{2n}\Big(2n(f_1 - f_3) - c - n + \frac{1}{2}\Big)\Big]h(U, Y). \end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.5) we get,

$$\begin{aligned} &\Big(f_1 - f_3 - \frac{1}{2}\Big)\phi(f_1 - f_3) - \frac{3}{2} - \frac{1}{2n}(2n(f_1 - f_3) - c - n \\ &+ \frac{1}{2})h(U, Y) = f[h(U, Y). \end{aligned} \tag{4.8}$$

That is, $h(U, Y) = 0$ implies M is totally geodesic provided,

$$f \neq \Big[\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{1}{2n}\Big(2n(f_1 - f_3) - c - n + \frac{1}{2}\Big)\Big]. \tag{4.9}$$

Conversely, If M is totally geodesic, then we get M satisfies $\overline{W}(X, Y) \cdot h = fQ(g, h)$. This completes the proof. □

§5. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{V}(X, Y) \cdot h = fQ(g, h)$

Theorem 5.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then, M satisfies $\overline{V}(X, Y) \cdot h = fQ(g, h)$ iff M is totally geodesic, provided,*

$$\begin{aligned} f \neq &\Big(f_1 - f_3 - \frac{1}{2}\Big) + \phi(f_1 - f_3) - \frac{3}{2} \\ &- \frac{2}{2n-1}\Big(2n(f_1 - f_3) - c - n + \frac{1}{2}\Big) + \Big(\frac{r}{2n(2n-1)}\Big). \end{aligned} \tag{5.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{V}(X,Y)\cdot h = fQ(g,h). \tag{5.2}$$

Notice that (5.2) can be written as,

$$\begin{aligned}&R^{\perp}(X,Y)h(U,V) - h(\overline{V}(X,Y)U,V) - h(U,\overline{V}(X,Y)V)\\ &\qquad = -f[h((X\wedge_g Y),V) + h(U,(X\wedge_g Y)V)].\end{aligned} \tag{5.3}$$

Putting $X = V = \xi$ and using (2.9) we get,

$$\begin{aligned}&R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{V}(\xi,Y)U,\xi) - h(U,\overline{V}(\xi,Y)\xi)\\ &\quad = -f[g(Y,U)h(\xi,\xi) - g(\xi,U)h(Y,\xi) + g(Y,\xi)h(U,\xi) - g(\xi,\xi)h(U,Y)].\end{aligned} \tag{5.4}$$

Substituting (2.11) in (5.4) we obtain,

$$-h(\overline{V}(\xi,Y)U,\xi) - h(U,\overline{V}(\xi,Y)\xi) = f[h(U,Y)]. \tag{5.5}$$

By virtue of (2.14), (2.15), (2.16), (2.17), (2.18) and (2.19) we get

$$h(\overline{V}(\xi,Y)U,\xi) = 0 \tag{5.6}$$

and

$$\begin{aligned}-h(U,\overline{V}(\xi,Y)\xi) =&\Bigg[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2}\\ &- \frac{2}{2n-1}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)\\ &+ \left(\frac{r}{2n(2n-1)}\right)\Bigg]h(U,Y).\end{aligned} \tag{5.7}$$

Substituting (5.6) and (5.7) in (5.5) we get,

$$\begin{aligned}&\Bigg[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{2}{2n-1}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)\\ &\qquad + \left(\frac{r}{2n(2n-1)}\right)\Bigg]h(U,Y) = f[h(U,Y)].\end{aligned} \tag{5.8}$$

That is, $h(U,Y) = 0$ implies M is totally geodesic provided,

$$\begin{aligned}f \neq &\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{2}{2n-1}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)\\ &\qquad + \left(\frac{r}{2n(2n-1)}\right).\end{aligned} \tag{5.9}$$

Conversely, If M is totally geodesic, then we obtain M comply with

$$\overline{V}(X,Y)\cdot h = fQ(g,h). \qquad \square$$

§6. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{C}(X,Y)\cdot h = fQ(S,h)$

Theorem 6.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{C}(X,Y)\cdot h = fQ(S,h)$ iff. M is totally geodesic provided,*

$$f \neq \frac{1}{2n(f_1-f_3)-c-n+\frac{1}{2}}\left[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3) - \frac{3}{2}-\left(\frac{r}{2n(2n+1)}\right)\right]. \tag{6.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{C}(X,Y)\cdot h = fQ(S,h). \tag{6.2}$$

Notice that (6.1) can be written as

$$R^{\perp}(X,Y)h(U,V) - h(\overline{C}(X,Y)U,V) - h(U,\overline{C}(X,Y)V) = -f[h((X\wedge_S Y),V) + h(U,(X\wedge_S Y)V)]. \tag{6.3}$$

Putting $X = V = \xi$ and using (2.9) we get,

$$R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{C}(\xi,Y)U,\xi) - h(U,\overline{C}(\xi,Y)\xi) = -f[\tilde{\overline{S}}(Y,U)h(\xi,\xi) - \tilde{\overline{S}}(\xi,U)h(Y,\xi) + \tilde{\overline{S}}(Y,\xi)h(U,\xi) - \tilde{\overline{S}}(\xi,\xi)h(U,Y)]. \tag{6.4}$$

Substituting (2.11), (2.12) in (6.4) we obtain,

$$\left[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3)-\frac{3}{2}-\left(\frac{r}{2n(2n+1)}\right)-f(\tilde{\overline{S}}(\xi,\xi))\right]h(U,Y) = 0. \tag{6.5}$$

That is, $h(U,Y) = 0$ implies M is totally geodesic provided,

$$f \neq \frac{1}{2n(f_1-f_3)-c-n+\frac{1}{2}}\left[\left(f_1-f_3-\frac{1}{2}\right) + \phi(f_1-f_3)-\frac{3}{2}-\left(\frac{r}{2n(2n+1)}\right)\right]. \tag{6.6}$$

This completes the proof. $\square$

§7. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{W}(X,Y).h = fQ(S,h)$

Theorem 7.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{W}(X,Y)\cdot h = fQ(S,h)$ iff M is totally geodesic provided,*

$$f \neq \frac{1}{\left(2n(f_1-f_3)-c-n+\frac{1}{2}\right)}\left[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3)-\frac{3}{2}\right]-\frac{1}{2n}. \tag{7.1}$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{W}(X,Y).h = fQ(S,h). \tag{7.2}$$

We have

$$\begin{aligned}R^{\perp}(X,Y)h(U,V) - h(\overline{W}(X,Y)U,V) - h(U,\overline{W}(X,Y)V)\\ = -f[h((X\wedge_S Y),V)+h(U,(X\wedge_S Y)V)].\end{aligned} \tag{7.3}$$

Taking $X=V=\xi$ and using (2.9) we acquire

$$\begin{aligned}&R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{W}(\xi,Y)U,\xi) - h(U,\overline{W}(\xi,Y)\xi)\\ &= -f[\tilde{\overline{S}}(Y,U)h(\xi,\xi)-\tilde{\overline{S}}(\xi,U)h(Y,\xi)+\tilde{\overline{S}}(Y,\xi)h(U,\xi)-\tilde{\overline{S}}(\xi,\xi)h(U,Y)].\end{aligned} \tag{7.4}$$

Substituting (2.11), (2.13) in (7.4) we obtain

$$\begin{aligned}\Bigg[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3)-\frac{3}{2}-\frac{1}{2n}\left(2n(f_1-f_3)-c-n+\frac{1}{2}\right)\\ -f(\tilde{\overline{S}}(\xi,\xi))\Bigg]h(U,Y)=0.\end{aligned} \tag{7.5}$$

We now have $h(U,Y)=0$ implies M^{2n+1} is totally geodesic provided,

$$f \neq \frac{1}{\left(2n(f_1-f_3)-c-n+\frac{1}{2}\right)}\left[\left(f_1-f_3-\frac{1}{2}\right)+\phi(f_1-f_3)-\frac{3}{2}\right]-\frac{1}{2n}. \tag{7.6}$$

This completes the proof. □

§8. Invariant Submanifolds of Generalised Sasakian Space Form Satisfying $\overline{V}(X,Y).h = fQ(S,h)$

Theorem 8.1 *Let M be an I.S.M. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then*

M satisfies $\overline{V}(X,Y).h = fQ(S,h)$ *iff. M is totally geodesic provided,*

$$f \neq \frac{1}{\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)} \Bigg[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} + \frac{r}{2n(2n-1)} \Bigg] - \frac{2}{2n-1}. \quad (8.1)$$

Proof Let M be an I.S.M. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{V}(X,Y).h = fQ(S,h). \quad (8.2)$$

Notice that (8.2) can be written as

$$R^{\perp}(X,Y)h(U,V) - h(\overline{V}(X,Y)U,V) - h(U,\overline{V}(X,Y)V) = -f[h((X\wedge_S Y),V) + h(U,(X\wedge_S Y)V)]. \quad (8.3)$$

Putting $X = V = \xi$ and using (2.9), we have

$$R^{\perp}(\xi,Y)h(U,\xi) - h(\overline{V}(\xi,Y)U,\xi) - h(U,\overline{V}(\xi,Y)\xi) = -f[\tilde{\tilde{S}}(Y,U)h(\xi,\xi) - \tilde{\tilde{S}}(\xi,U)h(Y,\xi) + \tilde{\tilde{S}}(Y,\xi)h(U,\xi) - \tilde{\tilde{S}}(\xi,\xi)h(U,Y)]. \quad (8.4)$$

By putting (2.14), (2.11) and (2.19) in (8.4) we get

$$\Bigg[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{2}{2n-1}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) + \frac{r}{2n(2n-1)} - f(\tilde{\tilde{S}}(\xi,\xi))\Bigg] h(U,Y) = 0. \quad (8.5)$$

That is, $h(U,Y) = 0$ implies M is totally geodesic provided

$$f \neq \frac{1}{\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)} \Bigg[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} + \frac{r}{2n(2n-1)} \Bigg] - \frac{2}{2n-1}. \quad (8.6)$$

This completes the proof. □

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