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$(LCS)_n$-Manifold Endowed with Torseforming Vector Field and Concircular Curvature Tensor

R.T.Naveen Kumar[1], P.Siva Kota Reddy[2*], Venkatesha[3] and Khaled A.A.Alloush[4]

1. Department of Mathematics, Siddaganga Institute of Technology, Tumakuru-572 103, Karnataka, India
2. Department of Mathematics, Sri Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru-570 006, Karnataka, India
3. Department of Mathematics, Kuvempu University, Shankaraghatta - 577451, Shimoga, Karnataka, India
4. Department of Mathematics, Dar Al-Uloom University, Riyadh–13314, Saudia Arabia

E-mail: rtnaveenkumar@gmail.com, naveenrt@sit.ac.in

pskreddy@jsstuniv.in, pskreddy@sjce.ac.in, vensmath@gmail.com, khaledindia@gmail.com

Abstract: This paper reviews few curvature properties of $(LCS)_n$-manifold endowed with torseforming vector field and concicrular curvature tensor. Moreover, we have proved several interesting results in this study.

Key Words: $(LCS)_n$-manifold, torseforming vector field, concircular vector field, concircular curvature tensor, scalar curvature.

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§1. Introduction

In 2003, Shaikh [14] has first developed and studied the structure of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with several examples, which generalizes the concept of LP-Sasakian manifolds given by Matsumoto [9] and by Mihai and Rosca [10]. Later on Shaikh et al., [16] proved the existence of ϕ-recurrent $(LCS)_n$-manifolds. Recently the same author studied invariant submanifolds of $(LCS)_n$-manifolds. The notion of $(LCS)_n$-manifolds have been intensively studied by several geometers such as Hui and Atceken [7], Prakasha [13], Venkatesha et. al., [28] and many others.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle is called a concircular transformation [31]. A concircular transformation is always a conformal transformation [8]. An invariant of a concircular transformation is the concircular curvature tensor C given by [31]

$$C(X,Y)U = R(X,Y)U - \frac{r}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{1.1}$$

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Now we can easily obtained From (1.1) that

$$(\nabla_W C)(X,Y)U = (\nabla_W R)(X,Y)U - \frac{dr(W)}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{1.2}$$

The study of torseforming vector field has a long history starting in 1925 by the work of Brinkmann [6], Shirokov [17] and Yano [30, 31]. Torseforming vector field in a Riemannian manifold has been introduced by Yano in 1944 [30] and the complex analogue of a torseforming vector field was introduced by Yamaguchi [29] in 1979. The geometry of torseforming vector field in a Riemannian manifold with different structures have been studied extensively by many geometers such as Bagewadi et. al., [5].

The paper is organized in the following way: In Section 2, we recall the basic definitions and formulas of $(LCS)_n$-manifold needed throughout the paper. The next Section is devoted to the study of $(LCS)_n$-manifold admitting unit torseforming vector field. Here we have shown that an $(LCS)_n$-manifold admits a concircular vector field. In Section 4, we consider globally ϕ-Concircularly symmetric $(LCS)_n$-manifold. Thus, we have obtain that the manifold is of constant scalar curvature provided $2\alpha\rho = \beta$.

For readers who are unfamiliar with terminology, notations, recent overviews and introductions, we suggest the auhthors to refer the papers [1, 2, 3, 4, 11, 12, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

§2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,\cdots,+)$, where T_pM denotes the tangent vector space of M at p and R is the real number space.

A Lorentzian manifold endowed with a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold, gives

$$g(\xi,\xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that

$$g(V,\xi) = \eta(X), \tag{2.2}$$

from which the following equation holds

$$(\nabla_U \eta)(V) = \alpha[g(U,V) + \eta(U)\eta(V)], \quad (\alpha \neq 0) \tag{2.3}$$

for all vector fields U, V, where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g and α is a non-zero scalar function satisfying

$$\nabla_V \alpha = (V\alpha) = d\alpha(V) = \rho\eta(V), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi V = \frac{1}{\alpha}\nabla_V \xi, \tag{2.5}$$

then from (2.3) and (2.4), we have

$$\phi V = V + \eta(V)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor field. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [14]. In a $(LCS)_n$-manifold, the following relations hold ([14, 15]):

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi V) = 0, \tag{2.7}$$

$$g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V), \tag{2.8}$$

$$R(U,V)W = (\alpha^2 - \rho)[g(V,W)U - g(U,W)V], \tag{2.9}$$

$$S(U,\xi) = (n-1)(\alpha^2 - \rho)\eta(U), \tag{2.10}$$

$$S(\phi U, \phi V) = S(U,V) + (n-1)(\alpha^2 - \rho)\eta(U)\eta(V), \tag{2.11}$$

$$Q\xi = (n-1)(\alpha^2 - \rho)\xi. \tag{2.12}$$

for any vector fields U, V, W, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

§3. Torseforming Vector Field in a $(LCS)_n$-Manifold

Definition 3.1 *A vector field γ on a Riemannian manifold is said to be torseforming vector field if the 1-form $\omega(V) = g(V,\gamma)$ satisfies the equation of the form*

$$(\nabla_U \omega)(V) = \beta g(U,V) + \pi(U)\omega(V), \tag{3.1}$$

where β is a non-vanishing scalar and π is a non zero 1-form given by $\pi(V) = g(V,P)$.

Let us consider an $(LCS)_n$ manifold admitting a unit torseforming vector field $\tilde{\gamma}$ corresponding to the torseforming vector field γ. Suppose $g(V,\tilde{\gamma}) = T(V)$, then we have

$$T(V) = \frac{\omega(V)}{\sqrt{\omega(\gamma)}}. \tag{3.2}$$

Now by considering the relation (3.1), we have

$$\frac{(\nabla_U \omega)(V)}{\sqrt{\omega(\gamma)}} = \frac{\beta}{\sqrt{\omega(\gamma)}} g(U,V) + \frac{\pi(U)}{\sqrt{\omega(\gamma)}} \omega(V).$$

Using (3.2) in the above equation, we obtain

$$(\nabla_U T)(V) = ag(U,V) + \pi(U)T(V), \tag{3.3}$$

where $a = \frac{\beta}{\sqrt{\omega(\gamma)}}$.

Plugging $Y = \tilde{\gamma}$ in (3.3) and using $T(\tilde{\gamma}) = g(\tilde{\gamma}, \tilde{\gamma}) = 1$, we get

$$\pi(U) = -aT(U), \tag{3.4}$$

and hence relation (3.3) can be written in the form

$$(\nabla_U T)(V) = a[g(U,V) - T(U)T(V)], \tag{3.5}$$

which implies that T is closed.

Taking covariant differential of (3.5) with respect to W and using Ricci identity, we get

$$\begin{aligned} -T(R(U,V)W) &= (Ua)[g(V,W) - T(V)T(W)] \\ &\quad -(Va)[g(U,W) - T(U)T(W)] \\ &\quad +a^2[g(V,W)T(U) - g(U,W)T(V)]. \end{aligned} \tag{3.6}$$

Replacing $W = \xi$ in (3.6) and then by considering (2.5), we obtain

$$\begin{aligned} -(\alpha^2 - \rho)T(\eta(V)U - \eta(U)V) &= (Ua)[\eta(V) - T(V)T(\xi)] \\ &\quad -(Va)[\eta(U) - T(U)T(\xi)] \\ &\quad +a^2[\eta(V)T(U) - \eta(U)T(V)]. \end{aligned} \tag{3.7}$$

Again, replacing $U = \tilde{\gamma}$ in (3.7) and since $T(\tilde{\gamma}) = g(\tilde{\gamma}, \tilde{\gamma}) = 1$, we have

$$(\alpha^2 - \rho + a^2 + \tilde{\gamma}a)[\eta(V) - \eta(\tilde{\gamma})T(V)] = 0. \tag{3.8}$$

Thus, we can state the following result.

Theorem 3.1 *If a $(LCS)_n$-manifold endowed with a unit torseforming vector field $\tilde{\gamma}$, then the following conditions are occur:*

$$\begin{aligned} \eta(V) - \eta(\tilde{\gamma})T(V) &= 0, && \textit{(I)} \\ (\alpha^2 - \rho + a^2 + \tilde{\gamma}a) &= 0. && \textit{(II)} \end{aligned}$$

We first begin with the case where the condition (I) holds true, from which it follows that

$$\eta(V) = \eta(\tilde{\gamma})T(V).$$

Plugging $V = \xi$ in above equation, gives

$$\eta(\xi) = \eta(\tilde{\gamma})^2,$$

and thus $\eta(\tilde{\gamma}) = \pm\sqrt{-1}$, since $\eta(\xi) = -1$, we get

$$\eta(V) = \pm\sqrt{-1}T(V). \tag{3.9}$$

Using (3.9) in (2.3) and by virtue of (3.5), we have

$$\alpha[g(U,V) - T(U)T(V)] = \pm\sqrt{-1}a(g(U,V) - T(U)T(V)).$$

This implies that $a = \pm\sqrt{-1}\alpha$ and hence the expression (3.4) reduces to

$$\pi(V) = \pm\sqrt{-1}\alpha T(V). \tag{3.10}$$

If we consider $\alpha = 1$, above equation yields

$$\pi(V) = \pm\sqrt{-1}T(V). \tag{3.11}$$

Since T is closed, π is also closed. Hence we can state the following result.

Lemma 3.1 *In an $(LCS)_n$-manifold satisfying condition (I), the unit torseforming vector field $\tilde{\gamma}$ reduces to cocircular vector field provided the manifold becomes LP Sasakian Structure.*

Next, we claim that the case where the condition (II) holds true, then the case (I) does not occur. That is, it follows that

$$\eta(V) - \eta(\tilde{\gamma})T(V) \neq 0. \tag{3.12}$$

Now it can be easily obtained from (3.6) that

$$-(\alpha^2 - \rho)T(QU) = (n-1)aU - (aU) + (\tilde{\gamma}a)T(U) + a^2(n-1)T(U). \tag{3.13}$$

By considering $U = \xi$ in (3.13) and making use of (2.10), we obtain

$$a\xi = -(\alpha^2 - \rho + a^2)\eta(\tilde{\gamma}). \tag{3.14}$$

Plugging $V = \xi$ in (3.7) and in the view of (3.14) and $T(\xi) = \eta(\tilde{\gamma})$, we get

$$aU = -(\alpha^2 - \rho + a^2)T(U). \tag{3.15}$$

Now it can be seen from (3.4) that

$$V\pi(U) = -[(Va)T(U) + a(VT(U))].$$

By considering the equation (3.15), we follows that

$$V\pi(U) = -[-(\alpha^2 - \rho + a^2)T(V)T(U) + a(VT(U))], \tag{3.16}$$

$$U\pi(V) = -[-(\alpha^2 - \rho + a^2)T(U)T(V) + a(UT(V))] \tag{3.17}$$

from which we can easily obtained that

$$\pi([U, V]) = -aT([U, V]). \tag{3.18}$$

Now by using (3.16), (3.17) and (3.18), we have

$$(d\pi)(U, V) = -a[(dT)(U, V)].$$

Since T is closed, π is also closed. Hence we can state that

Lemma 3.2 *In an $(LCS)_n$-manifold satisfying condition (II), the unit torseforming vector field $\tilde{\gamma}$ reduces to cocircular vector field.*

§4. Globally ϕ-Concircularly Symmetric $(LCS)_n$-Manifold

Definition 4.1 *An $(LCS)_n$-manifold M is said to be globally ϕ-concirularly symmetric if the concircular curvature tensor C satisfies*

$$\phi^2((\nabla_X C)(U, V)W) = 0,$$

for all vector fields $X, U, V, W \in \chi(M)$.

If X, U, V and W are horizontal vector fields then the manifold is called locally ϕ-concircularly symmetric.

Let us consider an globally ϕ-concircularly symmetric $(LCS)_n$-manifold, then we have

$$\phi^2((\nabla_X C)(U, V)W) = 0. \tag{4.1}$$

Using (2.6) in equation (4.1), gives

$$(\nabla_X C)(U, V)W + \eta((\nabla_X C)(U, V)W)\xi = 0,$$

from which it follows that

$$g((\nabla_X R)(U,V)W,Y) - \frac{drX}{n(n-1)}[g(V,W)g(U,Y) - g(U,W)g(V,Y)]$$
$$+\eta((\nabla_X R)(U,V)W)\eta(Y) - \frac{drX}{n(n-1)}[g(V,W)\eta(U) - g(U,W)\eta(V)]\eta(Y) = 0. \tag{4.2}$$

Plugging $U = Y = e_i$, where e_i is an orthonormal basis and taking summation over i, we get

$$\begin{aligned}(\nabla_X S)(V,W) &- \frac{drX}{n}g(V,W) + \eta((\nabla_X R)(e_i,V)W)\eta(e_i)\\ &+ \frac{drX}{n(n-1)}[g(V,W) + \eta(V)\eta(W)] = 0.\end{aligned}$$

Considering $W = \xi$ in the above equation, gives

$$(\nabla_X S)(V,\xi) - \frac{drX}{n}\eta(V) + \eta((\nabla_X R)(e_i,V)\xi)\eta(e_i) = 0. \tag{4.3}$$

By considering the expression

$$\eta((\nabla_X R)(e_i,V)\xi)\eta(e_i) = g((\nabla_X R)(e_i,V)\xi,\xi)g(e_i,\xi). \tag{4.4}$$

Now above equation takes the form

$$\begin{aligned}g((\nabla_X R)(e_i,V)\xi,\xi) &= g(\nabla_X R(e_i,V)\xi,\xi) - g(R(\nabla_X e_i,V)\xi,\xi)\\ &\quad -g(R(e_i,\nabla_X V)\xi,\xi) - g(R(e_i,V)\nabla_X\xi,\xi).\end{aligned} \tag{4.5}$$

Since e_i is an orthonormal basis and by virtue of (2.9), we find that

$$g((\nabla_X R)(e_i,V)\xi,\xi) = g(\nabla_X R(e_i,V)\xi,\xi) - g(R(e_i,V)\nabla_X\xi,\xi), \tag{4.6}$$

$$g(\nabla_X R(e_i,V)\xi,\xi) + g(R(e_i,V)\xi,\nabla_X\xi) = 0. \tag{4.7}$$

By employing (4.7) in (4.6), gives

$$g((\nabla_X R)(e_i,V)\xi,\xi) = 0. \tag{4.8}$$

Taking an account of (4.4) and (4.8) in (4.3), turns into

$$(\nabla_X S)(V,\xi) = \frac{drX}{n}\eta(V). \tag{4.9}$$

If we take $V = \xi$ in (4.9), we found that $dr(X) = 0$. Then, from (4.9), we have $\nabla_X S(V,\xi) = 0$. This implies that

$$dr(X) = n(n-1)(2\alpha\rho - \beta). \tag{4.10}$$

Next claim that if $2\alpha\rho = \beta$, we get $dr(X) = 0$ and hence the scalar curvature r is constant.

This leads to the result following.

Theorem 4.2 *A globally ϕ-concircularly symmetric $(LCS)_n$-manifold is of constant scalar curvature provided $2\alpha\rho = \beta$.*

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References

[1] P. G. Angadi, G. S. Shivaprasanna, G. Somashekhara and P. Siva Kota Reddy, Ricci-Yamabe Solitons on Submanifolds of Some Indefinite Almost Contact Manifolds, *Adv. Math., Sci. J.*, 9(11) (2020), 10067–10080.

[2] P. G. Angadi, G. S. Shivaprasanna, R. Rajendra, P. Siva Kota Reddy and G. Somashekhara, η-Yamabe soliton on 3-dimensional α-Para Kenmotsu manifold, *Chinese Journal of Mathematical Sciences*, 1(1) (2021), 29–37.

[3] P. G. Angadi, G. S. Shivaprasanna, G. Somashekhara and P. Siva Kota Reddy, Ricci solitons on (LCS)-manifolds under D-homothetic deformation, *Italian Journal of Pure & Applied Mathematics*, 46 (2021), 672-683.

[4] P. G. Angadi, P. Siva Kota Reddy, G. S. Shivaprasanna and G. Somashekhara, On weakly symmetric generalized (k, μ)-space forms, *Proceedings of the Jangjeon Math. Soc.*, 25(2) (2022), 133-144.

[5] C. S. Bagewadi, D. G. Prakasha and Venkatesha, Torseforming vector fields in a 3-dimensional contact metric manifold, *General Mathematics*, 16(1) (2008), 83-91.

[6] H. W. Brinkmann, Enstein spaces which mapped conformally on each other, *Math. Ann.*, 94 (1925).

[7] S. K. Hui and M. Atceken, Contact warped product semi-slant submanifolds of $(LCS)_n$-manifolds, *Acta Univ. Sapientiae, Mathematica*, 3(2) (2011), 212-224.

[8] W. Kuhnel, Conformal transformations between Einstein spaces, conformal geometry (Bonn, 1985/1986), 105-146, *Aspects Math.*, E12, Vieweg, Braunschweig, 1988.

[9] K. Matsumoto, On Lorentzian almost paracontact manifolds, *Bull. Yamagata Univ. Natur. Soc.*, 12 (1989), 151-156.

[10] I. Mihai and R. Rosca, *On Lorentzian Para-Sasakian Manifolds*, World Scientific Publication, Singapore, 1992, 155-169.

[11] B. Phalaksha Murthy, R. T. Naveen Kumar, P. Siva Kota Reddy and Venkatesha, On $N(k)$-contact metric manifold endowed with pseudo-quasi-conformal curvature tensor, *Adv. Math., Sci. J.*, 10(4) (2021), 1969–1982.

[12] B. Phalaksha Murthy, R. T. Naveen Kumar, P. Siva Kota Reddy and Venkatesha, Extended pseudo projective curvature tensor on $(LCS)_n$-manifold, *Proceedings of the Jangjeon Math. Soc.*, 25(3) (2022), 347-354.

[13] D. G. Prakasha, On Ricci η-recurrent $(LCS)_n$-manifolds, *Acta Universitatis Apulensis*, 24 (2010), 109-118.

[14] A. A. Shaikh, Lorentzian almost paracontact manifolds with a structure of concircular type, *Kyungpook Math. J.*, 43(2) (2003), 305-314.

[15] A. A. Shaikh, Some results on $(LCS)_n$-manifolds, *J. Korean Math. Soc.*, 46(3) (2009), 449-461.

[16] A. A. Shaikh, T. Basu and S. Eyasmin, On the existence of ϕ-recurrent $(LCS)_n$-manifolds, *Extracta Mathematicae*, 23(1) (2008), 71-83.

[17] P. A. Shirokov, *Collected Works of Geometry*, Kazan Univ. Press, 1966.

[18] G. Somashekhara, N. Pavani and P. Siva Kota Reddy, Invariant sub-manifolds of LP-Sasakian manifolds with semi-symmetric connection, *Bull. Math. Anal. Appl.*, 12(2) (2020), 35-44.

[19] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifold, *Bull. Int. Math. Virtual Inst.*, 11(1) (2021), 91-98.

[20] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Indefinite Sasakian manifold with quarter-symmetric metric connection, *Proceedings of the Jangjeon Math. Soc.*, 24(1) (2021), 91-98.

[21] G. Somashekhara, P. Siva Kota Reddy, N. Pavani and G. J. Manjula, η-Ricci-Yamabe solitons on submanifolds of some indefinite almost contact manifolds, *J. Math. Comput. Sci.*, 11(3) (2021), 3775-3791.

[22] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Conformal Ricci soliton in an indefinite Trans-Sasakian manifold, *Vladikavkaz Math. J*, 23(3) (2021), 43-49.

[23] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Ricci solitons and generalized weak symmetries under D-homothetically deformed LP-Sasakian manifolds, *Italian Journal of Pure & Applied Mathematics*, 46 (2021), 684-695.

[24] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Conformal η-Ricci solitons in Lorentzian para-Sasakian manifold admitting semi-symmetric metric connection, *Italian Journal of Pure & Applied Mathematics*, 46 (2021), 1008-1019.

[25] G. Somashekhara, P. Siva Kota Reddy, K. Shivashankara and N. Pavani, Slant submanifolds of generalized Sasakian-space-forms, *Proceedings of the Jangjeon Math. Soc.*, 25(1) (2022), 83-88.

[26] G. Somashekhara, P. Siva Kota Reddy and N. Pavani, Semi-invariant submanifolds of generalized Sasakian-space-forms, *International J. Math. Combin.*, 2 (2022), 47-55.

[27] G. Somashekhara, S. Girish Babu, P. Siva Kota Reddy and K. Shivashankara, On LP-Sasakian manifolds admitting generalized symmetric metric connection, *Proceedings of the Jangjeon Math. Soc.*, 25(3) (2022), 287-296.

[28] Venkatesha and R. T. Naveen Kumar, Some symmetric properties on $(LCS)_n$-manifolds, *Kyungpook Math. J.*, 55 (2015), 149-156.

[29] S. Yamaguchi, On Kaehlerian torse-forming vector fields, *Kodai Math. J.*, 2 (1979), 103-115.

[30] K. Yano, On the torseforming direction in Riemannian spaces, *Proc. Imp. Acad. Tokyo*,

20(1994), 340 - 345.

[31] K. Yano, Concircular geometry, *I*, *Proc. Imp. Acad. Tokyo*, 16(1940), 195 - 200.