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Roman Domination Polynomial of Cycles

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Abstract: A Roman dominating function on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value

$$W(f(V)) = \sum_{u \in V(G)} f(u).$$

The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G and is denoted by $\gamma_R(G)$. In [9], we have introduced and established the study of the Roman domination polynomial of graphs and obtained some important properties about the polynomial and we have computed the polynomial for some specific graphs and graph operations. In this paper, we study the Roman domination polynomial of a cycle C_n on n vertices. Exact formula for the polynomial, important properties of its coefficients and interesting results have obtained.

Key Words: Domination polynomial of cycles, Roman domination polynomial of graphs, Roman domination polynomial of cycles, Smarandachely dominating set.

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§1. Introduction

Let $G = (V, E)$ be a simple graph, where V and E are the set of vertices and edges of G, respectively. The open neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are defined by $N(v) = \{u \in V(G) : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The cardinality of $N(v)$ is called the degree of the vertex v and denoted by $deg(v)$ in G. For more terminology and notations about graph, the reader is referred to [6,10].

A subset D of $V(G)$ is a *dominating set* of G, if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that v is adjacent to u and a subset D is *Smarandachely dominating set* of G on a complete subgraph $K_s \prec G, s \geq 0$ if for every vertex $v \in V - V(H) - D$ there exists a vertex $u \in D$ such that v is adjacent to u but for vertices $w \in V(K_s)$ there are at least 2 vertices $u_1, u_2 \in D$ such that $wu_1, wu_2 \in E(G)$. Clearly, if $s = 0$, a Smarandachely dominating set of G is nothing else but the usual dominating set. A dominating set of G of cardinality

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$\gamma(G)$ is called the domination number of G. For more details about domination of graphs, we refer to [11].

The domination polynomial $D(G,x)$ of a graph G is defined by

$$D(G,x)=\sum_{i=\gamma(G)}^{n} d(G,i)x^i,$$

where $d(G,i)$ is the number of all the dominating sets of G of size i [5]. The dominating sets and the domination polynomial of graphs have been studied extensively in [5, 3, 4, 2]. Recently, the injective domination polynomial of graphs has been studied in [1].

A Roman dominating function of a graph $G=(V,E)$ (or in brief RDF of G) is a function $f:V(G)\to\{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. The weight of a Roman dominating function is

$$W(f(V))=\sum_{u\in V(G)} f(u).$$

A Roman dominating function of a graph G with weight $\gamma_R(G)$ is called the Roman domination number of G. For more details about Roman domination and its properties, the reader is referred to [7]. The next proposition showed that the exact value of γ_R of a path P_n and a cycle C_n on n vertices is $\left\lceil \frac{2n}{3} \right\rceil$.

Proposition 1.1([7]) *For the classes of paths P_n and cycles C_n,*

$$\gamma_R(P_n)=\gamma_R(C_n)=\left\lceil \frac{2n}{3} \right\rceil.$$

In [8], we have introduced the Roman domination polynomial of graph as

$$R(G,x)=\sum_{j=\gamma_R(G)}^{2n} r(G,j)x^j,$$

where $r(G,j)$ is the number of Roman dominating functions of G of weight j. We have established this study by obtaining some important properties of the polynomial and its coefficients, and determining the exact formula of the polynomial for some families of graphs and graph operations.

In the next proposition, we obtain some important properties of $R(G,x)$ of a graph G which we need to use in this paper.

Proposition 1.2([9]) *Let G be a non trivial graph on n vertices. Then,*

(*i*) *$R(G,x)$ has no constant term;*

(*ii*) *$R(G,x)$ has no term of degree one;*

(*iii*) *Zero is a root of $R(G,x)$, with multiplicity $\gamma_R(G)$;*

(iv) *$R(G,x)$ never equal x^p for any $2 \leq p \leq 2n$;*

(v) For any graph G, $r(G,2n) = 1$ and $r(G,2n-1) = n$;

(vi) *$r(G,j) = 0$ if and only if $j < \gamma_R(G)$ or $j > 2n$;*

(vii) *$R(G,x)$ is a strictly increasing function in $[0,\infty)$;*

(viii) *The only polynomial of degree two can $R(G,x)$ be equal is $x^2 + x$ if and only if $G \cong K_1$;*

(ix) *Let H be any induced subgraph of G. Then*

$$deg\big(R(G,x)\big) \geq deg\big(R(H,x)\big).$$

In this paper, we study the Roman domination polynomial of a cycle C_n on n vertices. Exact formula for $R(C_n,x)$, important properties and relations between the coefficient of $R(C_n,x)$ are obtained.

§2. Roman Domination Polynomial of a Cycle

In [3], Alikhani and Peng have showed that the number of all dominating sets with cardinality i of a cycle C_n equal to the sum of the number of all dominating sets of the cycle C_{n-1} with cardinality $i-1$, the cycle C_{n-2} with cardinality $i-1$ and the cycle C_{n-3} with cardinality $i-1$, and then they have found the exact formula of the domination polynomial of cycles, as follows:

Theorem 2.1([3]) *(i) If $\mathcal{C}_n^i$ is the family of all dominating sets with cardinality i of a cycle C_n, then*

$$\left|\mathcal{C}_n^i\right| = \left|\mathcal{C}_{n-1}^{i-1}\right| + \left|\mathcal{C}_{n-2}^{i-1}\right| + \left|\mathcal{C}_{n-3}^{i-1}\right|;$$

(ii) For every $n \geq 4$,

$$D(C_n,x) = x\big[D(C_{n-1},x) + D(C_{n-2},x) + D(C_{n-3},x)\big],$$

with initial values $D(C_1,x) = x$, $D(C_2,x) = x^2 + 2x$ and $D(C_3,x) = x^3 + 3x^2 + 3x$.

In this section, we find the Roman domination polynomial of a cycle C_n on n vertices, and then we study some of its properties, and finally, we illustrate in a table the coefficients of all Roman domination polynomials of cycles C_n with $n \leq 10$.

Let $\mathbb{C}_n^j$ be the set of all RDFs of C_n with weight j. Actually, to find a RDF of C_n, we do not need to consider RDFs of C_{n-4} with weight $j-2$ (weight $j-1$ is not possible here), we will show this in the next lemma. Note that, when we talk about a RDF f with weight $j-1$ or $j-2$ in C_{n-r}, where $r = 1,2,3$ such that $f \in \mathbb{C}_n^j$, we mean a RDF f of C_n minus only one vertex $v \in C_n \setminus C_{n-r}$ taking a value $f(v) = 1$ or $f(v) = 2$, respectively.

Lemma 2.2 *Let $f \in \mathbb{C}_n^j$. Then, if $f \in \mathbb{C}_{n-4}^{j-2}$, this implies that $f \in \mathbb{C}_{n-3}^{j-2}$.*

Proof Consider $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Suppose $f \in \mathbb{C}_{n-4}^{j-2}$. Then we have two cases following.

Case 1. If $f(v_1) = 0$ or 1, then the vertex v_{n-4} must take the value 2 under the function f (shortly $f(v_{n-4}) = 2$) because otherwise $f \notin \mathbb{C}_n^j$, a contradiction. Therefore, $f \in \mathbb{C}_{n-3}^{j-2}$.

Case 2. If $f(v_1) = 2$, then whatever the vertex v_{n-4} taking under the function f ($f(v_{n-4}) = 0$ or 1 or 2) $\Rightarrow f \in \mathbb{C}_{n-3}^{j-2}$. □

In the following theorem, according to Theorem 2.1 part (i) (since every RDF of a graph G it just a labeling on some dominating set of the graph G itself) and Lemma 2.2, we state the Roman domination polynomial of C_n in terms of the Roman domination polynomial of C_{n-1}, C_{n-2} and C_{n-3}.

Theorem 2.3 *Let C_n be a cycle on $n \geq 4$ vertices. Then*

$$R(C_n, x) = (x^2 + x)R(C_{n-1}, x) + x^2 R(C_{n-2}, x) + (x^3 + x^2)R(C_{n-3}, x),$$

with initial values $R(C_3, x) = x^6 + 3x^5 + 6x^4 + 7x^3 + 3x^2$, $R(C_2, x) = x^4 + 2x^3 + 3x^2$ and $R(C_1, x) = x^2 + x$.

Proof Consider $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let $f \in \mathbb{C}_n^j$. Then we have the following cases.

Case 1. Suppose that $f \in \mathbb{C}_{n-1}^{j-1}$ or $f \in \mathbb{C}_{n-1}^{j-2}$ (this means that, for the last vertex v_n either $f(v_n) = 1$ or $f(v_n) = 2$, respectively). Then we get the term $(x^2 + x)R(C_{n-1}, x)$.

Case 2. Suppose that $f \in \mathbb{C}_{n-2}^{j-1}$ or $f \in \mathbb{C}_{n-2}^{j-2}$. Thus, we have the following subcases.

Subcase 2.1 Suppose $f(v_{n-1}) = 1$ and $f(v_n) = 0$. Then the vertex v_1 should take the value 2. Hence, we get the term $x^3 R(C_{n-3}, x)$.

Subcase 2.2 Suppose $f(v_{n-1}) = 0$ and $f(v_n) = 1$. This case included in Case 1.

Subcase 2.3 Suppose $f(v_{n-1}) = 2$ and $f(v_n) = 0$. Then we get the term $x^2 R(C_{n-2}, x)$.

Subcase 2.4 Suppose $f(v_{n-1}) = 0$ and $f(v_n) = 2$. This situation has some connection with Case 1, so to avoid the repetition, we will take only the situations when $\mathbb{C}_{n-1}^{j-2} = \phi$. Therefore, we will choose $f(v_{n-2}) = 0$. Then the vertex v_{n-3} should take the value 2 ($f(v_{n-3}) = 2$), but if the vertex v_1 take the value 2 also, we get a repetition with Case 1. Hence, we get the term $x^4 R(C_{n-4}, x) - x^6 R(C_{n-5}, x)$.

Case 3. Suppose now $f \in \mathbb{C}_{n-3}^{j-2}$, where $f(v_{n-2}) = 2$, $f(v_{n-1}) = 0$ and $f(v_n) = 0$. Then we obtain the term $x^4 R(C_{n-4}, x)$.

Case 4. In this case, we have remaining the situation when $f(v_{n-2}) = 0$, $f(v_{n-1}) = 2$ and $f(v_n) = 0$ such that $\mathbb{C}_{n-2}^{j-2} = \phi$. Therefore, we have the term $x^2 R(C_{n-3}, x)$ but there are two possibilities of repetition, when $f(v_{n-3}) = 2$ or $f(v_n) = 2$. Thus, we have remove the term $2x^4 R(C_{n-4}, x)$. But while we remove the term $2x^4 R(C_{n-4}, x)$ we will miss the situation when $f(v_{n-3}) = 2$ and $f(v_n) = 2$ which give us the term $x^6 R(C_{n-5}, x)$. Hence in the end of this case we obtain the term

$$x^2 R(C_{n-3}, x) - 2x^4 R(C_{n-4}, x) + x^6 R(C_{n-5}, x).$$

The proof is completed. □

Using Theorem 2.3, we obtain $r(C_n, j)$ for $1 \leq n \leq 10$ as shown in Table 1.

j	1	2	3	4	5	6	7	8	9	10
n										
1	1	1								
2	0	3	2	1						
3	0	3	7	6	3	1				
4	0	0	4	15	16	10	4	1		
5	0	0	0	10	31	40	30	15	5	1
6	0	0	0	3	24	69	96	84	50	21
7	0	0	0	0	7	56	155	231	224	154
8	0	0	0	0	0	20	128	351	552	584
9	0	0	0	0	0	3	54	297	799	1314
10	0	0	0	0	0	0	10	140	690	1833

j	11	12	13	14	15	16	17	18	19	20
n										
6	6	1								
7	77	28	7	1						
8	448	258	112	36	8	1				
9	1494	1257	810	405	156	45	9	1		
10	3120	3770	3430	2430	1362	605	210	55	10	1

Table 1 $r(C_n, j)$, the number of Roman dominating functions of C_n with cardinality j.

In the following theorem, we obtain some important properties about the coefficients of the Roman domination polynomial of a cycle C_n.

Theorem 2.4 *The following properties are satisfied for the Roman domination polynomial $R(C_n, x)$ of a cycle C_n:*

(*i*) $r(C_n, j) = r(C_{n-1}, j-1) + r(C_{n-1}, j-2) + r(C_{n-2}, j-2) + r(C_{n-3}, j-2) + r(C_{n-3}, j-3)$;

(*ii*) $r(C_{3k}, 2k) = 3$, *where* $n = 3k$ *for some* $k \in \mathbb{N}$;

(*iii*) *If* $n = 3k+1$ *for some* $k \in \mathbb{N}$, *then* $r(C_{3k+1}, 2k+1) = 3k+1$;

(*iv*) *If* $n = 3k+2$ *for some* $k \in \mathbb{N}$, *then* $r(C_{3k+2}, 2k+2) = \dfrac{(3k+2)(k+3)}{2}$;

(*v*) *If* $n = 3k$ *for some* $k \in \mathbb{N}$, *then* $r(C_{3k}, 2k+1) = \dfrac{k(k+1)(k+6)}{2}$;

(*vi*) *If* $n = 3k+1$ *for some* $k \in \mathbb{N}$, *then* $r(C_{3k+1}, 2k+2) = \dfrac{(3k+1)(k+4)(k^2+11k+6)}{24}$;

(vii) If $n = 3k+2$ *for some* $k \in \mathbb{N}$*, then*

$$r(C_{3k+2}, 2k+3) = \frac{(3k+2)(k+3)(k^3+23k^2+122k+40)}{120};$$

(viii) If $n = 3k$ *for some* $k \in \mathbb{N}$*, then*

$$r(C_{3k}, 2k+2) = \frac{k(k^5+35k^4+365k^3+1165k^2+234k-360)}{240};$$

(ix) $r(C_n, 2n) = 1$*;*

(x) $r(C_n, 2n-1) = n$*;*

(xi) $r(C_n, 2n-2) = \dfrac{n(n+1)}{2}$*;*

(xii) $r(C_n, 2n-3) = \dfrac{n(n-1)(n+4)}{6}$*;*

(xiii) $r(C_n, 2n-4) = \dfrac{n(n+1)(n^2+5n-18)}{24}$*;*

(xiv) $r(C_n, 2n-5) = \dfrac{n(n-1)(n^3+11n^2-14n-144)}{120}$*;*

(xv) For every $k \in \mathbb{N}$*,*

$$1 = r(C_k, 2k) < r(C_{k+1}, 2k) < r(C_{k+2}, 2k) < \cdots < r(C_{2k}, 2k) > \cdots > r(C_{3k-1}, 2k) > r(C_{3k}, 2k) = 3;$$

(xvi) For every $k \in \mathbb{N}$*,*

$$k+1 = r(C_{k+1}, 2k+1) < r(C_{k+2}, 2k+1) < r(C_{k+3}, 2k+1) < \cdots < r(C_{2k+1}, 2k+1) > \cdots > r(C_{3k}, 2k+1) > r(C_{3k+1}, 2k+1) = 3k+1;$$

(xvii) If $\alpha_n = \sum\limits_{j=\lceil \frac{2n}{3} \rceil}^{2n} r(C_n, j)$*, then for every* $n \geq 4$*,* $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2} + 2\alpha_{n-3}$*, with initial values* $\alpha_1 = 2$*,* $\alpha_2 = 6$ *and* $\alpha_3 = 20$*.*

(xviii) For $j \geq 2$*,*

$$\sum_{i=j}^{3j} r(C_i, 2j) = \sum_{i=j}^{3j-2} r(C_i, 2j-1) + 3\sum_{i=j-1}^{3j-3} r(C_i, 2j-2) + \sum_{i=j-1}^{3j-5} r(C_i, 2j-3);$$

(xix) For $j \geq 3$*,*

$$\sum_{i=j}^{3j-2} r(C_i, 2j-1) = \sum_{i=j-1}^{3j-3} r(C_i, 2j-2) + 3\sum_{i=j-1}^{3j-5} r(C_i, 2j-3) + \sum_{i=j-2}^{3j-6} r(C_i, 2j-4).$$

Proof Let C_n be a path on n vertices with $V(C_n) = \{v_1, v_2, \cdots, v_n\}$.

(i) The proof of this result is straightforward from Theorem 2.3.

(ii) Let $n = 3k$ for some $k \in \mathbb{N}$. Since $\mathcal{C}_{3k}^k = \big\{\{v_1, v_4, \cdots, v_{3k-5}, v_{3k-2}\}, \{v_2, v_5, \cdots, v_{3k-4},$

$v_{3k-1}\}, \{v_3, v_6, \cdots, v_{3k-3}, v_{3k}\}\}$, then we have only three RDF of C_n, in this case, such that each vertex taking the value 2. Hence, $r(P_{3k}, 2k) = 3$.

(iii) Proof by induction on k. If $k = 1$, then $r(C_4, 3) = 4$ (see Table 1). Therefore, the result is true for $k = 1$. Now, suppose the result is true for all natural numbers less than or equal $k-1$. We will prove that the result still true for k. By parts (i) and (ii), the induction hypothesis and Proposition 1.2 part (vi), we get.

$$\begin{aligned} r(C_{3k+1}, 2k+1) &= r(C_{3k}, 2k) + r(C_{3k}, 2k-1) + r(C_{3k-1}, 2k-1) \\ &\quad + r(C_{3k-2}, 2k-1) + r(C_{3k-2}, 2k-2) \\ &= 3 + 0 + 0 + r\big(C_{3(k-1)+1}, 2(k-1)+1\big) + 0 \\ &= 3k+1. \end{aligned}$$

(iv) By induction on k. If $k = 1$, then $r(C_5, 4) = 10 = \frac{(3+2)(1+3)}{2}$ (see Table 1). Suppose now the result is true for all natural numbers less than or equal $k-1$. Then by parts (i), (ii) and (iii) and Proposition 1.2 part (vi), we have.

$$\begin{aligned} r(C_{3k+2}, 2k+2) &= r(C_{3k+1}, 2k+1) + r(C_{3k+1}, 2k) + r(C_{3k}, 2k) \\ &\quad + r(C_{3k-1}, 2k) + r(C_{3k-1}, 2k-1) \\ &= 3k + 1 + 0 + 3 + r\big(C_{3(k-1)+2}, 2(k-1)+2\big) + 0 \\ &= 3k + 4 + \frac{(3k-1)(k+2)}{2} = \frac{(3k+2)(k+3)}{2}. \end{aligned}$$

(v) Proof by induction on k. If $k = 1$, then $r(C_3, 3) = 7 = \frac{1(1+1)(1+6)}{2}$ (see Table 1). Suppose the result is true for all natural numbers less than k. Then by using parts (i), (ii), (iii) and (iv) and Proposition 1.2 part (vi), we obtain.

$$\begin{aligned} r(C_{3k}, 2k+1) &= r(C_{3(k-1)+2}, 2(k-1)+2) + r(C_{3(k-1)+2}, 2(k-1)+1) \\ &\quad + r(C_{3(k-1)+1}, 2(k-1)+1) + r(C_{3(k-1)}, 2(k-1)+1) + r(C_{3(k-1)}, 2(k-1)) \\ &= \frac{(3k-1)(k+2)}{2} + 0 + 3k - 2 + \frac{(k-1)(k)(k+5)}{2} + 3 = \frac{k(k+1)(k+6)}{2}. \end{aligned}$$

(vi) By induction on k. When $k = 1$, $r(C_4, 4) = 15 = \frac{(3+1)(1+4)(1+11+6)}{24}$ (see Table 1). Suppose the result is true for all natural numbers less than k. Then by using parts (i), (ii), (iii), (iv) and (v), we get.

$$\begin{aligned} r(C_{3k+1}, 2k+2) &= r(C_{3k}, 2k+1) + r(C_{3k}, 2k) + r(C_{3(k-1)+2}, 2(k-1)+2) \\ &\quad + r(C_{3(k-1)+1}, 2(k-1)+2) + r(C_{3(k-1)+1}, 2(k-1)+1) \\ &= \frac{k(k+1)(k+6)}{2} + 3 + \frac{(3k-1)(k+2)}{2} \\ &\quad + \frac{(3k-2)(k+3)\big((k-1)^2 + 11(k-1) + 6\big)}{24} + 3k - 2 \\ &= \frac{(3k+1)(k+4)(k^2+11k+6)}{24}. \end{aligned}$$

(vii) By induction on k. If $k=1$, then $r(C_5,5)=31=\frac{(3+2)(1+3)(1+23+122+40)}{120}$ (see Table 1). Now, suppose the result is true for all natural numbers less than k. Then by using parts (i), (iii), (iv), (v) and (vi), we get.

$$\begin{aligned}
r(C_{3k+2},2k+3) &= r(C_{3k+1},2k+2)+r(C_{3k+1},2k+1)+r(C_{3k},2k+1)\\
&\quad +r(C_{3(k-1)+2},2(k-1)+3)+r(C_{3(k-1)+2},2(k-1)+2)\\
&= \frac{(3k+1)(k+4)(k^2+11k+6)}{24}+3k+1+\frac{k(k+1)(k+6)}{2}\\
&\quad +\frac{(3k-1)(k+2)\big((k-1)^3+23(k-1)^2+122(k-1)+40\big)}{120}+\frac{(3k-1)(k+2)}{2}\\
&= \frac{(3k+2)(k+3)(k^3+23k^2+122k+40)}{120}.
\end{aligned}$$

$(viii)$ By induction on k. If $k=1$, then $r(C_3,4)=6=\frac{1(1+35+365+1165+234-360)}{240}$ (see Table 1). Now, suppose the result is true for all natural numbers less than k. Then by using parts (i), (iv), (v), (vi) and (vii), we get.

$$\begin{aligned}
r(C_{3k},2k+2) &= r(C_{3(k-1)+2},2(k-1)+3)+r(C_{3(k-1)+2},2(k-1)+2)\\
&\quad +r(C_{3(k-1)+1},2(k-1)+2)+r(C_{3(k-1)},2(k-1)+2)\\
&\quad +r(C_{3(k-1)},2(k-1)+1)\\
&= \frac{(3k-1)(k+2)\big((k-1)^3+23(k-1)^2+122(k-1)+40\big)}{120}+\frac{(3k-1)(k+2)}{2}\\
&\quad +\frac{(3k-2)(k+3)\big((k-1)^2+11(k-1)+6\big)}{24}+\frac{(k-1)(k)(k+5)}{2}\\
&\quad +\frac{(k-1)\Big[(k-1)^5+35(k-1)^4+365(k-1)^3+1165(k-1)^2+234(k-1)-360\Big]}{240}\\
&= \frac{k(k^5+35k^4+365k^3+1165k^2+234k-360)}{240}.
\end{aligned}$$

(ix) The proof is clear.

(x) Clearly, for every vertex $v\in V(C_n$ the function $f:V(C_n)\to\{0,1,2\}$ with $f(v)=1$ and weight $W(f(V))=2n-1$ is a Roman dominating function of G. Hence, $r(C_n,2n-1)=\binom{n}{1}=n$.

(xi) By induction on n. The result is true for $n=2$, since $r(C_2,2)=3$ (see Table 1). Suppose the result is true for every natural number less than n. Then by parts (i), (ix) and (x) and Proposition 1.2 part (vi), we have.

$$\begin{aligned}
r(C_n,2n-2) &= r\big(C_{n-1},2(n-1)-1\big)+r\big(C_{n-1},2(n-1)-2\big)+r\big(C_{n-2},2(n-2)\big)\\
&\quad +r\big(C_{n-3},2(n-3)+2\big)+r\big(C_{n-3},2(n-3)+1\big)\\
&= n-1+\frac{n(n-1)}{2}+1+0+0=\frac{n(n+1)}{2}.
\end{aligned}$$

(xii) By induction on n. The result is true for $n=3$, since $r(C_3,3)=7$ (see Table 1).

Suppose the result is true for every natural number less than n. Then by parts (i), (ix), (x) and (xi) and Proposition 1.2 part (vi), we have.

$$\begin{aligned}
r(C_n, 2n-3) &= r\big(C_{n-1}, 2(n-1)-2\big) + r\big(C_{n-1}, 2(n-1)-3\big) + r\big(C_{n-2}, 2(n-2)-1\big) \\
&\quad + r\big(C_{n-3}, 2(n-3)+1\big) + r\big(C_{n-3}, 2(n-3)\big) \\
&= \frac{n(n-1)}{2} + \frac{(n-1)(n-2)(n+3)}{6} + n - 2 + 0 + 1 \\
&= \frac{n(n-1)(n+4)}{6}.
\end{aligned}$$

$(xiii)$ By induction on n. If $n = 5$, then $r(C_5, 6) = 40$. Therefore, the result is true for $n = 4$(see Table 1). Suppose now the result is true for every natural number less than n. Then by parts (i), (ix), (x), (xi) and (xii), we have.

$$\begin{aligned}
r(C_n, 2n-4) &= r\big(C_{n-1}, 2(n-1)-3\big) + r\big(C_{n-1}, 2(n-1)-4\big) + r\big(C_{n-2}, 2(n-2)-2\big) \\
&\quad + r\big(C_{n-3}, 2(n-3)\big) + r\big(C_{n-3}, 2(n-3)-1\big) \\
&= \frac{(n-1)(n-2)(n+3)}{6} + \frac{(n-1)(n)\big[(n-1)^2 + 5(n-1) - 18\big]}{24} \\
&\quad + \frac{(n-1)(n-2)}{2} + 1 + n - 3 \\
&= \frac{n(n+1)(n^2+5n-18)}{24}.
\end{aligned}$$

(xiv) By induction on n. The result is true for $n = 5$, since $r(C_5, 5) = 31$ (see Table 1). Suppose now the result is true for every natural number less than n. Then by parts (i), (x), (xi), (xii) and $(xiii)$, we have.

$$\begin{aligned}
r(C_n, 2n-5) &= r\big(C_{n-1}, 2(n-1)-4\big) + r\big(C_{n-1}, 2(n-1)-5\big) + r\big(C_{n-2}, 2(n-2)-3\big) \\
&\quad + r\big(C_{n-3}, 2(n-3)-1\big) + r\big(C_{n-3}, 2(n-3)-2\big) \\
&= \frac{(n-1)(n)\big[(n-1)^2 + 5(n-1) - 18\big]}{24} \\
&\quad + \frac{(n-1)(n-2)\big[(n-1)^3 + 11(n-1)^2 - 14(n-1) - 144\big]}{120} \\
&\quad + \frac{(n-2)(n-3)(n+2)}{6} + n - 3 + \frac{(n-3)(n-2)}{2} \\
&= \frac{n(n-1)(n^3+11n^2-14n-144)}{120}.
\end{aligned}$$

(xv) We need to prove that for every $k \in \mathbb{N}$, $r(C_i, 2k) < r(C_i, 2k)$ for $k \le i \le 2k-1$ and $r(C_i, 2k) > r(C_i, 2k)$ for $2k \le i \le 3k$. By induction on k, the result is true for $k = 1$. Now, suppose that the result is true for every i less than or equal k. We will prove it for $i = k+1$ which means $r(C_i, 2k+2) < r(C_{i+1}, 2k+2)$ for $k+1 \le i \le 2k+1$. By part (i) and the induction hypothesis, we have

$$\begin{aligned} r(C_i, 2k+2) &= r(C_{i-1}, 2k+1) + r(C_{i-1}, 2k) + r(C_{i-2}, 2k) \\ &\quad + r(C_{i-3}, 2k) + r(C_{i-3}, 2k-1) \\ &< r(C_i, 2k+1) + r(C_i, 2k) + r(C_{i-1}, 2k) \\ &\quad + r(C_{i-2}, 2k) + r(C_{i-2}, 2k-1) \\ &= r(C_{i+1}, 2k+2). \end{aligned}$$

Similarly, we prove for the other inequality.

(xvi) Similar to the prove of part (xv), we will prove that for every $k \in \mathbb{N}$, $r(C_i, 2k+1) < r(C_i, 2k+1)$ for $k+1 \le i \le 2k$ and $r(C_i, 2k+1) > r(C_i, 2k+1)$ for $2k+1 \le i \le 3k+1$. By induction on k, the result is true for $k=1$. Now, suppose that the result is true for every i less than or equal $k+1$. We will prove it for $i = k+2$ which means $r(C_i, 2k+3) < r(C_{i+1}, 2k+3)$ for $k+2 \le i \le 2k+2$. By part (i) and the induction hypothesis, we have

$$\begin{aligned} r(C_i, 2k+3) &= r(C_{i-1}, 2k+2) + r(C_{i-1}, 2k+1) + r(C_{i-2}, 2k+1) \\ &\quad + r(C_{i-3}, 2k+1) + r(C_{i-3}, 2k) \\ &< r(C_i, 2k+2) + r(C_i, 2k+1) + r(C_{i-1}, 2k+1) \\ &\quad + r(C_{i-2}, 2k+1) + r(C_{i-2}, 2k) = r(C_{i+1}, 2k+3). \end{aligned}$$

Similarly, we prove for the other inequality.

$(xvii)$ By Theorem 2.3, we have

$$\begin{aligned} R(C_n, x) &= \sum_{j=\gamma_R(C_n)}^{2n} r(C_n, j)\, x^j = (x^2+x)R(C_{n-1}, x) + x^2 R(C_{n-2}, x) \\ &\quad + (x^3+x^2)R(C_{n-3}, x) \\ &= \sum_{j=\lceil \frac{2n-2}{3} \rceil}^{2n-2} r(C_{n-1}, j)\big[x^{j+2} + x^{j+1}\big] + \sum_{j=\lceil \frac{2n-4}{3} \rceil}^{2n-4} r(C_{n-2}, j) x^{j+2} \\ &\quad + \sum_{j=\lceil \frac{2n-6}{3} \rceil}^{2n-6} r(C_{n-3}, j)\big[x^{j+3} + x^{j+2}\big]. \end{aligned}$$

Now, if $\alpha_n = \sum_{j=\lceil \frac{2n}{3} \rceil}^{2n} r(C_n, j)$, we can see that all the coefficients of $R(C_{n-1}, x)$ and $R(C_{n-3}, x)$ counted twice and all the coefficients of $R(C_{n-2}, x)$ counted once in α_n. Hence, $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2} + 2\alpha_{n-3}$.

$(xviii)$ If $j = 2$, then

$$\begin{aligned} \sum_{i=2}^{6} r(C_i, 4) &= \sum_{i=2}^{4} r(C_i, 3) + 3\sum_{i=1}^{3} r(C_i, 2) + \sum_{i=1}^{1} r(C_i, 1) \\ 35 &= 13 + 3(7) + 1 = 35. \end{aligned}$$

By part (i), we have

$$\sum_{i=j}^{3j} r(C_i, 2j) = \sum_{i=j}^{3j} r(C_{i-1}, 2j-1) + \sum_{i=j}^{3j} r(C_{i-1}, 2j-2) + \sum_{i=j}^{3j} r(C_{i-2}, 2j-2) + \sum_{i=j}^{3j} r(C_{i-3}, 2j-2) + \sum_{i=j}^{3j} r(C_{i-3}, 2j-3).$$

Now, by Proposition 1.2 part (vi), we have

$$\begin{aligned}
\sum_{i=j}^{3j} r(C_{i-1}, 2j-1) &= \sum_{i=j-1}^{3j} r(C_i, 2j-1) = \sum_{i=j}^{3j-2} r(C_i, 2j-1),\\
\sum_{i=j}^{3j} r(C_{i-1}, 2j-2) &= \sum_{i=j-1}^{3j} r(C_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(C_i, 2j-2),\\
\sum_{i=j}^{3j} r(C_{i-2}, 2j-2) &= \sum_{i=j-2}^{3j} r(C_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(C_i, 2j-2),\\
\sum_{i=j}^{3j} r(C_{i-3}, 2j-2) &= \sum_{i=j-3}^{3j} r(C_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(C_i, 2j-2)
\end{aligned}$$

and

$$\sum_{i=j}^{3j} r(C_{i-3}, 2j-3) = \sum_{i=j-3}^{3j} r(C_i, 2j-3) = \sum_{i=j-1}^{3j-5} r(C_i, 2j-3).$$

(xix) The proof is similar to the proof of part $(xviii)$. □

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