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On Skew-Sum Eccentricity Energy of Digraphs

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Abstract: In this paper we introduce the concept of skew-sum eccentricity energy of directed graphs. We obtain upper and lower bounds for skew-sum eccentricity energy of digraphs. Then we compute the skew-sum eccentricity energy of some graphs such as star digraph, complete bipartite digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

Key Words: Energy, eccentricity, skew-sum eccentricity energy, digraph.

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§1. Introduction

In 2018, B. Sharada and Mohammad Issa Ahmed Sowaity [5] introduced the sum eccentricity energy of a simple graph G as follows. The sum eccentricity adjacency matrix of G is the $n \times n$ matrix (a_{ij}), where

$$a_{ij} = \begin{cases} e(v_i) + e(v_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The sum eccentricity energy of G is defined as the sum of absolute values of the eigenvalues of the sum eccentricity adjacency matrix of G.

In 2010, Adiga, Balakrishnan and Wasin So [1] introduced the skew energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = (s_{ij})$ where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D)$ and $s_{ij} = 0$ otherwise. Hence $S(D)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and λ is a real number. The skew energy of G is the sum of the absolute values of eigenvalues of $S(D)$.

Motivated by these works, we introduce the concept of skew-sum eccentricity energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. Then the skew-sum eccentricity adjacency matrix of D is the $n \times n$ matrix

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$A_{sse} = (a_{ij})$ where,

$$a_{ij} = \begin{cases} e(v_i) + e(v_j), & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\left(e(v_i) + e(v_j)\right), & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then the skew-sum eccentricity energy $E_{sse}(D)$ of D is defined as the sum of the absolute values of eigenvalues of A_{sse}. For example, let D be the directed circle on 4 vertices with the arc set $\{(1,2),(2,3),(3,4),(4,1)\}$. Then

$$A_{sse} = \begin{pmatrix} 0 & 4 & 0 & -4 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \\ 4 & 0 & -4 & 0 \end{pmatrix}.$$

with the characteristic equation $\lambda^4 + 64\lambda^2 = 0$. Its eigenvalues are $8i$, 0, 0, $-8i$ and the skew-sum eccentricity energy of D is 16.

In section 2 of this paper we obtain the upper and lower bounds for skew-sum eccentricity energy of digraphs. In Section 3 we compute the skew-sum eccentricity energy of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

§2. Upper and Lower Bounds for Skew-Sum Eccentricity Energy

Theorem 2.1 *Let D be a simple digraph of order n. Then*

$$E_{sse}(D) \le \sqrt{2n \sum_{j\sim k} \left(e(v_j) + e(v_k)\right)^2}.$$

Proof Let $i\lambda_1, i\lambda_2, i\lambda_3, \cdots, i\lambda_n$, be the eigenvalues of A_{sse}, where $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4 \ge \cdots \ge \lambda_n$. Since

$$\sum_{j=1}^{n} (i\lambda_j)^2 = tr(A_{sse}^2) = -\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^2 = -2\sum_{j\sim k} \left(e(v_j) + e(v_k)\right)^2,$$

we have

$$\sum_{j=1}^{n} |\lambda_j|^2 = 2\sum_{j\sim k} \left(e(v_j) + e(v_k)\right)^2. \tag{1}$$

Applying the Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \le \left(\sum_{j=1}^{n} a_j^2\right) \cdot \left(\sum_{j=1}^{n} b_j^2\right)$$

with $a_j = 1$, $b_j =\mid \lambda_j \mid$, we obtain

$$\begin{aligned} E_{sse}(D) &= \sum_{j=1}^{n} |\lambda_j| = \sqrt{(\sum_{j=1}^{n} |\lambda_j|)^2} \leq \sqrt{n\sum_{j=1}^{n} |\lambda_j|^2} \\ &= \sqrt{2n\sum_{j\sim k} (e(v_j)+e(v_k))^2}. \end{aligned}$$

This completes the proof. □

Theorem 2.2 *Let D be a simple digraph of order n. Then*

$$E_{sse}(D) \geq \sqrt{2\sum_{j\sim k} (e(v_j)+e(v_k))^2 + n(n-1)p^{\frac{2}{n}}}, . \tag{2}$$

where $p = |detA_{sg}| = \prod_{j=1}^{n} |\lambda_j|$.

Proof Notice that

$$(E_{sse}(D))^2 = \left(\sum_{j=1}^{n} |\lambda_j|\right)^2 = \sum_{j=1}^{n} |\lambda_j|^2 + \sum_{1\leq j\neq k\leq n} |\lambda_j||\lambda_k|.$$

By the arithmetic-geometric mean inequality, we get

$$\begin{aligned} \sum_{1\leq j\neq k\leq n} |\lambda_j||\lambda_k| &= |\lambda_1|(|\lambda_2|+|\lambda_3|+\cdots+|\lambda_n|) \\ &\quad +|\lambda_2|(|\lambda_1|+|\lambda_3|+\cdots+|\lambda_n|) \\ &\quad +\cdots+|\lambda_n|(|\lambda_1|+|\lambda_2|+\cdots+|\lambda_{n-1}|) \\ &\geq n(n-1)(|\lambda_1||\lambda_2|\cdots|\lambda_n|)^{\frac{1}{n}}(|\lambda_1|^{n-1}|\lambda_2|^{n-1}\cdots|\lambda_n|^{n-1})^{\frac{1}{n(n-1)}} \\ &= n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{1}{n}}\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{1}{n}} = n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}}. \end{aligned}$$

Thus

$$(E_{sse}(D))^2 \geq \sum_{j=1}^{n} |\lambda_j|^2 + n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}}.$$

From the equation (1), we get

$$(E_{sse}(D))^2 \geq 2\sum_{j\sim k} (e(v_j)+e(v_k))^2 + n(n-1)p^{\frac{2}{n}},$$

which gives the inequality (2). □

§3. Skew-Sum Eccentricity Energies of Some Graph Families

We begin with some basic definitions and notations.

Definition 3.1 *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 3.2 *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 3.3 *A crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_iv_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore S_n^0 coincides with complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Definition 3.4 *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l)|v_iv_k \in E(S_m)$ and $v_jv_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Definition 3.5 *A graph G is said to be strong vertex graceful if there exists a bijective mapping $f : V(G) \to \{1, 2, \cdots, n\}$ such that for the induced labeling $f^+ : E(G) \to \mathbb{N}$ defined by $f^+(e) = f(u) + f(v)$, where $e = uv$, the set $f^+(E(G))$ consists of consecutive integers.*

Now we compute skew-sum eccentricity energies of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

Theorem 3.6 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of complete bipartite digraph $K_{m,n}(m > 1)$ be respectively given by $V(D) = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(u_i, v_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$. Then, the skew-sum eccentricity energy of $K_{m,n}$ is $8\sqrt{mn}$.*

Proof The skew-sum eccentricity matrix of complete bipartite digraph is given by

$$A_{sse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 4 & 4 & \cdots & 4 \\ 0 & 0 & \cdots & 0 & 4 & 4 & \cdots & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 4 & 4 & \cdots & 4 \\ -4 & -4 & \cdots & -4 & 0 & 0 & \cdots & 0 \\ -4 & -4 & \cdots & -4 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & -4 & \cdots & -4 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then, its characteristic polynomial is

$$|\lambda I - A_{sse}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & -4 & -4 & \cdots & -4 \\ 0 & \lambda & \cdots & 0 & -4 & -4 & \cdots & -4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -4 & -4 & \cdots & -4 \\ 4 & 4 & \cdots & 4 & \lambda & 0 & \cdots & 0 \\ 4 & 4 & \cdots & 4 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & \cdots & 4 & 0 & 0 & \cdots & \lambda \end{vmatrix} = \begin{vmatrix} \lambda I_m & -4J^T \\ 4J & \lambda I_n \end{vmatrix},$$

where, J is an $n \times m$ matrix with all the entries are equal to 1. Hence, its characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -4J^T \\ 4J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m| \left| \lambda I_n - (4J) \frac{I_m}{\lambda} \left(-4J^T\right) \right| = 0.$$

By simplification, we obtain

$$16\lambda^{m-n} \left| \frac{\lambda^2}{16} I_n + JJ^T \right| = 0,$$

which can be written as

$$16\lambda^{m-n} P_{JJ^T}\left(\frac{-\lambda^2}{16}\right) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$16\lambda^{m-n} \left(\frac{\lambda^2}{16} + mn\right) \left(\frac{\lambda^2}{16}\right)^{n-1} = 0,$$

which is the same as

$$\lambda^{m+n-2}(\lambda^2 + 16mn) = 0.$$

Hence,

$$Spec\,(D) = \begin{pmatrix} 0 & i4\sqrt{mn} & -i4\sqrt{mn} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Therefore, the skew-sum eccentricity energy of the complete bipartite digraph is

$$E_{sse}(D) = 8\sqrt{mn},$$

as desired. □

Theorem 3.7 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of star digraph S_n be respectively given by $V(D) = \{v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(v_1, v_j) \,|\, 2 \leq j \leq n\}$. Then, the skew-sum eccentricity energy of D is $6\sqrt{n-1}$.*

Proof The skew-sum eccentricity matrix of the star digraph D is given by

$$A_{sse} = \begin{pmatrix} 0 & 3 & 3 & \cdots & 3 & 3 \\ -3 & 0 & 0 & \cdots & 0 & 0 \\ -3 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -3 & 0 & 0 & \cdots & 0 & 0 \\ -3 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence, its characteristic polynomial is given by

$$|\lambda I - A_{sse}| = \begin{vmatrix} \lambda & -3 & -3 & \cdots & -3 \\ 3 & \lambda & 0 & \cdots & 0 \\ 3 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 0 & 0 & \cdots & \lambda \end{vmatrix} = 3^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where $\mu = \frac{\lambda}{3}$. Then, $|\lambda I - A_{sse}| = 3^n \phi_n(\mu)$ with

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu^{n-2} + \mu\phi_{n-1}(\mu)).$$

Iterating by this formula, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 + n - 1).$$

Therefore

$$|\lambda I - A_{sse}| = 3^n \left[\left(\frac{\lambda^2}{9} + (n-1) \right) \left(\frac{\lambda}{3} \right)^{n-2} \right].$$

Consequently, the characteristic equation is given by

$$\lambda^{n-2} \left(\lambda^2 + 9(n-1) \right) = 0.$$

Hence

$$Spec\,(D) = \begin{pmatrix} 0 & i3\sqrt{n-1} & -i3\sqrt{n-1} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Thus, the skew-sum eccentricity energy of D is $E_{sse}(D) = 6\sqrt{n-1}$. □

Theorem 3.8 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of crown digraph $S_n^0(n > 2)$ be respectively given by $V(D) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(u_i, v_j) \,|\, 1 \le i \le n, 1 \le j \le n, i \ne j\}$. Then, the skew-sum eccentricity energy of the crown digraph is* $16(n-1)$.

Proof The skew-sum eccentricity matrix of crown digraph is given by

$$A_{sse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 4 & \cdots & 4 \\ 0 & 0 & \cdots & 0 & 4 & 0 & \cdots & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 4 & 4 & \cdots & 0 \\ 0 & -4 & \cdots & -4 & 0 & 0 & \cdots & 0 \\ -4 & 0 & \cdots & -4 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & -4 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then, its characteristic polynomial is

$$|\lambda I - A_{sse}| = \begin{vmatrix} \lambda I_n & -4K^T \\ 4K & \lambda I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -4K^T \\ 4K & \lambda I_n. \end{vmatrix} = 0.$$

This is same as

$$|\lambda I_n| \left| \lambda I_n - (4K) \frac{I_n}{\lambda} \left(-4K^T\right) \right| = 0,$$

which can be written as

$$16^n P_{KK^T}(-\frac{\lambda^2}{16}) = 0,$$

where $P_{KK^T(\lambda)}$ is the characteristic polynomial of the matrix KK^T. Thus, we have

$$16^n \left[\frac{\lambda^2}{16} + (n-1)^2\right] \left[\frac{\lambda^2}{16} + 1\right]^{n-1} = 0,$$

which is same as

$$\left(\lambda^2 + 16(n-1)^2\right) \left(\lambda^2 + 16\right)^{n-1} = 0.$$

Therefore

$$Spec\,(D) = \begin{pmatrix} i4(n-1) & -i4(n-1) & i4 & -i4 \\ 1 & 1 & n-1 & n-1 \end{pmatrix}$$

. Hence the skew-sum eccentricity energy of crown digraph is

$$E_{sse}(D) = 16(n-1)$$

as desired. □

Theorem 3.9 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of digraph $(S_m \wedge P_2)(m > 1)$ be respectively given by*

$$V(D) = \{v_1, v_2, \cdots, v_{2m+2}\},$$
$$\Gamma(D) = \{(v_1, v_j), (v_{m+2}, v_k) \,|\, 2 \leq k \leq m+1, m+3 \leq j \leq 2m+2\}.$$

Then the skew-sum eccentricity energy of D is $12\sqrt{n-1}$.

Proof The skew-sum eccentricity matrix of $(S_m \wedge P_2)$ digraph is given by

$$A_{sse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 3 & \cdots & 3 \\ 0 & 0 & \cdots & 0 & -3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -3 & 0 & \cdots & 0 \\ 0 & 3 & \cdots & 3 & 0 & 0 & \cdots & 0 \\ -3 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -3 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m+1=n$. Then, its characteristic polynomial is given by

$$|\lambda I - A_{sse}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -3 & \cdots & -3 \\ 0 & \lambda & \cdots & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 3 & 0 & \cdots & 0 \\ 0 & -3 & \cdots & -3 & \lambda & 0 & \cdots & 0 \\ 3 & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence, the characteristic equation is given by

$$3^{2n} \begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where $\Lambda = \frac{\lambda}{3}$. Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}.$$

$$= \quad (-1)^{2n+2n}\Lambda \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

and let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Applying the properties of the determinants, we obtain

$$\Psi_{2n-1}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda),$$

after some simplifications, where

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 1 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}.$$

Then, we have

$$\phi_{2n}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as the above, we obtain

$$\begin{aligned} \phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+1}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= \Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda). \end{aligned}$$

and continuous like this, we finally obtain

$$\phi_{2n}(\Lambda) = (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda)$$

at the $(n-1)^{th}$ step, where

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}.$$

$$\begin{aligned} \phi_{2n}(\Lambda) &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= ((n-1)\Lambda^{n-2} + \Lambda^n)\Theta_n(\Lambda). \end{aligned}$$

Using the properties of the determinants, we get that

$$\Theta_n(\Lambda) = (n-1)\Lambda^{n-2} + \Lambda^n.$$

Therefore

$$\phi_{2n}(\Lambda) = ((n-1)\Lambda^{n-2} + \Lambda^n)^2.$$

Hence, the characteristic equation becomes

$$3^{2n}\phi_{2n}(\Lambda) = 0,$$

which is the same as

$$3^{2n}((n-1)\Lambda^{n-2} + \Lambda^n)^2 = 0$$

and can be reduced to

$$\lambda^{2n-4}((n-1) + \frac{\lambda}{9}^2)^2 = 0.$$

Therefore

$$Spec\,(D) = \begin{pmatrix} 0 & i3\sqrt{n-1} & -i3\sqrt{n-1} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the skew-sum eccentricity energy of $(S_m \wedge P_2)$ digraph is

$$E_{sse}(D) = 12\sqrt{n-1}.$$

□

Theorem 3.10 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of strong vertex graceful digraph $(n, 2n-3)$ $D = K_2 + \overline{K}_{n-2}(n > 3)$ be respectively given by $V(D) = \{v_1, v_2, \cdots, v_n\}$ and $\Gamma(D) = \{(v_1, v_j)\,|\,2 \le j \le n\} \cup \{(v_j, v_n)\,|\,2 \le j \le n-1\}$. Then, the skew-sum eccentricity energy of D is $2\sqrt{4+18(n-2)}$.*

Proof The skew-sum eccentricity matrix of the graph is given by

$$A_{sse} = \begin{pmatrix} 0 & 3 & 3 & \cdots & 2 \\ -3 & 0 & 0 & \cdots & 3 \\ -3 & 0 & 0 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -3 & -3 & \cdots & 0 \end{pmatrix}$$

and its characteristics polynomial is

$$|\lambda I - A_{sse}| \quad = \quad 3^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -\gamma \\ 1 & \mu & 0 & \cdots & 0 & -1 \\ 1 & 0 & \mu & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & -1 \\ \gamma & 1 & 1 & \cdots & 1 & \mu \end{vmatrix},$$

where $\mu = \frac{\lambda}{3}$ and $\gamma = \frac{2}{3}$. Using the properties of the determinants, we obtain

$$|\lambda I - A_{sse}| = 3^n \left[(-1)^{2n+1}\left(\mu^2 - \gamma^2\right)\mu^{n-2} + (-1)^{2n} 2\mu\phi_{n-1}(\mu)\right] \quad (3)$$

after some simplifications, where

$$\phi_{n-1}(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 \\ 1 & \mu & 0 & \cdots & 0 \\ 1 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & \mu \end{vmatrix}_{(n-1)\times(n-1)}.$$

Now, as in the proof of the Theorem 3.7, we obtain

$$\phi_{n-1}(\mu) = \mu^{n-3} + \mu\phi_{n-2}(\mu).$$

Iterating with this formula, we obtain

$$\phi_{n-1}(\mu) = \mu^{n-3}(\mu^2 + n - 2). \tag{4}$$

Substituting (4) in (3) and using $\mu = \frac{\lambda}{3}$, we obtain

$$\begin{aligned} |\lambda I - A_{sse}| &= 3^n \left[-\left(\mu^2 - \gamma^2\right)(\mu)^{n-2} + 2(\mu)^{n-2}(\mu^2 + n - 2)\right] \\ &= 9\lambda^{n-2}\left(\mu^2 + \gamma^2 + 2(n-2)\right). \end{aligned}$$

Thus, the characteristic equation is given by

$$\lambda^{n-2}\left(\frac{\lambda^2}{9} + \frac{4 + 18(n-2)}{9}\right) = 0.$$

Hence

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{4 + 18(n-2)} & -i\sqrt{4 + 18(n-2)} \\ n-2 & 1 & 1 \end{pmatrix}.$$

So, the skew-sum eccentricity energy of D is $E_{sse}(D) = 2\sqrt{4 + 18(n-2)}$ as desired. □

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