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# On the Modular Graphic Family of a Graph

J. Kok

Independent Mathematics Researcher

City of Tshwane, South Africa & Visiting Faculty at CHRIST (Deemed to be a University), Bangalore, India

E-mail: jacotype@gmail.com

**Abstract**: For a finite, connected simple graph $G$ of order $n$ with degree sequence, $s = (d_i : 1 \leq i \leq n, d_i = deg(v_i)$ and, $d_j \geq d_{j+1}, 1 \leq j \leq n-1)$ the *modular sequences*, $s(mod\ k)$, $1 \leq k \leq n$ are introduced. Those modular sequences which are graphic constitute the *modular graphic family* of a graph. Numerous introductory results are presented.

**Key Words**: Degree sequence, graphic sequence, modular sequence, modular graph family.

**AMS(2010)**: 05C15, 05C38, 05C62, 05C75, 05C85.

## §1. Introduction

It is assumed that the reader is familiar with most of the classical concepts in graph theory. Unless stated otherwise, only finite, connected simple graphs will be considered. For more on general notation and concepts in graphs see [1,4,8].

Recall that the vertices of a graph $G$ of order $n$ can be labeled $v_1, v_2, v_3, \cdots, v_n$ such that a sequence $s = (d_i : 1 \leq i \leq n, d_i = deg(v_i)$ and, $d_j \geq d_{j+1}, 1 \leq j \leq n-1)$, can be defined. This sequence is called the *degree sequence* of $G$. We say that the sequence is of order $n$. Conversely, a finite sequence of non-increasing, non-negative integers, $s = (d_1, \geq d_2, \geq d_3, \geq \cdots, \geq d_n)$ is said to be *simple graphic* (graphic or graphical for brevity) if there exists a finite, simple graph $G$ (not necessarily connected) with degree sequence corresponding to $s$. It is said that $G$ is a graphical realization of $s$. The notion of graphic integer sequences has been studied extensively. Characterizations of graphic integer sequences are found in [2,3,5].

Let $s = (d_1, d_2, d_3, \cdots, d_n)$ be the degree sequence of graph $G$. The *modular sequences* of $G$ are, $s(mod\ k) = (d_1(mod\ k), d_2(mod\ k), d_3(mod\ k), \ldots, d_n(mod\ k))$, $1 \leq k \leq n$. For a given $k$ a modular sequence is abbreviated as, $s(mod\ k) = (d_1, d_2, d_3, \ldots, d_n)(mod\ k)$. Two integer sequences both of order $n$ are said to be distinct if after say, arranging both as non-increasing sequences there exists at least one entry say, the $i^{th}$ entry in each sequence which are not equal. The set of distinct modular sequences which are graphic are called the *graphic family* of $G$ and is denoted by, $\mathfrak{M}_{od}(G)$. Clearly, $|\mathfrak{M}_{od}(G)| \leq n$. Also, $deg_G(v_i) = d_i \leq d_i(mod\ k)$. Note that the realizations of graphic modular sequences need not be connected.

It is agreed that if the set of integers $X = \{b_1, b_2, b_3, \cdots, b_m\}$ is added (or inserted) to an

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integer sequence $s$ it is denoted by, $s \cup X$.

The motivation for this study is firstly, that it is in principle acceptable to do mathematics for the sake of mathematics. Secondly, studying graph theoretic properties and parameters as it may relate between the realizations of modular sequences and the graph $G$ could potentially find application in mathematical chemistry. Thirdly, it could find application to the family of molecular structures which can derive when say, a virus Type A mutates to virus Type B. In the context of the emphasized virology research on the virus, SARS-CoV-2 (causing Covid-19) this notion could be relevant. Other applications can also be conceptualized.

## §2. Main Results

### 2.1 Preliminary Results

It is known that $n(mod\ 1) = 0$, $\forall n$. Hence, the modular sequence $(\underbrace{0,0,0,\cdots,0}_{n\ entries})$ which corresponds to the degree sequence of the null graph (or edgeless graph) of order $n$ is always in $\mathfrak{M}_{od}(G)$ despite the fact that $G$ is a connected simple graph. Furthermore, since $\Delta(G) \leq n-1$ the modular sequence $(d_1, d_2, d_3, \cdots, d_n)(mod\ n) = (d_1, d_2, d_3, \cdots, d_n)$ for graphs of order $n$.

**Proposition** 2.1 *Any connected simple graph $G$ of order $n \geq 2$ has, $2 \leq |\mathfrak{M}_{od}(G)| \leq n$.*

*Proof* For any null graph $\mathfrak{N}_n$ it follows that, $(\underbrace{0,0,0,\ldots,0}_{n\ entries})(mod\ k) = (\underbrace{0,0,0,\cdots,0}_{n\ entries})$, $1 \leq k \leq n$ thus, $|\mathfrak{M}_{od}(\mathfrak{N}_n)| = 1$, $\forall n$. However, for any graph $G$ of order $n \geq 2$ and $\delta(G) \geq 1$ the result is trivial because both $(\underbrace{0,0,0,\cdots,0}_{n\ entries})$ and the degree sequence of $G$ are graphic. □

In fact, besides repetition of graphic modular sequences some graphs have modular sequences which are not graphic. For such graphs, $2 \leq |\mathfrak{M}_{od}(G)| < n$. For example, a triangle with each vertex on $C_3$ joined to a distinct pendent vertex is of order 6. The degree sequence of such a graph is $(3,3,3,1,1,1)$ and $(3,3,3,1,1,1)(mod\ 3) = (0,0,0,1,1,1)$. Clearly, $(1,1,1,0,0,0)$ is not graphical.

Two necessary conditions for a non-negative integer sequence to be graphic are

$(i)$ $d_i \leq n-1$ and

$(ii)$ $\sum\limits_{i=1}^{n} d_i$ is even.

Modular sequences as defined for the degree sequence of a graph always satisfy (i) but not always satisfy (ii). A fundamental avenue for research is, to characterize graphs which yield modular sequences which all satisfy condition (i) and (ii). The next step will be to validate which of the aforesaid modular sequences also satisfy the sufficient conditions to be graphic. We present a family of graphs from [6] which have an unexpected property. Let the family be denoted by $\mathcal{F}$. For $k \in \mathbb{N}$, $k \geq 2$ a graph $G \in \mathcal{F}$ is defined as follows.

**Definition** 2.2 *Let the complete graph be on vertices $v_1, v_2, v_3, \cdots, v_k$, $k \geq 2$. For each $v_i$,*

*$i = 2, 3, 4, \cdots, k$ add a distinct pendent vertex $u_i$. Add the edges $u_iv_j$, $i = 2, 3, 4, \cdots, (k-1)$, $j = i+1, i+2, i+3, \cdots, k$ to obtain $G \in \mathcal{F}$.*

**Lemma** 2.3 *If a sequence of positive integers, after possible arrangement, is a sequence of consecutive decreasing integers say, $s = (t, t-1, t-2, t-3, \cdots, 1)$ then the modular sequences, $s(mod\ k)$, $k = 1, 2, 3, \cdots, t$ are distinct.*

*Proof* Let $s_1 = (1)$. Clearly, $s_1(mod\ 1) = (0)$ which is distinct. Let $s_2 = (2, 1)$ then, $s(mod\ 1) = (0, 0)$ and $s_2(mod\ 2) = (0, 1)$. Also for $s_3 = (3, 2, 1)$ we have, $s_3(mod\ 1) = (0, 0, 0)$, $s_3(mod\ 2) = (1, 0, 1)$ and $s_3(mod\ 3) = (0, 2, 1)$. The results holds for $1 \leq t \leq 3$. Assume the results holds for $1 \leq t \leq \ell$.

Consider $t = \ell + 1$. Let $s' = (\ell+1, \ell, \ell-1, \ell-2, \cdots, 1)$. Clearly, $s' = s \cup \{\ell+1\}$. It follows immediately that since $s_i(mod\ i) \neq s_j(mod\ j)$ for all distinct pairs, $1 \leq i, j \leq \ell$ then, $s_i(mod\ i) \cup \{(\ell+1)(mod\ i)\} \neq s_j(mod\ j) \cup \{(\ell+1)(mod\ j)\}$. Hence, the result follows through induction. □

**Corollary** 2.4 *If the maximum sub-sequence of distinct positive integers in a non-negative integer sequence $s$ of order $n$, after possible re-arrangement, is the sequence of consecutive decreasing integers say, $(t, t-1, t-2, t-3, \cdots, 1)$, $t \leq n$ then the modular sequences, $s(mod\ k)$, $k = 1, 2, 3, \cdots, n$ are distinct.*

*Proof* The result is a direct consequence of Lemma 2.3. □

Observe that if a sequence $s = (a_1, a_2, a_3, \cdots, a_n)$ is graphic then, $s' = (a_1, a_2, a_3, \cdots, a_n, 0, 0, 0, \cdots, 0)$ is also graphic. We recall the useful result which is independently due to [3, 5]. It is called the Havel-Hakimi theorem. Also see [7].

**Theorem** 2.5([3, 5]) *Let $s = (a_1, a_2, a_3, \cdots, a_n)$ be a sequence of non-increasing, non-negative integers. Then $s$ is graphic if and only if the sequence $s' = (a_2-1, a_3-1, \cdots, a_{a_1+1}-1, a_{a_1+2}, \cdots, a_n)$ is graphic.*

**Theorem** 2.6 *If $G \in \mathcal{F}$ then, $|\mathfrak{M}_{od}(G)| = n$.*

*Proof* Observe that $G \in \mathcal{F}$ is a connected simple graph. It must be shown that the $n$ modular sequences are distinct and graphic. Clearly, for $k \in \mathbb{N}$ the graph $G$ has order $n = 2k-1$. Also, $\Delta(G) = 2(k-1) = t$. The degree sequence is $s = (t, t-1, t-2, \ldots, t-k, t-k, t-(k+1), t-(k+2), \cdots, 1)$. It follows from Lemma 2.3 (or Corollary 2.4) that the modular sequences are all distinct. Hence, $n = 2k-1$ such distinct modular sequences are yielded.

We must show that each modular sequence is graphic. It is known that all modular sequences, satisfy condition (i). Also it follows easily that if the number of odd integers in the sub-sequence $(t, t-1, t-2, \cdots, t-k)$ is odd then the number of odd integers in the sub-sequence $(t-k, t-(k+1), t-(k+2), \cdots, 1)$ is odd and conversely. Hence, same is found in any modular sequence $s(mod\ r)$, $1 \leq r \leq n$. Therefore all modular sequences satisfy condition (ii).

We are left to show sufficiency. Applying the Havel-Hakimi theorem to the sequence $s = (t, t-1, t-2, \cdots, t-k, t-k, t-(k+1), t-(k+2), \cdots, 1)$ give the derived sequence, $s' = (t-1, t-$

$2, t-3, \ldots, t-k-1, t-k-1, t-(k+2), t-(k+3), \cdots, 0)$. For the graphic property the zero entry may be deleted. Note that $s'' = (t-1, t-2, t-3, \cdots, t-k-1, t-k-1, t-(k+2), t-(k+3), \cdots, 1)$ is the degree sequence of $G' \in \mathcal{F}$ of order $n-2$. Furthermore, all modular sequences of $G'$ are distinct and graphic, and the union of each with $\{t(mod\ k)\}$, $k = (n-1), n$ respectively does not change the graphic property and adds two more distinct modular sequences. The aforesaid is true because $(n, n-1)(mod\ n) = (0, n-1)$ and $(n, n-1)(mod\ (n-1)) = (1, 0)$. Hence, it follows through immediate induction that the results holds in general. □

The graph defined in Definition 2.2 was first constructed to prove a result related to degree tolerant coloring [6]. Following Theorem 2.5 it is proposed to refer to this family of graphs as the *Havel-Hakimi graphs*, the main topic of next section. It is observed that if both non-negative sequences $s_1, s_2$ are graphic then, $s_1 \cup s_2$ is graphic. It follows because the degree sequence $s$ of the disjoint union $G_1 \cup G_2$ which is graphic by definition is, $s = s_1 \bigcup s_2$.

### 2.2 Disjoint Union of Graphs

Let graphs $G$ and $H$ have modular families $\mathfrak{M}_{od}(G)$, $\mathfrak{M}_{od}(G)$, respectively. When we consider the disjoint union $G \bigcup H$ the definition of modular sequences is relaxed to permit the derivative Cartesian product, $\mathfrak{M}_{od}(G) \times \mathfrak{M}_{od}(H) = s_i \bigcup s_j$, $s_i \in \mathfrak{M}_{od}(G)$, $s_j \in \mathfrak{M}_{od}(H)$.

**Proposition** 2.7 *For $G_1, G_2 \in \mathcal{F}$ of order $n$ and $m$ respectively, it follows that*

$$|\mathfrak{M}_{od}(G_1 \bigcup G_2)| \geq nm.$$

*Proof* From the $n$ and $m$ distinct and graphical modular sequences a total of $nm$ sequences can be *cup'ed*. From each such $(n+m)$-sequence a total of $\binom{n+m}{n}$ sequences of the form, ($n$-sequence) $\bigcup$ ($m$-sequence), can be constructed. At least $nm$ have been shown to be distinct and graphic by Theorem 2.5. Hence,

$$|\mathfrak{M}_{od}(G_1 \bigcup G_2)| \geq nm$$ □

Proposition 2.7 implies a generalization for disconnected graphs.

**Corollary** 2.8 *Let $G$ be a simple graph with $\ell$ components and each component has order $m_i$, $i = 1, 2, 3, \cdots, \ell$. Then*

$$|\mathfrak{M}_{od}(G)| \geq \prod_{i=1}^{\ell} |\mathfrak{M}_{od}(G_i)|.$$

### 2.2 Complement Graph

This subsection presents a result of interest. The result appears to be trivial but it suggests a deeper problem which we pose in the next section.

**Theorem** 2.9 *Consider a connected simple graph $G$ and $\overline{G}$ with degree sequences $s_1, s_2$, respectively. For the pairs of modular sequences, $s_1(mod\ k)$, $s_2(mod\ k)$, $k = 1, 2, 3, \cdots, n$ it follows* *that, $s_1(mod\ k) = s_2(mod\ k)$ if and only if $G$ is self-complementary.*

*Proof* Let $G$ be self-complementary. It is obvious that $s_1(mod\ k) = s_2(mod\ k)$, $k = 1, 2, 3, \cdots, n$. Now assume $G$ is not self-complementary. Let $s_1$, $s_2$ be the degree sequences of respectively, $G$ and $\overline{G}$. Also, assume that, $s_1(mod\ k) = s_2(mod\ k)$, $k = 1, 2, 3, \cdots, n$. Since $G$ is not self-complementary there exists at least one $v \in V(G)$ such that $deg_G(v) \neq deg_{\overline{G}}(v)$. It implies that amongst the modular sequences at least, $s_1(mod\ n) \neq s_2(mod\ n)$. This contradiction suffices to settle the result. □

### 2.3 Complete Graphs $\mathfrak{M}_{od}(K_n)$, $k \geq 2$

A complete graph $K_n$ is the graph of smallest order which permits a $(n-1)$-regular graph. Studying the complete graph with regards to the graph parameter, $\mathfrak{M}_{od}(K_n)$, $k \geq 2$ serves as a basis to study same for $k$-regular graphs in general.

**Theorem** 2.10 *For a complete graph, $K_n$, $n \geq 2$:*

$$|\mathfrak{M}_{od}(K_n)| = \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is even;} \\ t - \lceil \frac{t}{2} \rceil + 1, t = \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

*Proof* The proof is divided into two cases.

**Case 1.** $n$ is even.

It follows immediately that $s = (\underbrace{n-1, n-1, n-1, \cdots, n-1}_{n-entries})(mod\ \frac{n}{2})$ results in the modular sequence, $(\underbrace{\frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}-1, \cdots, \frac{n}{2}-1}_{n-entries})$.

Similarly through immediate induction, $s_i = (\underbrace{n-1, n-1, n-1, \cdots, n-1}_{n-entries})(mod\ (\frac{n}{2} + i)) =$ $(\underbrace{\frac{n}{2}-(i+1), \frac{n}{2}-(i+1), \cdots, \frac{n}{2}-(i+1)}_{n-entries})$, $i = 0, 1, 2, \ldots, \frac{n}{2} - 1$. Together with the degree sequence, $s(mod\ n) = (\underbrace{n-1, n-1, n-1, \cdots, n-1}_{n-entries})$. All distinct modular sequences have been obtained. The reason is that any $s(mod\ t)$, $t < \frac{n}{2}$ is a repetition as some modular sequence.

We are left to show all distinct modular sequences are graphic. It is easy to verify that all modular sequences satisfy both necessary conditions. finally applying the Havel-Hakimi theorem recursively to each modular sequence results in a sequence of one's which is graphic. Hence, by the Havel-Hakimi theorem all modular sequences are graphic. Therefore, $|\mathfrak{M}_{od}(K_n)| = \frac{n}{2} + 1$, if $n$ is even.

**Case 2.** $n$ is odd.

It follows through similar reasoning as Case 1 with the following exception. All distinct modular sequences with odd entries are non-graphic because such sequence prescribes an odd number of odd degrees. It is easy to verify that $\lceil \frac{n+1}{2} \rceil - 1$ such sequences exist. Hence, $|\mathfrak{M}_{od}(K_n)| = t - \lceil \frac{t}{2} \rceil + 1$, $t = \frac{n+1}{2}$, if $n$ is odd. □

Theorem 2.10 leads to a useful corollary.

**Corollary** 2.11 *For a connected simple k-regular graph G, $n \geq k+1$, it follows that*

$$|\mathfrak{M}_{od}(G)| = \begin{cases} \frac{k+3}{2}, & \text{if } k+1 \text{ is even;} \\ t - \lceil \frac{t}{2} \rceil + 1, t = \frac{k+2}{2}, & \text{if } k+1 \text{ and } n \text{ are odd;} \\ \frac{k+2}{2}, & \text{if } k+1 \text{ is odd and } n \text{ is even} \end{cases}$$

*Proof* Since the degree sequence $s$ of $G$ has the modular sequences, $s(mod\ k_1)$, either $1 \leq k_1 \leq \frac{k}{2}$, $k$ even, or $1 \leq k_1 \leq \frac{k-1}{2}$, $k$ odd, and $s(mod\ k_2)$, $k+2 \leq k_2 \leq n$ as repetitions, the result is a direct consequence of Theorem 2.10. □

## §3. Conclusion

The paper serves as an introduction to the notion of modular sequences of graphs. The scope for further research is evidently, enormous. Problems which could be worthy to research are listed below.

**Problem** 3.1 *Besides the Havel-Hakimi graphs which other graphs have, $|\mathfrak{M}_{od}(G)| = n$?*

**Problem** 3.2 *If possible characterize graphs which have, $|\mathfrak{M}_{od}(G)| < n$.*

**Problem** 3.3 *Can Proposition 2.7 be improved to $|\mathfrak{M}_{od}(G_1 \bigcup G_2)| = nm$?*

The cycle $C_4$ has distinct modular sequences $(0,0,0,0)$, $(2,2,2,2)$. The complement, $\overline{C_4} = P_2 \bigcup P_2$ cannot be realized by any of the modular sequences. However, the butterfly graph, $G$ has distinct modular sequences, $(0,0,0,0,0)$, $(1,2,2,2,2)$, $(0,2,2,2,2)$, $(4,2,2,2,2)$. Note that $(4,2,2,2,2)(mod\ 4) = (0,2,2,2,2)$ which realizes $\overline{G} = C_4 \bigcup K_1$. See

*www.graphclasses.org/smallgraphs.html*

for details.

**Problem** 3.4 *If possible, characterize the graphs other than self-complementary graphs, which has a modular sequence which realization is the complement graph.*

Note that $K_3$ (or $C_3$), has exactly one modular graphic sequence i.e $(2,2,2)$ with the realization which is Hamiltonian. For $K_n$, $n \geq 4$ the $\mathfrak{M}_{od}(K_n)$ has at least two modular sequences of which the realizations are Hamiltonian. The two typical modular sequences are, $(\underbrace{2,2,2,\cdots,2}_{n-entries})$ and $(\underbrace{n-1,n-1,n-1,\ldots,n-1}_{n-entries})$. The modular sequence $(\underbrace{2,2,2,\cdots,2}_{n-entries})$ has an Eulerian realization whilst $(\underbrace{n-1,\cdots,n-1}_{n-entries})$ is an Eulerian realization if and only if $n$ is odd.

**Problem** 3.5 *Find all Hamiltonian and Eulerian realizations in* $\mathfrak{M}_{od}(K_n)$*,* $n \geq 4$*.*

Finding results for graph operations and finding relations between other graph parameters of $G$ and those for graphic modular sequence realizations would be of great interest.

## References


[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications,* Macmillan Press, London, 1976.

[2] P. Erdös and T. Gallai,Graphs with prescribed degrees of vertices, *Mat. Lapok*, Vol.11 (1960), 264-274.

[3] S.I. Hakimi,On the realizability of a set of integers as degrees of the vertices of a graph, *J. SIAM Appl. Math.*, Vol.10 (1962), 496-506.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Reading MA, 1969.

[5] V. Havel,A remark on the existence of finite graphs, *Časopis Pěst. Mat.*, Vol.80 (1955), 477-480.

[6] J. Kok and N.K. Sudev, Degree tolerant coloring of graphs, Communicated.

[7] A. Tripathi and H. Tyagi,A simple criterion on degree sequences of graphs, *Discrete Appl. Math.*, Vol.156 (2008), 3513-3517.

[8] B. West, *Introduction to Graph Theory*, Prentice-Hall, Upper Saddle River, 1996.