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On n-Polynomial P-Function and Related Inequalities

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Abstract: In this paper, we introduce and study the concept of n-polynomial P-function and establish Hermite-Hadamard's inequalities for this type of functions. In addition, we obtain some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is n -polynomial P-function by using Hölder and power-mean integral inequalities. Some applications to special means of real numbers are also given.

Key Words: n-polynomial convexity, n-polynomial P-function, Hermite-Hadamard inequality.

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§1. Preliminaries

Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double inequality is well known as the Hermite-Hadamard inequality []5. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f.

In [4], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1.1 *A nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality*

$$f(tx+(1-t)y) \leq f(x)+f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Theorem 1.1 *Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)\,dx \leq 2\left[f(a)+f(b)\right]. \tag{1.2}$$

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In [10], Tekin et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1.2 *Let $n \in \mathbb{N}$. A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial convex function if for every $x, y \in I$ and $t \in [0,1]$,*

$$f\left(tx+(1-t)y\right) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^{s}\right]f(x)+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^{s}\right]f(y).$$

Theorem 1.2([10]) *Let $f : [a,b] \to \mathbb{R}$ be a n-polynomial convex function. If $a < b$ and $f \in L[a,b]$, then the following Hermite-Hadaamrd type inequalities hold:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b}f(x)dx \leq \left(\frac{f(a)+f(b)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}.$$

The main purpose of this paper is to introduce the concept of n-polynomial P-function which is connected with the concepts of P-function and n-polynomial convex function and establish some new Hermite-Hadamard type inequality for these classes of functions. In recent years many authors have studied error estimations Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [1, 2, 3, 4, 6, 7, 8, 9, 10].

§2. Definition of n-Polynomial P-Function

In this section, we introduce a new concept, which is called n-polynomial P-function and we give by setting some algebraic properties for the n-polynomial P-function, as follows:

Definition 2.1 *Let $n \in \mathbb{N}$. A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial P-function if for every $x, y \in I$ and $t \in [0,1]$,*

$$f\left(tx+(1-t)y\right) \leq \frac{1}{n}\sum_{s=1}^{n}\left[2-t^{s}-(1-t)^{s}\right]\left[f(x)+f(y)\right]. \tag{2.1}$$

We will denote by $POLP(I)$ the class of all n-polynomial P-functions on interval I. Notice that every n-polynomial P-function is a h-convex function with the function

$$h(t)=\frac{1}{n}\sum_{s=1}^{n}\left[2-t^{s}-(1-t)^{s}\right].$$

Therefore, if $f, g \in POLC(I)$, then

(i) $f+g \in POLCP(I)$ *and for* $c \in \mathbb{R}$ $(c \geq 0)$ $cf \in POLCP(I)$ (see [11], Proposition 9).

(ii) If f and g be a similarly ordered functions on I , then $fg \in POLCP(I)$.(see [11], Proposition 10).

Also, if $f : I \to J$ is a convex and $g \in POLCP(J)$ and nondecreasing, then $g \circ f \in POLCP(I)$ (see [11], Theorem 15).

Remark 2.1 We note that if f satisfies (2.1), then f is a nonnegative function. Indeed, if we rewrite the inequality (2.1) for $t = 0$, then

$$f(y) \leq f(x) + f(y)$$

for every $x, y \in I$. Thus we have $f(x) \geq 0$ for all $x \in I$.

Proposition 2.1 *Every nonnegative P-function is also a n-polynomial P-function.*

Proof The proof is clear from the following inequalities

$$t \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right] \text{ and } 1-t \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right]$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. In this case, we can write

$$1 \leq \frac{1}{n}\sum_{s=1}^{n}\left[2-t^s-(1-t)^s\right].$$

Therefore, the desired result is obtained. □

We can give the following corollary for every nonnegative convex function is also a P-function.

Corollary 2.1 *Every nonnegative convex function is also a n-polynomial P-function.*

Theorem 2.1 *Let $b > a$ and $f_\alpha : [a,b] \to \mathbb{R}$ be an arbitrary family of n-polynomial P-function and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a,b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is a n-polynomial P-function on J.*

Proof Let $t \in [0,1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned} f(tx+(1-t)y) &= \sup_\alpha f_\alpha(tx+(1-t)y) \\ &\leq \sup_\alpha \left[\frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[f_\alpha(x)+f_\alpha(y)\right]\right] \\ &\leq \frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[\sup_\alpha f_\alpha(x)+\sup_\alpha f_\alpha(y)\right] \\ &= \frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[f(x)+f(y)\right] < \infty. \end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a n-polynomial P-function on J. □

§3. Hermite-Hadamard Inequality for n-Polynomial P-Functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for n-polynomial P-functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 3.1 *Let $f:[a,b]\to\mathbb{R}$ be a n-polynomial P-function. If $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type inequalities hold*

$$\frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\left(\frac{f(a)+f(b)}{n}\right)\sum_{s=1}^{n}\frac{2s}{s+1}. \tag{3.1}$$

Proof From the property of the n-polynomial P-function of f, we get

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{[ta+(1-t)b]+[(1-t)a+tb]}{2}\right)\\
&= f\left(\frac{1}{2}[ta+(1-t)b]+\frac{1}{2}[(1-t)a+tb]\right)\\
&\le \frac{1}{n}\sum_{s=1}^{n}\left[2-2\left(\frac{1}{2}\right)^s\right][f(ta+(1-t)b)+f((1-t)a+tb)].
\end{aligned}$$

By taking integral in the last inequality with respect to $t\in[0,1]$, we deduce that

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{n}\sum_{s=1}^{n}\left[2-2\left(\frac{1}{2}\right)^s\right]\left[\int_0^1 f(ta+(1-t)b)\,dt+\int_0^1 f((1-t)a+tb)\,dt\right]\\
&= \frac{4}{b-a}\left(\frac{n+2^{-n}-1}{n}\right)\int_a^b f(x)dx.
\end{aligned}$$

By using the property of the n-polynomial P-function of f, if the variable is changed as $x=ta+(1-t)b$, then

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)du &= \int_0^1 f(ta+(1-t)b)\,dt\\
&\le \int_0^1\left[\frac{1}{n}\sum_{s=1}^{n}[2-t^s-(1-t)^s][f(a)+f(b)]\right]dt\\
&= \frac{f(a)+f(b)}{n}\sum_{s=1}^{n}\int_0^1[2-t^s-(1-t)^s]\,dt\\
&= \left[\frac{f(a)+f(b)}{n}\right]\sum_{s=1}^{n}\frac{2s}{s+1},
\end{aligned}$$

where

$$\int_0^1[2-t^s-(1-t)^s]\,dt=\frac{2s}{s+1}.$$

This completes the proof of theorem. □

Remark 3.1 In case of $n = 1$, the inequality (3.1) coincides with the the inequality (1.2).

§4. New Inequalities for n-Polynomial P-Functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is n-polynomial P-function. Dragomir and Agarwal [3] used the following lemma.

Lemma 4.1([3]) *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt.$$

Theorem 4.1 it Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is n-polynomial P-function on interval $[a, b]$, then the following inequality

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}\right] A\left(\left|f'(a)\right|, \left|f'(b)\right|\right) \quad (4.1)$$

holds for $t \in [0, 1]$.

Proof Using Lemma 4.1 and the inequality

$$\left|f'(ta+(1-t)b)\right| \le \frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[|f'(a)|+|f'(b)|\right],$$

we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt\right| \\
&\le \frac{b-a}{2}\int_0^1 |1-2t|\left(\frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[|f'(a)|+|f'(b)|\right]\right)dt \\
&\le \frac{b-a}{2n}\left[|f'(a)|+|f'(b)|\right]\sum_{s=1}^{n}\int_0^1 |1-2t|\left(2-t^s-(1-t)^s\right)dt \\
&= \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}\right] A\left(\left|f'(a)\right|, \left|f'(b)\right|\right),
\end{aligned}$$

where

$$\int_0^1 |1-2t|\left(2-t^s-(1-t)^s\right)dt = \frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}$$

and A is the arithmetic mean. This completes the proof of theorem. □

Corollary 4.1 *If we take $n = 1$ in the inequality (4.1), we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right). \tag{4.2}$$

Theorem 4.2 *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$. If $|f'|^q$, $q > 1$, is an n-polynomial P-function on interval $[a,b]$, then the following inequality holds for $t \in [0,1]$.*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right),\end{aligned} \tag{4.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and A is the arithmetic mean.

Proof Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$|f'(ta+(1-t)b)|^q \leq \frac{1}{n}\sum_{s=1}^{n}\left(2-t^s-(1-t)^s\right)\left[|f'(a)|^q+|f'(b)|^q\right]$$

which is the n-polynomial P-function of $|f'|^q$, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2}\left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{n}\int_0^1\sum_{s=1}^{n}\left[2-t^s-(1-t)^s\right]dt\right)^{\frac{1}{q}}\end{aligned}$$

$$=\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right)$$

where

$$\int_0^1|1-2t|^p\,dt=\frac{1}{p+1},\qquad \int_0^1\left[2-t^s-(1-t)^s\right]dt=\frac{2s}{s+1}$$

This completes the proof of theorem. □

Corollary 4.2 *If we take $n=1$ in the inequality (4.3), we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left[\frac{1}{2(p+1)}\right]^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).$$

Theorem 4.3 *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$ and assume that $f' \in L[a,b]$. If $|f'|^q, q \geq 1$, is an n-polynomial P-function on the interval $[a,b]$, then the following inequality holds for $t \in [0,1]$.*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right). \qquad (4.4)\end{aligned}$$

Proof From Lemma 4.1, well known power-mean integral inequality and the property of the n-polynomial P-function of $|f'|^q$, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{n}\int_0^1\sum_{s=1}^{n}|1-2t|\left[2-t^s-(1-t)^s\right]dt\right)^{\frac{1}{q}} \\ &= \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\end{aligned}$$

where

$$\begin{aligned}\int_0^1 |1-2t|dt &= \frac{1}{2}, \\ \int_0^1 |1-2t|\left[1-(1-t)^s\right]dt &= \frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}.\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.3 *Under the assumption of Theorem 4.3, If we take $q=1$ in the inequality (4.4), then we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{n}\left(\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^s}\right) A\left(|f'(a)|, |f'(b)|\right)$$

This inequality coincides with the inequality (4.1).

Corollary 4.4 *Under the assumption of Theorem* 4.3*, If we take* $n = 1$ *in the inequality (*4.4*), then we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2^{2-\frac{1}{q}}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

which is identical to the inequality in [1, Theorem 2.3].

Corollary 4.5 *Under the assumption of Theorem* 4.3*, If we take* $n = 1$ *and* $q = 1$ *in the inequality (*4.4*), then we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right).$$

*This inequality coincides with the inequality (*4.2*).*

§5. Applications for Special Means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a, b with $b > a$.

1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a,b \geq 0,$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a,b \geq 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0,$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases} ; \quad a,b > 0$$

5. The p-logaritmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R}\setminus\{-1,0\} \\ a, & a = b \end{cases} ; \quad a,b > 0.$$

6. The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b > 0,$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships

$$H \le G \le L \le I \le A.$$

are known in the literature. It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 5.1 *Let $a,b \in [0,\infty)$ with $a < b$ and $n \in (-\infty,0) \cup [1,\infty) \setminus \{-1\}$. Then, the following inequalities are obtained:*

$$\frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right) A^n(a,b) \le L_n^n(a,b) \le A(a^n,b^n)\frac{2}{n}\sum_{s=1}^{n}\frac{2s}{s+1}.$$

Proof The assertion follows from the inequalities (3.1) for the function

$$f(x) = x^n, \quad x \in [0,\infty). \qquad \square$$

Proposition 5.2 *Let $a,b \in (0,\infty)$ with $a < b$. Then, the following inequalities are obtained*

$$\frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right) A^{-1}(a,b) \le L^{-1}(a,b) \le \frac{2}{n}H^{-1}(a,b)\sum_{s=1}^{n}\frac{2s}{s+1}.$$

Proof The assertion follows from the inequalities (3.1) for the function

$$f(x) = x^{-1}, \quad x \in (0,\infty). \qquad \square$$

Proposition 5.3 *Let $a,b \in (0,1]$ with $a < b$. Then, the following inequalities are obtained*

$$\frac{2\ln G(a,b)}{n}\sum_{s=1}^{n}\frac{2s}{s+1} \le \ln I(a,b) \le \frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right)\ln A(a,b).$$

Proof The assertion follows from the inequalities (3.1) for the function

$$f(x) = -\ln x, \quad x \in (0,1]. \qquad \square$$

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