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Separation Axioms (T_1) on Fuzzy Bitopological Spaces in Quasi-Coincidence Sense

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Abstract: In this paper, we introduce some new definitions of T_1 separation on fuzzy bitopological space in quasi-coincidence sense and establish relations among them and their counterparts. We show that the notions satisfy good extension, hereditary, productive and projective properties. We present their one-one, onto, fuzzy open and fuzzy continuous mappings. In addition, we also discuss the initial and final fuzzy bitopological spaces in quasi-coincidence sense.

Key Words: Fuzzy bitopological space, quasi-coincidence, fuzzy T_1 bitopological space, good extension, mapping, initial and final fuzzy topology.

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§1. Introduction

The fuzzy set was first explored in [36] and this concept extended to fuzzy topological spaces in [4]. Much research has been done to extend the theory of fuzzy topological spaces in various directions; in particular, fuzzy normality [11, 23], fuzzy uniformity [12], fuzzy regularity [1], fuzzy topological representation [5], separations on fuzzy topological spaces [2, 9, 20, 22], fuzzy topological groups [6], fuzzy bitopological spaces [2, 3, 10, 14, 26], product of fuzzy topological spaces [13], strong-separation and strong countability on fuzzy topological spaces [31], supra fuzzy topological spaces [7, 8, 18] and infra fuzzy topological spaces [28, 33]. One of the important topics in fuzzy mathematics is fuzzy bitopological space with separation axioms, which continuously attracted significant international attention.

The research for fuzzy bitopological spaces started in early nineties [14]. The fuzzy bitopological spaces with separation axioms has become attractive as these spaces possess many desirable properties and can be found throughout various areas in fuzzy topologies. Recent progress has been made constructing separation axioms on fuzzy bitopological spaces in [14,

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27]. One most studied in separation axioms on fuzzy bitopological spaces is T_1 separation [27].

The purpose of this paper is to further contribute to the development of fuzzy bitopological spaces, especially on fuzzy T_1 bitopological spaces in quasi-coincidence sense. In this paper, we define fuzzy T_1 bitopological space in quasi-coincidence sense [19, 21, 27]. We show that the definitions of the T_1 separation satisfy the good extension property. We also present the hereditary, order preserving, productive, and projective properties of these new concepts. In addition, we discuss the initial and final fuzzy bitopologies of the T_1 separation.

§2. Basic Notions and Preliminary Results

In this section, we review some concepts, which will be needed in the sequel. In this paper, X and Y are always presented non-empty sets.

Definition 2.1([36]) *A function u from X into the unit interval I is called a fuzzy set in X. For every $x \in X, u(x) \in I$ is called the grade of membership of x in u. Some authors say that u is a fuzzy subset of X instead of saying that u is a fuzzy set in X. The class of all fuzzy sets from X into the closed unit interval I is denoted by I^X.*

Definition 2.2([24]) *A fuzzy set u in X is called a fuzzy singleton if and only if $u(x) = r, 0 < r \leq 1$, for a certain $x \in X$ and $u(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in X will be denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in u$ if and only if $r \leq u(x)$*

Definition 2.3([35]) *A fuzzy set u in X is called a fuzzy point if and only if $u(x) = r, 0 < r < 1$, for a certain $x \in X$ and $u(y) = 0$ for all points y of X except x. The fuzzy point is denoted by x_r and x is its support.*

Definition 2.4([14]) *A fuzzy singleton x_r is said to be quasi-coincidence with u, denoted by $x_r qu$ if and only if $u(x) + r > 1$. If x_r is not quasi-coincidence with u, we write $x_r \bar{q} u$ and defined as $u(x) + r \leq 1$.*

Definition 2.5([4]) Let f be a mapping from a set X into a set Y and v be a fuzzy subset of Y. Then the inverse of v written as $f^{-1}(v)$ is a fuzzy subset of X defined by $f^{-1}(v)(x) = v(f(x))$, for $x \in X$.

Definition 2.6([25]) The function $f : (X, t) \to (Y, s)$ is called fuzzy continuous if and only if for every $v \in s, f^{-1}(v) \in t$, the function f is called fuzzy homeomorphic if and only if f is bijective and both f and f^{-1} are fuzzy continuous.

Definition 2.7([17]) The function $f : (X, t) \to (Y, s)$ is called fuzzy open if and only if for every open fuzzy set u in $(X, t), f(u)$ is open fuzzy set in (Y, s).

Definition 2.8([29]) Let f be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real α, then f is called lower semi continuous function.

Definition 2.9([4]) A fuzzy topology t on X is a collection of members of I^X which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair (X, t) is called a fuzzy topological space (in short fts) and members of t are called $t-open$ fuzzy sets. A fuzzy set μ is called a $t-$ closed (or simply closed) fuzzy set if $1-\mu \in t$.

Definition 2.10([30]) *A bitopological space (X, S, T) is called pairwise$-T_1(PT_1$ in short) if for all $x, y \in X, x \neq y$, there exist $U \in S, V \in T$ such that $x \in U, y \notin U$ and $x \notin V, y \in V$.*

A fuzzy bitopological property P is called hereditary if each subspace of a fuzzy bitopological space with property P, also has property P.

Definition 2.11([34]) *Let $\{(X_i, s_i, t_i) : i \in \Lambda\}$ is a family of fuzzy bitopological spaces. Then the space $(\prod X_i, \prod s_i, \prod t_i)$ is called the product fuzzy bitopological space of the family $\{(X_i, s_i, t_i) : i \in \Lambda\}$, where $\prod s_i$ and $\prod t_i$ denote the usual product fuzzy topologies of the families $\{\prod s_i : i \in \Lambda\}$ and $\{\prod t_i : i \in \Lambda\}$ of the fuzzy topologies respectively on X.*

A fuzzy bitopological property P is called productive if the product of fuzzy bitopological spaces of a family of fuzzy bitopological space, each having property P, has property P.

A fuzzy bitopological property P is called projective if for a family of fuzzy bitopological space $\{(X_i, s_i, t_i) : i \in \Lambda\}$, the product fuzzy bitopological space $(\prod X_i, \prod s_i, \prod t_i)$ has property P implies that each coordinate space has property P.

Definition 2.12([15]) Let (X, T) be an ordinary topological space. The set of all lower semi continuous functions from (X, T) into the closed unit interval I equipped with the usual topology constitutive a fuzzy topology associated with (X, T) and is denoted by $(X, \omega(T))$.

Definition 2.13([16]) The initial fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i, t_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : X \to (X_i, t_i)\}_{i\in\Lambda}$ is the smallest fuzzy topology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in t_i\}_{i\in\Lambda}$.

Definition 2.14([16]) The final fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i, t_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : (X_i, t_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy topology on X making each f_i fuzzy continuous.

Definition 2.15([26]) A function f from a fuzzy bitopological space (X, s, t) into a fuzzy bitopological space (Y, s_1, t_1) is called fuzzy $FP-$continuous if and only if $f : (X, s) \to (Y, s_1)$ and $f : (X, t) \to (Y, t_1)$ are both fuzzy continuous.

Theorem 2.1([3]) A bijective mapping from an fts (X, t) to an fts (Y, s) preserves the value of a fuzzy singleton (fuzzy point).
Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

§3. **Fuzzy T_1 Bitopological Space**

In this section, we present some new notions on fuzzy T_1 bitopological spaces and their relevant results. We also discuss existing some well-known properties using these new concepts and establish relationships between these new notions and the relevant existing notions.

Definition 3.1 *A fuzzy bitopological space* (X, s, t) *is called*

(a) $FPT_1(i)$ *if and only if for any pair* $x_m, y_n \in S(X)$ *with* $x \neq y$*, there exist* $u, v \in s \cup t$ *such that* $x_m q u, y_n \bar{q} u$ *and* $y_n q v, x_m \bar{q} v$*;*

(b) $FPT_1(ii)$ *if and only if for any pair* $x_m, y_n \in S(X)$ *with* $x \neq y$*, there exist* $u, v \in s \cup t$ *such that* $x_m q u, y_n \cap u = 0$ *and* $y_n q v, x_m \cap v = 0$*;*

(c) $FPT_1(iii)$ *if and only if for any pair of fuzzy points* x_m, y_n *in* X *with* $x \neq y$*, there exist* $u, v \in s \cup t$ *such that* $x_m \in u, y_n \bar{q} u$ *and* $y_n \in v, x_m \bar{q} v$*;*

(d) $FPT_1(iv)$ *if and only if for any pair of distinct fuzzy points* p, q *in* X*, there exists a fuzzy set* $u, v \in s \cup t$ *such that* $p \in u, q \cap u = 0$ *or* $q \in u, p \cap u = 0$*;*

(e) $FPT_1(v)$ *if and only if for all* $x, y \in X$ *with* $x \neq y$*, there exist* $u, v \in s \cup t$ *such that* $u(x) = 1, u(y) = 0$ *and* $v(y) = 1, v(x) = 0$.

Here it is mentioned that $FPT_1(iii)$ and $FPT_1(iv)$ are according to Sufiya et al.[32], $FPT_1(i)$ is according to A A Nouh [27], and $FPT_1(v)$ is according to M. Srivastava and R. Srivastava [30].

The examples of definitions of $FPT_1(i)$ and $FPT_1(ii)$ are as follows:

Example 3.1 Let $X = \{x, y\}$, $u, v \in I^X$ with $u(x) = 1$, $u(y) = 0$ and $v(y) = 1$, $v(x) = 0$ and t be the fuzzy topology on X generated by $\{0, u, v, 1\}$ and s be the fuzzy topology on X generated by $\{constants\}$. Also, let $x_m, y_n \in S(X)$ with $x \neq y$, then $u(x) + m > 1$ and $u(y) + s \leq 1$ for $m, n \in (0, 1]$. Thus $x_m q u, y_n \bar{q} u$. Similarly, $y_n q v, x_m \bar{q} v$. Hence (X, s, t) is $FPT_1(i)$ as $u, v \in s \cup t$. Also, as $u(y) = 0, y_n \cap u = 0$ and similarly $x_m \cap v = 0$. Therefore, (X, s, t) is $FPT_1(ii)$.

Theorem 3.1 *Let* (X, s, t) *be a fuzzy bitopological space and* $(X, s \cup t)$ *be a fuzzy topological space. If* (X, s, t) *is* FPT_1 *then* $(X, s \cup t)$ *is fuzzy* T_1 *topological space.*

Proof Let (X, s, t) be FPT_1. Since $s \subseteq s \cup t$ and $t \subseteq s \cup t$, it follows immediately that $(X, s \cup t)$ is FT_1. □

Theorem 3.2 *If the fuzzy topological space* (X, s) *and* (X, t) *are both fuzzy* $T_1(j)$ *topological spaces, then their corresponding fuzzy bitopological space* (X, s, t) *is* $FPT_1(j)$*, for* $j = i, ii$*. But the converse is not true in general.*

Proof Let (X, s) and (X, t) are both $FT_1(j)$. Then their corresponding fuzzy bitopological space (X, s, t) is $FPT_1(j)$, for $j = i, iii$ as $s \subseteq s \cup t$ and $t \subseteq s \cup t$. To prove (X, s, t) is $FPT_1(j)$ does not imply (X, s) and (X, t) are both $FT_1(j)$, for $j = i, ii$, the following is its a counter example. □

Example 3.2 Let $X = \{x, y\}$, $u, v \in I^X$ and t be the fuzzy topology on X generated by $\{u, v\} \cup \{constants\}$, with $u(x) = 1$, $u(y) = 0$ and $v(y) = 1$, $v(x) = 0$. Also, let s be the fuzzy topology on X generated by $\{constants\}$. Then, for any $0 < m \leq 1$ and $0 < n \leq 1, u(x) + m > 1$ and $u(y) + n \leq 1$, which imply that $x_m q u, y_n \bar{q} u$. Similarly, $y_n q v, x_m \bar{q} v$. Also, $u(y) = 0 \Rightarrow y_n \cap u = 0$ and similarly $x_m \cap v = 0$. As $u, v \subseteq s \cup t, (X, s, t)$ is $FPT_1(j)$ but (X, s) is not $FT_1(j)$, for $j = i, ii$.

Theorem 3.3 *If a fuzzy bitopological space (X, s, t) is $FPT_1(j)$ then (X, s, t) is $FPT_0(j)$, for $j = i, ii, iii, iv, v$.*

Proof The proof is obvious. □

Theorem 3.4 *For a fuzzy bitopological space (X, s, t) the implications in Figure* 1 *are true.*

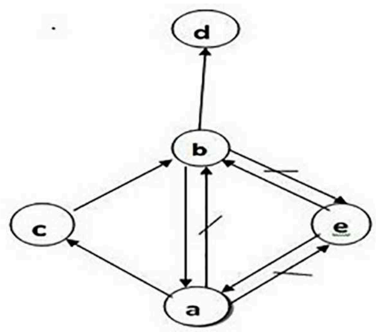

Figure 1

Proof $(c) \Rightarrow (b)$: Let (X, s, t) be $FPT_1(iii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Also let $r > 1 - m$ for $0 < m < 1$. Since (X, s, t) is $FPT_1(iii)$, there exist fuzzy sets $u, v \in s \cup t$ such that $x_r \in u, y_1 \bar{q} u$ and $y_1 \in v, x_r \bar{q} v$, where x_r and y_1 are distinct fuzzy points in X. Now, $x_r \in u \Rightarrow u(x) \geq r > 1 - m \Rightarrow u(x) + m > 1 \Rightarrow u(x) + m > 1$ for $0 < m \leq 1$ also. $\Rightarrow x_m q u$ when $x_m \in S(X)$ and $y_1 \bar{q} u \Rightarrow u(y) + 1 \leq 1 \Rightarrow u(y) \leq 1 - 1 = 0 \Rightarrow u(y) = 0 \Rightarrow y_n \cap u = 0$ for $0 < n \leq 1$.

Similarly, it is easy to prove that $y_n q v$ and $x_m \cap v = 0$. It follows that for any fuzzy singletons x_m, y_n in X with $x \neq y$ there exist $u, v \in s \cup t$ such that $x_m q u, y_n \cap u = 0$ and $y_n q v, x_m \cap v = 0$. Thus (X, s, t) is $FPT_1(ii)$.

$(b) \Rightarrow (d)$: Let x_m, y_n be distinct fuzzy points in X and $0 < r \leq 1, 0 < s \leq 1$ with $r \leq 1 - m, s \leq 1 - n$. Since (X, s, t) is $FPT_1(ii)$, there exist $u, v \in s \cup t$ such that $x_r q u, y_s \cap u = 0$ and $y_s q v, x_r \cap v = 0$.

Now, $x_r q u \Rightarrow u(x) + r > 1 \Rightarrow u(x) > 1 - r \geq m \Rightarrow u(x) \geq m \Rightarrow x_m \in u$ and $y_s \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow y_n \cap u = 0$. Similarly, we can prove that $y_n \in v$ and $x_m \cap v = 0$.

It follows that for any distinct fuzzy points x_m, y_n in X with $x \neq y$ there exist $u, v \in s \cup t$ such that $x_m \in u, y_n \cap u = 0$ and $y_n \in v, x_m \cap v = 0$. Thus (X, s, t) is $FPT_1(iv)$.

$(b) \Rightarrow (a)$: Let (X, s, t) be $FPT_1(ii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_1(ii)$, there exist fuzzy sets $u, v \in s \cup t$ such that $x_m q u, y_n \cap u = 0$ and $y_n q v, x_m \cap v = 0$. To prove (X, s, t) is $FPT_1(i)$, it is only needed to prove that $y_n \bar{q} u$ and $x_m \bar{q} v$.

Now, $y_n \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$ and similarly $x_m \bar{q} v$. Thus (X, s, t) is $FPT_1(i)$. To show $(a) \not\Rightarrow (b)$, we give a counter example in Example 3.3.

Example 3.3 Let $X = \{x, y\}$, $u, v \in I^X$ be given by $u(x) = 1$, $u(y) = 0.1$, $v(y) = 1$, $v(x) = 0.1$. Let us consider the fuzzy topology $s \cup t$ on X generated by $\{0, u, v, 1\}$. For $0 < m \leq 1, 0 < n < 0.9, u(x) + m > 1 \Rightarrow x_m q u$ and $u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. Similarly, $y_n q v$ and $x_m \bar{q} v$. Thus (X, s, t) is $FPT_1(i)$. But $u(y) \neq 0 \Rightarrow y_n \cap u \neq 0$. Also, $v(x) \neq 0 \Rightarrow x_m \cap v \neq 0$. Thus (X, s, t)

is not $FPT_1(ii)$.

$(e) \Rightarrow (a)$: Let (X, s, t) be $FPT_1(v)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_1(v)$, there exist fuzzy sets $u, v \in s \cup t$ such that $u(x) = 1, u(y) = 0$ and $v(y) = 1, v(x) = 0$. Now, $u(x) = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m q u$ and $u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. Similarly, it is easy to prove that $y_n q v$ and $x_m \bar{q} v$. Thus (X, s, t) is $FPT_1(i)$. To show $(a) \not\Rightarrow (e)$, we give a counter example in Example 3.4

Example 3.4 Let $X = \{x, y\}$, $u, v \in I^X$ be given by $u(x) = 1 - \gamma$, $u(y) = 0$, $v(y) = 1 - \delta$, $v(x) = 0$, where $\gamma = m/2, \delta = n/2$ for $m, n \in (0, 1]$. Let the fuzzy topology $s \cup t$ on X generated by $\{0, u, v, 1\} \cup \{constants\}$.

Now, $u(x) = 1 - \gamma \Rightarrow u(x) = 1 - m/2 \Rightarrow u(x) + m/2 = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m q u$ and $u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. In the similar way, $y_n q v$ and $x_m \bar{q} v$. Thus (X, s, t) is $FPT_1(i)$. But $u(x) \neq 1$ and $v(y) \neq 1$. Thus (X, s, t) is not $FPT_1(v)$.

$(a) \Rightarrow (c)$: As $(b) \Rightarrow (d)$ we can say that $(a) \Rightarrow (c)$.

$(e) \Rightarrow (b)$: Let (X, s, t) be $FPT_1(v)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_1(v)$, there exist fuzzy sets $u, v \in s \cup t$ such that $u(x) = 1, u(y) = 0$ and $v(y) = 1, v(x) = 0$. Now, $u(x) = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m q u$ and $u(y) = 0 \Rightarrow y_n \cap u = 0$. Similarly, we can show that $y_n q v$ and $x_m \cap v = 0$. Thus (X, s, t) is $FPT_1(ii)$. A counter example in Example 3.5 shows that $(b) \not\Rightarrow (e)$.

Example 3.5 Let $X = \{x, y\}$, $u, v \in I^X$ be given by $u(x) = 1 - \gamma$, $u(y) = 0$, $v(y) = 1 - \delta$, $v(x) = 0$, where $\gamma = m/2, \delta = n/2$ for $m, n \in (0, 1]$. Let the fuzzy topology $s \cup t$ on X generated by $\{0, u, v, 1\} \cup \{constants\}$.

Now, $u(x) = 1 - \gamma \Rightarrow u(x) = 1 - m/2 \Rightarrow u(x) + m/2 = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m q u$ and $u(y) = 0 \Rightarrow y_n \cap u = 0$. In the similar way, $y_n q v$ and $x_m \cap v = 0$. Thus (X, s, t) is $FPT_1(ii)$. But $u(x) \neq 1$ and $v(y) \neq 1$. Thus (X, s, t) is not $FPT_1(v)$. Thus proof is completed. □

Theorem 3.5 *Let (X, S, T) be a bitopological space. Then (X, S, T) is PT_1 if and only if $(X, \omega(S), \omega(T))$ is $FPT_1(j)$, where $j = i, ii, iii, iv, v$.*

Proof Let (X, S, T) be a PT_1 topological space. We shall prove that $(X, \omega(S), \omega(T))$ is $FPT_1(ii)$. Let x, y in X with $x \neq y$. Since (X, S, T) be a PT_1 topological space hence there exists $U, V \in S \cup T$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$. From the definition of lower semi continuous function, $1_U, 1_V \in (\omega(S) \cup \omega(T))$, i.e., $1_U \in \omega(S)$ or $1_U \in \omega(T)$. Then $1_U(x) = 1 \Rightarrow 1_U(x) + m > 1 \Rightarrow x_m q 1_U$ and $1_U(y) = 0 \Rightarrow y_n \cap 1_U = 0$.

Similarly, we can prove that $y_n q 1_V$ and $x_m \cap 1_V = 0$. Hence $(X, \omega(S), \omega(T))$ is $FPT_1(ii)$.

Conversely, let $(X, \omega(S), \omega(T))$ is $FPT_1(ii)$. It is required to prove that (X, S, T) be a PT_1 topological space. Let x, y in X with $x \neq y$. Since $(X, \omega(S), \omega(T))$ is $FPT_1(ii)$, we have for any fuzzy singletons x_m, y_n in X, there exist $u, v \in \omega(S) \cup \omega(T)$ such that $x_m q u, y_n \cap u = 0$ and $y_n q v, x_m \cap v = 0$.

Now, $x_m q u \Rightarrow u(x) + m > 1 \Rightarrow u(x) > 1 - m = \alpha \Rightarrow x \in u^{-1}(\alpha, 1]$ And $y_n \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow u(y) \leq 1 - n = \alpha \Rightarrow u(y) \leq \alpha \Rightarrow y \notin u^{-1}(\alpha, 1]$. Similarly, we can prove that $y \in v^{-1}(\alpha, 1]$ and $x \notin v^{-1}(\alpha, 1]$. Also, $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in S \cup T$. Hence (X, S, T)

be a PT_1 topological space. Proof for $j = i, iii, iv, v$ is similar to above. □

§4. Hereditary, Productive and Projective Properties

In this section, we describe the hereditary, productive and projective properties on our given concepts. The first theorem is on hereditary property and the second one is on productive and projective properties.

Theorem 4.1 *If (X, s, t) be a fuzzy bitopological space and $A \subseteq X, s_A = \{u/A : u \in s\}, t_A = \{v/A : v \in t\}$ and (X, s, t) is $FPT_1(j)$ then (A, s_A, t_A) is $FPT_1(j)$, where $j = i, ii, iii, iv, v$.*

Proof We first prove this theorem for $j = ii$ and remaining are similar. Let (X, s, t) is $FPT_1(ii)$ and x_m, y_n are fuzzy singletons in A with $x \neq y$. Since $A \subseteq X, x_m, y_n$ are also fuzzy singletons in X. Also since (X, s, t) is $FPT_1(ii)$, there exist $u, v \in s \cup t$ such that $x_m q u, y_n \cap u = 0$ and $y_n q v, x_m \cap v = 0$. For $A \subseteq X$, we have $u/A, v/A \in s_A \cup t_A$.

Now, $x_m q u \Rightarrow u(x) + m > 1, x \in X \Rightarrow u/A(x) + m > 1, x \in A \subseteq X \Rightarrow x_m q u/A$ and $y_n \cap u = 0 \Rightarrow u(y) = 0, y \in X \Rightarrow u/A(y) = 0, y \in A \subseteq X \Rightarrow y_n \cap u/A = 0$. Similarly, we can show that $y_n q v/A, x_m \cap v/A = 0$. Therefore, (A, s_A, t_A) is $FPT_1(ii)$. □

Theorem 4.2 *If $\{(X_i, s_i, t_i) : i \in \Lambda\}$ is a family of fuzzy bitopological spaces then the product fuzzy bitopological space $(\prod X_i, \prod s_i, \prod t_i) = (X, s, t)$ is $FPT_1(j)$ if and only if each coordinate space (X_i, s_i, t_i) is $FPT_1(j)$, where $j = i, ii, iii, iv, v$.*

Proof Let for all $i \in \Lambda, (X_i, s_i, t_i)$ is $FPT_1(ii)$ space. We have to prove that (X, s, t) is $FPT_1(ii)$. Let x_m, y_n be fuzzy singletons in X with $x \neq y$. Then $(x_i)_m, (y_i)_n$ are fuzzy singletons with $x_i \neq y_i$ for some $i \in \Lambda$. Since(X_i, s_i, t_i) is $FPT_1(ii)$, there exist $u_i, v_i \in s_i \cup t_i$ such that $(x_i)_m q u_i, (y_i)_n \cap u_i = 0$ and $(y_i)_n q v_i, (x_i)_m \cap v_i = 0$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$.

Now, $(x_i)_m q u_i \Rightarrow u_i(x_i) + m > 1 \Rightarrow u_i(\pi_i(x)) + m > 1 \Rightarrow (u_i \circ \pi_i)(x) + m > 1 \Rightarrow x_m q(u_i \circ \pi_i)$ and $(y_i)_n \cap u_i = 0 \Rightarrow u_i(y_i) = 0 \Rightarrow u_i(\pi_i(y)) = 0 \Rightarrow (u_i \circ \pi_i)(y) = 0 \Rightarrow y_n \cap (u_i \circ \pi_i) = 0$. Similarly, we can show that $y_n q(v_i \circ \pi_i), x_m \cap (v_i \circ \pi_i) = 0$. Hence (X, s, t) is $FPT_1(ii)$.

Conversely, let the product fuzzy bitopological space (X, s, t) is $FPT_1(ii)$. It is required to prove that for all $i \in \Lambda, (X_i, s_i, t_i)$ is $FPT_1(ii)$ space. Let a_i be a fixed element in X_i. Let $A_i = \{x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j \text{ for some } i \neq j\}$. Then A_i is a subset of X, and hence (A_i, s_{A_i}, t_{A_i}) is a subspace of (X, s, t). Since (X, s, t) is $FPT_1(ii)$, so (A_i, s_{A_i}, t_{A_i}) is $FPT_1(ii)$. Again, A_i is homeomorphic image of X_i. Therefore, for all $i \in \Lambda, (X_i, s_i, t_i)$ is $FPT_1(ii)$. Similarly, one can prove the others. □

§5. Mappings in Fuzzy T_1 Bitopological Space

We discuss in this section about order preserving property of the notions under one-one, onto, fuzzy open and fuzzy continuous mappings.

Theorem 5.1 *Suppose (X, s, t) and (Y, s_1, t_1) are two fuzzy bitopological spaces and $f : X \to Y$*

is bijective and fuzzy open map. If (X, s, t) is $FPT_1(j)$ then (Y, s_1, t_1) is $FPT_1(j)$, where $j = i, ii, iii, iv, v$.

Proof Let (X, s, t) is $FPT_1(ii)$ and x'_m, y'_n be fuzzy singletons in Y with $x' \neq y'$. Since f is onto then there exist $x, y \in X$ with $f(x) = x', f(y) = y'$ and x_m, y_n are fuzzy points in X with $x \neq y$ as f is one-one. Again, (X, s, t) is $FPT_1(ii)$, there exist $u, v \in s \cup t$ such that $x_m qu, y_n \cap u = 0$ and $y_n qv, x_m \cap v = 0$.

Now, $x_m qu \Rightarrow u(x) + m > 1$ and $y_n \cap u = 0 \Rightarrow u(y) = 0$. Again, $f(u)(x') = \{\sup u(x) : f(x) = x'\} \Rightarrow f(u)(x') = u(x)$ for some x and $f(u)(y') = \{\sup u(y) : f(y) = y'\} \Rightarrow f(u)(y') = u(y)$ for some y. Also, since f is a fuzzy open hence $f(u) \in s_1 \cup t_1$ as $u \in s \cup t$.

Again, $u(x) + m > 1 \Rightarrow (f(u))(x') + m > 1 \Rightarrow x'_m qf(u)$ and $u(y) = 0 \Rightarrow f(u)(y') = 0 \Rightarrow y'_n \cap f(u) = 0$. Similarly, it is easy to show that $y'_n qf(v), x'_m \cap f(v) = 0$. Thus, (Y, s_1, t_1) is $FPT_1(ii)$. Similarly, one can prove the others. □

Theorem 5.2 *Suppose (X, s, t) and (Y, s_1, t_1) are two fuzzy bitopological spaces and $f : X \to Y$ is one-one and fuzzy $FP-$continuous map. If (Y, s_1, t_1) is $FPT_1(j)$, then (X, s, t) is $FPT_1(j)$, where $j = i, ii, iii, iv, v$.*

Proof Let (Y, s_1, t_1) is $FPT_1(ii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Then $(f(x))_m, (f(y))_n$ are fuzzy singletons in Y with $f(x) \neq f(y)$ as f is one-one. Also, since (Y, s_1, t_1) is $FPT_1(ii)$, there exist $u, v \in s_1 \cup t_1$ such that $(f(x))_m qu, (f(y))_n \cap u = 0$ and $(f(y))_n qv, (f(x))_m \cap v = 0$.

Now, $(f(x))_m qu \Rightarrow u(f(x)) + m > 1 \Rightarrow f^{-1}(u(x)) + m > 1 \Rightarrow (f^{-1}(u))(x) + m > 1 \Rightarrow x_m q(f^{-1}(u))$ and $(f(y))_n \cap u = 0 \Rightarrow u(f(y)) = 0 \Rightarrow f^{-1}(u(y)) = 0 \Rightarrow (f^{-1}(u))(y) = 0 \Rightarrow y_n \cap (f^{-1}(u)) = 0$. Since f is fuzzy continuous and $u \in s_1 \cup t_1$ hence $f^{-1}(u) \in s \cup t$. In the same way, it is easy to prove that $y_n \cap q(f^{-1}(v))$ and $x_m \cap (f^{-1}(v)) = 0$. Therefore, (X, s, t) is $FPT_1(ii)$. The proof of other properties is similar to above. □

§6. Initial and Final Fuzzy T_1 Bitopological Space

We define and discuss the initial and final fuzzy bitopologies in this section.

Definition 6.1 *The initial fuzzy bitopology on a set X for the family of fuzzy bitopological spaces $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ and the family of functions $\{f_i : X \to (X_i, s_i \cup t_i)\}_{i\in\Lambda}$ is the smallest fuzzy bitopology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in s_i \cup t_i\}_{i\in\Lambda}$.*

Definition 6.2 *The final fuzzy bitopology on a set X for the family of fuzzy bitopological spaces $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ and the family of functions $\{f_i : (X_i, s_i \cup t_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy bitopology on X making each f_i fuzzy continuous.*

Theorem 6.1 *If $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ is a family of $FPT_1(j)$ fts and $\{f_i : X \to (X_i, s_i \cup t_i)\}_{i\in\Lambda}$, a family of one-one and fuzzy continuous functions, then the initial fuzzy bitopology on X for the family $\{f_i\}_{i\in\Lambda}$ is $FPT_1(j)$, for $j = i, ii, iii, iv, v$.*

Proof We shall prove the above theorem for $j = ii$ and the remaining is similar. Let t, s be the initial fuzzy topologies on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $f_i(x), f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as f_i is one-one. Scince (X_i, s_i, t_i) is $FPT_1(ii)$, then for any two distinct fuzzy singletons $(f_i(x))_r, (f_i(y))_s$ in X_i, there exist fuzzy sets $u_i, v_i \in s_i \cup t_i$ such that $(f_i(x))_r q u_i, (f_i(y))_s \cap u_i = 0$ and $(f_i(y))_s q v_i, (f_i(x))_r \cap v_i = 0$.

Now, $(f_i(x))_r q u_i \Rightarrow u_i(f_i(x)) + r > 1 \Rightarrow f_i^{-1}(u_i)(x) + r > 1$. This is true for every $i \in \Lambda$. So, $\inf f_i^{-1}(u_i)(x) + r > 1$ and $(f_i(y))_s \cap u_i = 0 \Rightarrow u_i(f_i(y)) = 0 \Rightarrow f_i^{-1}(u_i)(y) = 0$. This is true for every $i \in \Lambda$. So, $\inf f_i^{-1}(u_i)(y) = 0$. Let $u = \inf f_i^{-1}(u_i)$. Then $u \in s_i \cup t_i$ as f_i is fuzzy continuous. So $u(x) + r > 1$ and $u(y) = 0$. Hence $x_r q u$ and $y_s \cap u = 0$. Similarly, we can prove that $y_s q v$ and $x_r \cap v = 0$. Therefore, (X, s, t) is $FPT_1(ii)$. □

Theorem 6.2 *If $\{(X_i, s_i, t_i)\}_{i\in\Lambda}$ is a family of $FPT_1(j)$ fts and $\{f_i : (X_i, s_i \cup t_i) \to X\}_{i\in\Lambda}$, a family of fuzzy open and bijective function, then the final fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$ is $FPT_1(j)$, for $j = i, ii, iii, iv, v$.*

Proof We shall prove the above theorem for $j = ii$ and the remaining is similar. Let s, t be the final fuzzy topologies on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $f_i^{-1}(x), f_i^{-1}(y) \in X_i$ and $f_i^{-1}(x) \neq f_i^{-1}(y)$ as f_i is bijective. Since (X_i, s_i, t_i) is $FPT_1(ii)$, then for any two distinct fuzzy singletons $(f_i^{-1}(x))_r, (f_i^{-1}(y))_s$ in X_i, there exist fuzzy sets $u_i, v_i \in s_i \cup t_i$ such that $(f_i^{-1}(x))_r q u_i, (f_i^{-1}(y))_s \cap u_i = 0$ and $(f_i^{-1}(y))_s q v_i, (f_i^{-1}(x))_r \cap v_i = 0$.

Now, $(f_i^{-1}(x))_r q u_i \Rightarrow u_i(f_i^{-1}(x)) + r > 1 \Rightarrow f_i(u_i)(x) + r > 1$. This is true for every $i \in \Lambda$. So, $\inf f_i(u_i)(x) + r > 1$ and $(f_i^{-1}(y))_s \cap u_i = 0 \Rightarrow u_i(f_i^{-1}(y)) = 0 \Rightarrow f_i(u_i)(y) = 0$. This is true for every $i \in \Lambda$. So, $\inf f_i(u_i)(y) = 0$. Let $u = \inf f_i(u_i)$. Then $u \in s_i \cup t_i$ as f_i is fuzzy open. So, $u(x) + r > 1$ and $u(y) = 0$. Hence $x_r q u$ and $y_s \cap u = 0$. Similarly, we can prove that $y_s q v$ and $x_r \cap v = 0$. Therefore, (X, s, t) is $FPT_1(ii)$. □

§7. Conclusion

One of the main results of this paper is introducing some new definitions of fuzzy T_1 bitopological spaces in sense of quasi-coincidence. We present their good extension, hereditary, productive and projective properties. We compare the results with other existing notions and their counterparts' examples [27, 30, 32]. These concepts would be interesting to more expansion on fuzzy bitopological spaces [30] and extending to general fuzzy topological space [4].

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