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R_1 Concepts in Fuzzy Bitopological Spaces with Quasi-Coincidence Sense

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Abstract: In this paper, we have defined some new notions of R_1-separation in fuzzy bitopological spaces using quasi-coincidence sense. We have discuss the relations among our and other such notions. We have observed that all these notions satisfy good extension property. We have shown that these notions are preserved under the one-one, onto and FP-continuous mapping. Moreover, we have obtained some other properties of this new concept. Initial and final topologies are studied here.

Key Words: Quasi-coincidence, fuzzy bitopological spaces, fuzzy pairwise R_1 separations.

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§1. Introduction

The concept of R_1-property in classical topology first defined by Yang [23]. In fuzzy topology, the concept of fuzzy R_1 spaces was first introduced by Hutton and Relly [12] in 1980. Since then much attention has been paid to define such notion by many fuzzy topologist e.g., by Ali [4], Hossain and Ali [11], Caldas [9], Roy and Mukherjee [21], Keskin and Nori [16], Srivastava and Ali [3] and Petricevic [20]. In 2012, Ali And Azom [3] introduced some other definitions of fuzzy R_1-axioms in fuzzy topological spaces. In 1990, Kandil [13] introduced the concept of fuzzy bitopological spaces and in 1991, Kandil [13] first defined R_1-property in fuzzy bitopological spaces. After then Abu Safiya [1, 2], Kandil [14] and Nouh [19] defined several type of R_1-properties.

In this paper, we introduce four notions of R_1–property in fuzzy bitopological spaces by using quasi-coincident sense. We show that all these notions satisfy good extension property. Also hereditary is satisfied by these concepts. We have observed that all these concepts are preserved under one-one, onto and continuous mappings. Finally, we have showed that initial and final fuzzy bitopological spaces satisfy R_1-property.

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§2. Preliminaries

In this section, we recall some known definitions and results useful in the sequel. For details, we refer to [1]–[10].

We give some elementary concepts and results which will be used in the sequel. Throughout this paper, X will be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$, $I_1 = [0, 1)$ and FP(resp P) stands for fuzzy pairwise (resp pairwise). The class of all fuzzy sets on a universe X will be denoted by I^X and fuzzy sets on X will be denoted by u, v, w, etc. Crisp subset of X will be denoted by capital letters U, V, W etc. In this paper (X, t) and (X, s, t) will be denoted fuzzy topological space and fuzzy bitopological space respectively. $x_r q u$ denotes x_r is quasi-coincident with u and $x_r \bar{q} u$ denotes that x_r is not quasi-coincident with u throughout this paper.

We shall follow [5] for the definitions of fuzzy singleton, quasi-coincident, fuzzy topology, image of fuzzy set, the inverse images of a fuzzy set, fuzzy continuous mapping good extension property.

Definition 2.1([13]) *A fuzzy singleton x_r is said to be quasi-coincident with a fuzzy set μ, denoted by $x_r q\mu$ iff $r + \mu(x) > 1$. If x_r is not quasi-coincident with μ, we write $x_r \bar{q}\mu$.*

Definition 2.2([22]) *Let f be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real $\alpha \in I_1$, then f is called lower semi continuous function.*

C. L.Chang [10] have defined fuzzy topology and fuzzy continuous mapping.

Definition 2.3([10]) *A function f from a fuzzy topological space (X, t) into a fuzzy topological space (Y, s) is called fuzzy continuous if and only if for every $u \in s$, $f^{-1}(u) \in t$.*

Definition 2.4([11]) *A fuzzy topological space (X, t) is called*

(*a*) *$FR_1(i)$ iff for each pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever there exists a fuzzy set $\gamma \in t$ with $\gamma(x) \neq \gamma(y)$, then $\exists\, \mu, \lambda \in t$ such that $x_r q\mu$, $y_s q\lambda$ and $\mu\bar{q}\lambda$;*

(*b*) *$FR_1(ii)$ iff for each pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever there exists a fuzzy set $\gamma \in t$ with $\gamma(x) \neq \gamma(y)$, then $\exists\, \mu, \lambda \in t$ such that $x_r \in \mu$, $y_s \in \lambda$ and $\mu \cap \lambda = 0$;*

(*c*) *$FR_1(iii)$ iff for each pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever there exists a fuzzy set $\gamma \in t$ with $\gamma(x) \neq \gamma(y)$, then $\exists\, \mu, \lambda \in t$ such that $x_r \in \mu$, $y_s \in \lambda$ and $\mu \subseteq \lambda^c$;*

(*d*) *$FR_1(iv)$ iff for each pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever there exists a fuzzy set $\gamma \in t$ with $\gamma(x) \neq \gamma(y)$, then $\exists\, \mu, \lambda \in t$ such that $x_r q\mu$, $y_s q\lambda$ and $\mu \cap \lambda = 0$.*

Definition 2.5([15]) *Let X be any non empty set and S and T be any two general topologies on X then the triple (X, S, T) is called a bitopological space.*

Definition 2.6([13]) *A fuzzy bitopological space (fbts, in short) is a triple (X, s, t) where s and t are arbitrary fuzzy topologies on X.*

In previous works we have introduced the following definitions and discussed many related concepts among them.

Definition 2.7([13]) *A fuzzy bitopological space* (X, t_1, t_2) *is called* FPR_0 *if and only if* $x_t \bar{q} t_i.cl(y_r)$ *implies* $y_s \bar{q} t_j.cl(x_t)\,(i, j \in \{1, 2\},\, i \neq j)$.

Definition 2.8([2]) *A fbts* (X, t_1, t_2) *is said to be* PFR_0 *if and only if for any distinct fuzzy points* p *and* q *in* X*, whenever there exists* $\mu \in t_i$ *such that* $p \in \mu$ *and* $q \cap \mu = 0$*, then there exists* $\gamma \in t_j$ *such that* $p \cap \gamma = 0$ *and* $q \in \gamma\,(i,\, j = 1, 2,\, i \neq j)$.

Kelly defines bitopological space in his classical paper [15] as a bitopological space (X, S, T) is called pairwise-R_0 (PR_0, in short) if for all $x, y \in X$, $x \neq y$, whenever $\exists U \in S$ with $x \in U$, $y \notin U$, then $\exists V \in T$ such that $y \in V$, $x \notin V$.

In previous works [6], [7], we introduced the following definitions and discussed many related concepts among them.

Definition 2.9([6]) *A fbts* (X, s, t) *is called* FPT_0*-space iff for every pair of fuzzy singletons* x_p, y_r *in* X *with* $x \neq y$*, there exist fuzzy set* $\mu \in s \cup t$ *such that* $(x_p q \mu,\, y_r \cap \mu = 0)$ *or* $(y_r q \mu, x_p \cap \mu = 0)$.

Definition 2.10([7]) *A fbts* $(X,\, s,\, t)$ *is called* FPT_2 *iff for every pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, there exist fuzzy sets* $\mu \in s$, $\lambda \in t$ *such that* $x_r q \mu$, $y_s q \lambda$ *and* $\mu \cap \lambda = 0$.

§3. Main Results with Proofs

Definition 3.1 *A fbts* $(X,\, s,\, t)$ *is called*

(*a*) $FPR_1(i)$ *iff for each pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, whenever there exists a fuzzy set* $\gamma \in s \cup t$ *with* $\gamma(x) \neq \gamma(y)$*, then* $\exists\, \mu \in s$, $\lambda \in t$ *such that* $x_r q \mu$, $y_s q \lambda$ *and* $\mu \bar{q} \lambda$*;*

(*b*) $FPR_1(ii)$ *iff for each pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, whenever there exists a fuzzy set* $\gamma \in s \cup t$ *with* $\gamma(x) \neq \gamma(y)$*, then* $\exists\, \mu \in s$, $\lambda \in t$ *such that* $x_r \in \mu$, $y_s \in \lambda$ *and* $\mu \cap \lambda = 0$*;*

(*c*) $FPR_1(iii)$ *iff for each pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, whenever there exists a fuzzy set* $\gamma \in s \cup t$ *with* $\gamma(x) \neq \gamma(y)$*, then* $\exists\, \mu \in s$, $\lambda \in t$ *such that* $x_r \in \mu$, $y_s \in \lambda$ *and* $\mu \subseteq \lambda^c$*;*

(*d*) $FPR_1(iv)$ *iff for each pair of fuzzy singletons* x_r, y_s *in* X *with* $x \neq y$*, whenever there exists a fuzzy set* $\gamma \in s \cup t$ *with* $\gamma(x) \neq \gamma(y)$*, then* $\exists\, \mu \in s$, $\lambda \in t$ *such that* $x_r q \mu$, $y_s q \lambda$ *and* $\mu \cap \lambda = 0$.

In general it is true that union of fuzzy topologies is not a topology. But if union of two fuzzy topologies is again a topology then we have the following theorem.

Theorem 3.1 *Let* $(X,\, s,\, t)$ *be a fuzzy bitopological space and* $(X,\, s \cup t)$ *be a fuzzy topological space, then*

$$(X, s, t) \text{ is } FPR_1(j) \Rightarrow (X,\, s \cup t) \text{ is } FR_1(j),$$

where $j = i, ii, iii, iv$.

Proof First suppose that $(X,\, s,\, t)$ is $FPR_1(i)$. We have to prove that $(X,\, s \cup t)$ is $FR_1(i)$. Let x_r, y_s be two distinct fuzzy singletons in X and $\gamma \in s \cup t$ with $\gamma(x) \neq \gamma(y)$. Since $(X,\, s,\, t)$ is $FPR_1(i)$, then there exist $\mu \in s$, $\lambda \in t$ such that

$$x_r q \mu,\; y_s q \lambda \quad \text{and} \quad \mu \bar{q} \lambda.$$

But it follows that $\lambda \in s \cup t$ with $\gamma(x) \neq \gamma(y)$ and $\mu \in s \cup t$, $\lambda \in s \cup t$ such that

$$x_r q \mu,\; y_s q \lambda \quad \text{and} \quad \mu \bar{q} \lambda.$$

Hence the topological space $(X,\, s \cup t)$ is $FR_1(i)$. □

For non-implications, we have the the following counter example that will serve the purpose.

Example 3.1 Let $X = \{x,\, y\}$ and s be the discrete fuzzy topology on X. Again t be the indiscrete fuzzy topology on X. Then $(X,\, s \cup t)$ is $FR_1(i)$, $FR_1(ii)$, $FR_1(iii)$ and $FR_1(iv)$. On the other hand, $(X,\, s,\, t)$ is none of the $FPR_1(i)$, $FPR_1(ii)$, $FPR_1(iii)$ and $FPR_1(iv)$.

Remark 3.1 *Let $(X,\, s)$ and $(X,\, t)$ be two fuzzy topological space and $(X,\, s,\, t)$ be its corresponding bitopological space. Then "$(X,\, s,\, t)$ is $FPR_1(j)$" does not imply $(X,\, s)$ and $(X,\, t)$ are $FR_1(j)$ in general, where $j = i,\, ii,\, iii,\, iv$.*

Example 3.2 Let $X = \{x,\, y\}$ and s be the fuzzy topology on X generated by $\{x_1,\, x_{0.6}\} \cup \{\text{constants}\}$. Again t be the fuzzy topology on X generated by $\{y_1\} \cup \{\text{constants}\}$. Then $(X,\, s,\, t)$ is $FPR_1(i)$, $FPR_1(ii)$, $FPR_1(iii)$, and $FPR_1(iv)$. On the other hand, $(X,\, s)$ and $(X,\, t)$ are none of the $FR_1(i)$, $FR_1(ii)$, $FR_1(iii)$ and $FR_1(iv)$.

Remark 3.2 *Let $(X,\, s)$ and $(X,\, t)$ be two fuzzy topological space and $(X,\, s,\, t)$ be its corresponding bitopological space. Then "$(X,\, s)$ and $(X,\, t)$ are both $FR_1(j)$" does not imply $(X,\, s,\, t)$ is $FPR_1(j)$ in general, where $j = i,\, ii,\, iii,\, iv$.*

Example 3.3 Let $X = \{x,\, y\}$ and s be the fuzzy topology on X generated by $\{x_1,\, y_1\} \cup \{\text{constants}\}$. Again t be the fuzzy topology on X generated by $\{\text{constants}\}$. Then it is clear that $(X,\, s)$ and $(X,\, t)$ are both $FR_1(i)$, $FR_1(ii)$, $FR_1(iii)$ and $FR_1(iv)$. But on the other hand, the fuzzy bitopological space $(X,\, s,\, t)$ is none of $FPR_1(i)$, $FPR_1(ii)$, $FPR_1(iii)$, and $FPR_1(iv)$.

Theorem 3.2 *Let $(X,\, s,\, t)$ be an fbts. Then the following are equivalent:*

(i) $(X,\, s,\, t)$ *is* FPT_2*;*

(ii) $(X,\, s,\, t)$ *is* FPT_0 *and* FPR_1.

Proof $(ii) \Rightarrow (i)$: Let x_r, y_s be two fuzzy singletons in X with $x \neq y$. Since $(X,\, s,\, t)$ is FPT_0, there exists a fuzzy set $\mu \in s \cup t$ such that

$$x_r q \quad \text{and} \quad \mu \cap y_s = 0.$$

This implies that $\mu(x) \neq \mu(y)$. Again since $\mu(x) \neq \mu(y)$ and $(X,\, s,\, t)$ is FPR_1, there exist

$v \in s$, $w \in t$ such that

$$x_r q v,\, y_s q w \quad \text{and} \quad v \cap w = 0.$$

Hence (X, s, t) is FPT_2.

$(i) \Rightarrow (ii)$: $FPT_2 \Rightarrow FPR_1$ is obvious. We have to show that $FPT_2 \Rightarrow FPT_0$. Let $x_r, y_s \in S(X)$ with $x \neq y$. Since (X, s, t) is FPT_2, there exist a fuzzy sets $u \in s$, $v \in t$ such that

$$x_r q u,\, y_s q v \quad \text{and} \quad u \cap v = 0.$$

To show that (X, s, t) is FPT_0, it is enough to show that $x_r \cap v = 0$. Suppose that

$$x_r \cap v \neq 0.$$

This implies that $v(x) > 0$. Since $u \cap v = 0$, we have

$$u(x) = 0, \quad \text{that is, } x_r \bar{q} u$$

which is a contradiction. Hence $x_r \cap v = 0$. □

In the following theorem now we discuss about the good extension property of FPR_1 concepts given earlier. All the properties $FPR_1(i)$, $FPR_1(ii)$, $FPR_1(iii)$ and $FPR_1(v)$ are good extension of PR_1.

Theorem 3.3 *Let (X, S, T) be a bitopological space. Then (X, S, T) is $PR_1 \Leftrightarrow (X, \omega(S), \omega(T))$ is $FPR_1(j)$, for $j = i, ii, iii, iv$.*

Proof Let (X, S, T) be PR_1 space. Suppose $x_r, y_s \in S(X)$, with $x \neq y$ and $\gamma \in \omega(S) \cup \omega(T)$ with $\gamma(x) \neq \gamma(y)$. Then we have

$$\gamma(x) < \gamma(y) \quad \text{or} \quad \gamma(x) > \gamma(y).$$

Suppose $\gamma(x) < \gamma(y)$. Then $\gamma(x) < r < \gamma(y)$ for some $r \in I_0$. So, it is clear that

$$x \notin \gamma^{-1}(r, 1],\, y \in \gamma^{-1}(r, 1] \quad \text{and} \quad \gamma^{-1}(r, 1] \in S \cup T.$$

Since (X, S, T) is PR_1 space, then there exist $U \in S$, $V \in T$ such that

$$x \in U,\, y \in V \quad \text{and} \quad U \cap V = \phi.$$

So, by definition of lower semi-continuous, we get $1_U \in \omega(S)$ and $1_V \in \omega(S)$. Now, we have

$$1_U(x) = 1,\, 1_V(y) = 1 \quad \text{and} \quad 1_{U \cap V} = 0.$$

We know that $1_{U\cap V} = 0$ implies $1_U \cap 1_V = 0$. Therefore $x_r q 1_U$, $y_s q 1_V$ and $1_U \cap 1_V = 0$. Hence $(X, \omega(S), \omega(T))$ is $FPR_1(iv)$.

Conversely, suppose that $(X, \omega(S), \omega(T))$ is FPR_1. Let $x, y \in X$, $x \neq y$ and $M \in S \cup T$

with

$$x \in M,\, y \notin M \quad \text{or} \quad x \notin M,\, y \in M.$$

Suppose $x \in M$, $y \notin M$. But from definition of lower semi-continuous function, we have

$$1_M \in \omega(S) \cup \omega(T) \quad \text{and} \quad 1_M(x) = 1, 1_M(y) = 0.$$

So, $1_M(x) \neq 1_M(y)$. Since $(X,\, \omega(S),\, \omega(T))$ is $FPR_1(iv)$, then there exist $\mu \in \omega(S), \lambda \in \omega(T)$ such that

$$x_1 q\mu,\; y_1 q\lambda \quad \text{and} \quad \mu \cap \lambda = 0.$$

Now $x_1 q\mu$, $y_1 q\lambda$ implies that

$$\mu(x) > 0, \lambda(y) > 0.$$

So, $x \in \mu^{-1}(0,1]$, $y \in \lambda^{-1}(0,1]$.

To show that $\mu^{-1}(0,1] \cap \lambda^{-1}(0,1] = \phi$, suppose that $\mu^{-1}(0,1] \cap \lambda^{-1}(0,1] \neq \phi$. Then there exits $z \in \mu^{-1}(0,1] \cap \lambda^{-1}(0,1]$ such that

$$\mu(z) > 0,\; \lambda(z) > 0.$$

Consequently $(\mu \cap \lambda)(z) \neq 0$ which contradicts the fact that $\mu \cap \lambda = 0$. Hence $(X,\, S,\, T)$ is PR_1. Other proofs are similar. □

We discuss the hereditary and productive properties of $FPR_1(j)$, for $j = i,\, ii,\, iii,\, iv,\, v$ in the following two theorems respectively.

Theorem 3.4 *Let $(X,\, s,\, t)$ be a fuzzy bitopological space, $A \subseteq X$ and $S_A = \{u/A : u \in s\}$, $t_A = \{v/A : v \in t\}$. Then,*

(*a*) *$(X,\, s,\, t)$ is $FPR_1(i) \Longrightarrow (A,\, s_A,\, t_A)$ is $FPR_1(i)$;*
(*b*) *$(X,\, s,\, t)$ is $FPR_1(ii) \Longrightarrow (A,\, s_A,\, t_A)$ is $FPR_1(ii)$;*
(*c*) *$(X,\, s,\, t)$ is $FPR_1(iii) \Longrightarrow (A,\, s_A,\, t_A)$ is $FPR_1(iii)$;*
(*d*) *$(X,\, s,\, t)$ is $FPR_1(iv) \Longrightarrow (A,\, s_A,\, t_A)$ is $FPR_1(iv)$.*

Proof (*a*) First suppose that $(X,\, s,\, t)$ is $FPR_1(i)$. We have to prove that $(A,\, s_A,\, t_A)$ is $FPR_1(i)$. Let x_r, y_s be two distinct fuzzy singletons in A and $\gamma \in s_A \cup t_A$ with $\gamma(x) \neq \gamma(y)$. Then γ can be written as $\gamma = \sigma/A$, where $\sigma \in s \cup t$ with $\sigma(x) \neq \sigma(y)$. Since $(X,\, s,\, t)$ is $FPR_1(i)$, then there exist fuzzy sets $\mu \in s$, $\lambda \in t$ such that

$$x_r q\mu,\; y_s q\lambda \quad \text{and} \quad \mu\bar{q}\lambda.$$

Now $\mu/A \in t_A$, $\lambda/A \in t_A$ for every $\mu \in s$, $\lambda \in t$ respectively. So

$$x_r q(\mu/A),\; y_s q(\lambda/A) \quad \text{and} \quad (\mu/A)\bar{q}(\lambda/A).$$

Hence the fuzzy subspace bitopological space $(A,\, s_A,\, t_A)$ is $FPR_1(i)$. Proofs of others are similar. □

In the following two theorems, we observe the preservations of $FPR_1(j)$, $j = i, ii, iii, iv$ properties under continuous, one-one and open mappings.

Definition 3.2([18]) *A function f from a fuzzy bitopological space (X, s, t) into a fuzzy bitopological space (Y, s_1, t_1) is called FP-continuous if and only if $f : (X, s) \to (Y, s_1)$ and $f : (X, t) \to (Y, t_1)$ are both fuzzy continuous.*

Theorem 3.5 *Let $(X,\, s,\, t)$ and $(Y,\, s_1,\, t_1)$ be two fuzzy bitopological spaces and $f : X \to Y$ be bijective, FP-continuous and FP-open map, then*

$$(X,\, s,\, t) \text{ is } FPR_1(j) \Longrightarrow (Y,\, s_1,\, t_1) \text{ is } FPR_1(j),$$

where $j = i, ii, iii, iv$.

Proof Suppose $(X,\, s,\, t)$ is $FPR_1(iv)$. We shall prove that $(Y,\, s_1,\, t_1)$ is $FPR_1(iv)$. Let $a_r,\, b_p \in S(Y)$ with $a \neq b$ and $\gamma \in s_1 \cup t_1$ with $\gamma(x) \neq \gamma(y)$. Since f is bijective, then there exist $c_r,\, d_p \in S(X)$ such that

$$f(c) = a,\, f(d) = b \quad \text{and} \quad c \neq d.$$

Again $f^{-1}(\gamma) \in s \cup t$ as f is FP-continuous. We have

$$f^{-1}(\gamma)(c) = \gamma(f(c)) = \gamma(a),$$

$$f^{-1}(\gamma)(d) = \gamma(f(d)) = \gamma(b).$$

So $f^{-1}(\gamma)(c) \neq f^{-1}(\gamma)(d)$ as $\gamma(x) \neq \gamma(y)$.

Since $(X,\, s,\, t)$ is $FPR_1(iv)$, then there exist $\mu \in s$, $\lambda \in t$ such that

$$c_r q\mu,\, d_p q\lambda \quad \text{and} \quad \mu \cap \lambda = 0.$$

Then $c_r q\mu$, $d_p q\lambda$ implies that

$$\mu(c) + r > 1 \quad \text{and} \quad \lambda(d) + p > 1.$$

Now we have

$$f(\mu)(a) = f(\mu)(f(c)) = \sup \mu(c) = \mu(c)$$

and

$$f(\lambda)(b) = f(\lambda)(f(d)) = \sup \lambda(d) = \lambda(d)$$

because f is bijective. So we have

$$f(\mu)(a) + r = \mu(c) + r > 1 \quad \text{and} \quad f(\lambda)(b) + p > 1.$$

Therefore

$$a_r q f(\mu) \quad \text{and} \quad b_p q f(\lambda).$$

Again we have

$$f(\mu \cap \lambda)(y) = \{\sup(\mu \cap \lambda)(x) : f(x) = y\} = (\mu \cap \lambda)(x) = 0$$

as $\mu \cap \lambda = 0$. Also $f(\mu \cap \lambda) = 0 \Rightarrow f(\mu) \cap f(\lambda) = 0$. Since f is FP-open, then $f(\mu) \in s_1$, $f(\lambda) \in t_1$. Therefore, there exist $f(\mu) \in s_1$, $f(\lambda) \in t_1$ such that

$$a_r q f(\mu),\ b_p q f(\lambda) \ \text{ and } \ f(\mu) \cap f(\lambda) = 0.$$

Hence (Y, s_1, t_1) is $FPR_1(iv)$. □

Theorem 3.6 *Let* (X, s, t) *and* (Y, s_1, t_1) *be two fuzzy bitopological spaces and* $f : X \to Y$ *be* FP*-continuous,* FP*-open and injective map, then*

$$(Y, s_1, t_1) \text{ is } FPR_1(j) \Longrightarrow (X, s, t) \text{ is } FPR_1(j)$$

where $j = i, ii, iii, iv$.

Proof Suppose that (Y, s_1, t_1) is $FPR_1(iv)$. Let $x_r, y_p \in S(X)$, $x \neq y$ and $\gamma \in s \cup t$ with $\gamma(x) \neq \gamma(y)$. Since f is injective, then $f(x) \neq f(y)$. Also $f(\gamma) \in s_1 \cup t_1$ as f is FP-open.

We know that

$$f(\gamma)(f(x)) = \sup \gamma(x) = \gamma(x)$$

and

$$f(\gamma)(f(y)) = \sup \gamma(y) = \gamma(y).$$

Then we have

$$f(\gamma)(f(x)) \neq f(\gamma)(f(y)).$$

Since (Y, s_1, t_1) is $FPR_1(iv)$, then $\exists \mu \in s_1, \lambda \in t_1$ such that

$$\mu(f(x)) + r > 1,\ \lambda(f(y)) + p > 1 \quad \text{and} \quad \lambda \cap \mu = 0,$$

which implies that

$$f^{-1}(\mu)(x) + r > 1,\ f^{-1}(\lambda)(y) + p > 1$$

and $f^{-1}(\mu \cap \lambda) = 0$ implies that

$$f^{-1}(\mu) \cap f^{-1}(\lambda) = 0.$$

Since f is FP-continuous, then $f^{-1}(\mu) \in s$, $f^{-1}(\lambda) \in t$. So, there exist $f^{-1}(\mu) \in s$, $f^{-1}(\lambda) \in t$ such that

$$x_r q f^{-1}(\mu),\ y_p q f^{-1}(\lambda) \ \text{ and } \ f^{-1}(\mu) \cap f^{-1}(\lambda) = 0.$$

Therefore (X, s, t) is $FPR_1(iv)$. Other proofs are similar. □

In previous a work [5], we have introduced the following definitions and discussed many related concepts among them.

Definition 3.3([5]) *The initial fuzzy bitopology on a set X for the family of fbts $\{(X_i, s_i, t_i)\}_{i\in J}$ and the family of functions $\{f_i : X \longrightarrow (X_i, s_i\, t_i)\}_{i\in J}$ is smallest fuzzy topology on X making each f_i fuzzy continuous.*

Definition 3.4([5]) *The final fuzzy topology on a set X for the family of fts $\{(X_i, s_i, t_i)\}_{i\in J}$ and the family of functions $\{f_i : (X_i, s_i, t_i) \longrightarrow X\}_{i\in J}$ is finest fuzzy topology on X making each f_i fuzzy continuous.*

Theorem 3.7 *If $\{(X_i, s_i, t_i)\}_{i\in J}$ is family of $FPR_1(iv)$ fbts and $\{f_i : X \to (X_i, s_i, t_i)\}_{i\in J}$, a family of functions, then the initial fuzzy bitopology on X for the family $\{f_i\}_{i\in J}$ is $FPR_1(iv)$.*

Proof Let s and t be the initial fuzzy topologies on X. Let $x, y \in X$ with $x \neq y$ and let a fuzzy set $w \in s \cup t$ with $w(x) \neq w(y)$. So, there exists $r \in (0,1)$ such that

$$w(x) < r < w(y).$$

Let x_r and y_r be two fuzzy points of X. For any $\alpha \in (0, r)$, consider the fuzzy point y_α. Then $y_\alpha \in w$ and so it is possible to find a basic fuzzy s-open set, say

$$f_{i_1}^{-1}(u_{i_1}^{\alpha}) \cap f_{i_2}^{-1}(u_{i_2}^{\alpha}) \cap \ldots \cap f_{i_n}^{-1}(u_{i_n}^{\alpha}),\ u_{i_k}^{\alpha}\ (1 \leq k \leq n)$$

being s_{i_k}–open fuzzy set such that

$$y_\alpha \in \inf f_{i_k}^{-1}(u_{i_k}^{\alpha}) \subset w \tag{1}$$

So for all $\alpha \in (0, r)$,

$$\alpha < \inf f_{i_k}^{-1}(u_{i_k}^{\alpha})(y) \leq w(y)$$

or

$$\alpha < \inf u_{i_k}^{\alpha}(f_{i_k}(y)) \ \text{(for all } \alpha \in (0,\, r)).$$

Thus,

$$r = \sup inf u_{i_k}^{\alpha}(f_{i_k}(y)).$$

Now as $\forall \alpha \in (0, r)$,

$$u_{i_k}^{\alpha}(f_{i_k}(y)) \leq \sup u_{i_k}^{\alpha}(f_{i_k}(y)),$$

we have

$$\inf u_{i_k}^{\alpha}(f_{i_k}(y)) \leq \inf \sup\ u_{i_k}^{\alpha}(f_{i_k}(y)).$$

Hence

$$r = \sup \inf u_{i_k}^{\alpha}(f_{i_k}(y)) \leq \inf \sup u_{i_k}^{\alpha}(f_{i_k}(y)).$$

This implies that

$$\sup u_{i_k}^{\alpha}(f_{i_k}(y)) > r$$

for all k, $1 \leq k \leq n$. In particular,

$$\sup u^{\alpha}_{i_1}(f_{i_1}(y)) > r.$$

Now let $u_1 = \sup\ u^{\alpha}_{i_1}$. Then $u_1 \in s_{i_1} \cup t_{i_1}$ and $u_1(f_{i_1}(y)) > r$. Also as $w(x) < r$, from (1), we have

$$u^{\alpha}_{i_1}(f_{i_1}(x)) < r\, \forall \alpha \in (0, r).$$

Thus $u_1(f_{i_1}(x)) = r$. Hence $u_1(f_{i_1}(x)) \neq u_1(f_{i_1}(y))$.

Since $(X_{i_1}, s_{i_1}, t_{i_1})$ is $FPR_1(iv)$, then for every two distinct fuzzy points $(f_{i_1}(x))_r$, $(f_{i_1}(y))_r$ of X_{i_1}, there exist fuzzy sets $v_1 \in s_{i_1}$, $u_1 \in t_{i_1}$ such that

$$(f_{i_1}(x))_r q v_1,\ (f_{i_1}(y))_r q u_1 \quad \text{and} \quad u_1 \cap v_1 = 0.$$

Let $v_r = f^{-1}_{i_1}(v_1)$ and $u_r = f^{-1}_{i_1}(u_1)$. We have to show that $x_r q v_r$. For this, since $(f_{i_1}(x))_r q v_1$ we have

$$v_1(f_{i_1}(x)) + r > 1, \quad \text{that is } f^{-1}_{i_1}(v_1)(x) + r > 1,$$

i.e., $v_r(x) + r > 1$. Hence, it is true for $x_r q v_r$. Similarly, it is also true for $y_r q u_r$.

Now, we have to show that $u_r \cap v_r = 0$. Suppose $u_r \cap v_r \neq 0$, then there exists $z \in X$ with $u_r(f_{i_1}(z)) > 0$ and $v_r(f_{i_1}(z)) > 0$. Notice that $v_r(z) = f^{-1}_{i_1}(v_1)(z) = v_1(f_{i_1}(z)) > 0$ and similarly, $u_1(f_{i_1}(z)) > 0$ contradict that $u_1 \cap v_1 = 0$. Hence (X, s, t) is must FPR_1. □

Theorem 3.8 *If $\{(X_i, s_i, t_i)\}_{i \in J}$ is family of $FPR_1(iv)$ fbts and $\{f_i : (X_i, s_i, t_i) \to X\}_{i \in J}$, a family of FP-open and bijective functions, then the final fuzzy bitopology on X for the family $\{f_i\}_{i \in J}$ is $FPR_1(iv)$.*

Proof Let s and t be the final fuzzy topologies on X. Let $x, y \in X$ with $x \neq y$ and let a fuzzy set $w \in s \cup t$ with $w(x) \neq w(y)$. So, there exists $r \in (0, 1)$ such that

$$w(x) < r < w(y).$$

Let x_r and y_r be two distinct fuzzy points of X. For any $\alpha \in (0, r)$, consider the fuzzy point y_α. Then $y_\alpha \in w$ and so it is possible to find a basic fuzzy s–open set, say

$$f_{i_1}(u^{\alpha}_{i_1}) \bigcap f_{i_2}(u^{\alpha}_{i_2}) \bigcap \cdots \bigcap f_{i_n}(u^{\alpha}_{i_n}), \quad u^{\alpha}_{i_k},\ (1 \leq k \leq n)$$

being s_{i_k}–open fuzzy set such that

$$y_\alpha \in \inf f_{i_k}(u^{\alpha}_{i_k}) \subset u.$$

But $\forall \alpha \in (0, r)$,

$$\alpha < \inf f_{i_k}(u^{\alpha}_{i_k})(y) \leq u(y)$$

or

$$r = \sup \inf f_{i_k}(u_{i_k}^{\alpha})(y).$$

But as $\forall \alpha \in (0, r)$,

$$f_{i_k}(u_{i_k}^{\alpha})(y) \leq \sup f_{i_k}(u_{i_k}^{\alpha})(y).$$

We have $\forall \alpha \in (0, r)$,

$$\inf f_{i_k}(u_{i_k}^{\alpha})(y) \leq \inf \sup f_{i_k}(u_{i_k}^{\alpha})(y).$$

Hence

$$r = \sup \inf f_{i_k}(u_{i_k}^{\alpha})(y) \leq \inf \sup f_{i_k}(u_{i_k}^{\alpha})(y).$$

This implies that

$$\sup f_{i_k}(u_{i_k}^{\alpha})(y) > r, \quad k(1 \leq k \leq n)$$

or

$$\sup(u_{i_k}^{\alpha})(y_{i_k}) > r,$$

where $f_{i_k}(y_{i_k}) = y$, since f_{i_k} is bijective. In particular

$$\sup(u_{i_1}^{\alpha})(y_{i_1}) > r.$$

Now let $u_1 = \sup u_{i_1}^{\alpha}$. Then $u_1 \in s_{i_1} \cup t_{i_1}$ and $u_1(y_{i_1}) > r$. Also as $w(x) < r$, from (1), we get

$$f_{i_1}(u_{i_1}^{\alpha})(x) < r \quad \forall \alpha \in (0, r).$$

Thus

$$\sup f_{i_1}(u_{i_1}^{\alpha})(x) = r, \quad \forall \alpha \in (0, r).$$

or

$$\sup(u_{i_1}^{\alpha})(x_{i_1}) = r$$

where $f_{i_1}(x_{i_1}) = x$, since f_{i_1} is bijective. Hence $u_1(x_{i_1}) = r$. Therefore

$$u_1(x_{i_1}) \neq u_1(y_{i_1}).$$

Since $(X_{i_1}, s_{i_1}, t_{i_1})$ is $FPR_1(iv)$, then for every two distinct fuzzy points $(x_{i_1})_r$, $(y_{i_1})_r$ of X_{i_1}, there exist fuzzy sets $v_1 \in s_{i_1}$, $u_1 \in t_{i_1}$ such that

$$(x_{i_1})_r q v_1,\ (y_{i_1})_r q u_1 \text{ and } u_1 \cap v_1 = 0.$$

Let $v_r = f_{i_1}(v_1)$ and $u_r = f_{i_1}(u_1)$. Now we have to show that $(x_{i_1})_r q v_r$. For this, since $(x_{i_1})_r q v_1$ that is, $v_1(x_{i_1}) + r > 1$, we have

$$v_r(x_{i_1}) = f_{i_1}(v_1)(x_{i_1}) = v_1(x_{i_1}) > 1 - r.$$

So, $v_r(x_{i_1}) + r > 1$. Hence $x_{i_1} q v_r$. Similarly, $y_r q u_r$.

Now, to show that $u_r \cap v_r = 0$, suppose $u_r \cap v_r \neq 0$, then there exists $z \in X$ with $u_r(z) > 0$

and $v_r(z) > 0$. Notice that $v_r(z) = f_{i_1}(v_1)(z) = v_1(z_{i_1}) > 0$, where $f_{i_1}(z_{i_1}) = z$, as f_{i_1} is bijective. Similarly, we can prove that $u_1(z_{i_1}) > 0$ contradict that $u_1 \cap v_1 = 0$. Hence (X, s, t) is must $FPR_1(iv)$. □

§4. Conclusion

The main result of this paper is introducing some new concepts of fuzzy pairwise R_1 bitopological spaces. We discuss some features of these concepts and present their good extension, hereditary. Initial and final topologies introduced in FPR_1 spaces are interesting result.

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