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On Status Coindex Distance Sum and Status Connectivity Coindices of Graphs

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Abstract: The status of a vertex u in a connected graph G, denoted by $\sigma(u)$ is defined as the sum of the distance between u and all other vertices of a graph G. Let G be a connected graph of order $n \geq 3$ and size m. The first and second status coindices distance sum of graph G, denoted by $S_1^d(G)$ and $S_2^d(G)$, are defined as

$$\begin{aligned} S_1^d(G) &= \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v)] d(u,v), \\ S_2^d(G) &= \sum_{uv \notin E(G)} [\sigma(u)\sigma(v)] d(u,v) \end{aligned}$$

respectively. In this paper the first and second status coindex distance sum of some graphs are obtained. Status connectivity coindices of some standard graphs are computed. The bounds of the first and second status coindex distance sum and status connectivity coindices are established.

Key Words: Distance, status of a vertex, status coindex distance sum, status connectivity coindices.

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§1. Introduction

Let G be a connected graph with n vertices and m edges. Let $V(G)$ and $E(G)$ be its vertex and edge sets, respectively. The edge joining the vertices u and v is denoted by uv. The complement $\overline{G}$ of the graph G is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. The *degree* of a vertex u in a graph G is the number of edges joining to u and is denoted by $d(u)$ or d_u. The *distance* between the vertex u and v is the length of the shortest path joining u and v and is denoted by $d_G(u,v)$ [6]. For well known graph and terminology, we refer the books [6], [17].

The *status* of a vertex $u \in V(G)$, denoted by $\sigma_G(u)$ is defined as [8],

$$\sigma_G(u) = \sum_{v \in V(G)} d(u,v).$$

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The Wiener index $W(G)$ of a connected graph G is defined as [12],

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{v\in V(G)} \sigma_G(u).$$

The first and second Zagreb indices of a graph G are defined as [13]

$$M_1(G) = \sum_{u\in V(G)} d_u^2 \quad \text{and} \quad M_2(G) = \sum_{uv\in E(G)} d(u)d(v).$$

Results on the Zagreb indices can be found in [5, 18, 15, 22, 24, 20, 21].

The first and second Zagreb coindices of a graph G are defined as [15]

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} d(u)+d(v) \quad \text{and} \quad \overline{M}_2 = \sum_{uv\notin E(G)} d(u)d(v).$$

$M_1(G)$ can be written also as [25], [26]

$$M_1(G) = \sum_{uv\in E(G)} [d_u + d_v].$$

More results on Zagreb coindices can be found in [1], [2].

Furtula and Gutman [3] introduced the forgotten topological index of a graph G, also called as F-index, which is defined as

$$F(G) = \sum_{u\in V(G)} (d(u))^3.$$

The first status connectivity index, $S_1(G)$ and second status connectivity index, $S_2(G)$ of a connected graph is defined as [9]

$$\begin{aligned} S_1(G) &= \sum_{uv\in E(G)} [\sigma_G(u)+\sigma_G(v)], \\ S_2(G) &= \sum_{uv\in E(G)} [\sigma_G(u)\sigma_G(v)]. \end{aligned}$$

The first and second status connectivity coindex of a graph G are defined by [10]

$$\begin{aligned} \overline{S}_1(G) &= \sum_{uv\notin E(G)} [\sigma_G(u)+\sigma_G(v)], \\ \overline{S}_2(G) &= \sum_{uv\notin E(G)} [\sigma_G(u)\sigma_G(v)]. \end{aligned}$$

Definition 1.1([19]) *Let G be a connected graph of order $n \geq 3$. The first and second status coindex distance sum of G are defined as*

$$S_1^d(G) = \sum_{uv\notin E(G)} (\sigma(u)+\sigma(v))d(u,v)$$

and

$$S_2^d(G) = \sum_{uv \notin E(G)} \sigma(u)\sigma(v)d(u,v)$$

respectively.

§2. Status Coindex Distance Sum

In this section, we obtain status coindices distance sum of connected graphs in terms of Wiener index and also status coindices distance sum of complements of graphs.

Proposition 2.1 *Let G be a connected graph on n vertices with $diam(G) = 2$. Then,*

$$S_1^d(G) = 4(n-1)W(G) - 2S_1(G)$$

and

$$S_2^d(G) = 4(W(G))^2 - \sum_{u \in V(G)} (\sigma_G(u))^2 - 2S_2(G).$$

Proof By definition, we know that

$$\begin{aligned} S_1^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)]d(u,v) = \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)]2 \\ &= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u) + \sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u) + \sigma_G(v)] \right] 2 \\ &= [(n-1) \sum_{u \in V(G)} \sigma_G(u) - S_1(G)]2 \\ &= [2(n-1)W(G) - S_1(G)]2 = 4(n-1)W(G) - 2S_1(G) \end{aligned}$$

Also,

$$\begin{aligned} S_2^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u)\sigma_G(v)]d(u,v) = \sum_{uv \notin E(G)} [\sigma_G(u)\sigma_G(v)]2 \\ &= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u)\sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u)\sigma_G(v)] \right] 2 \\ &= \left[\frac{1}{2} \left(\left(\sum_{u \in V(G)} \sigma_G(u) \right)^2 - \sum_{u \in V(G)} \sigma_G(u)^2 \right) - S_2(G) \right] 2 \\ &= [2(W(G))^2 - \frac{1}{2} \sum_{u \in V(G)} (\sigma_G(u))^2 - S_2(G)]2. \end{aligned}$$

$$S_2^d(G) = 4(W(G))^2 - \sum_{u\in V(G)} (\sigma_G(u))^2 - 2S_2(G). \qquad \square$$

Proposition 2.2 *Let G be a graph of order n and size m. Let $\overline{G}$, the complement of G, be connected. Then*

$$S_1^d(\overline{G}) \geq 4m(n-1) + 2M_1(G) \tag{2.1}$$

and

$$S_2^d(\overline{G}) \geq 2m(n-1)^2 + 2(n-1)M_1(G) + 2M_2(G) \tag{2.2}$$

with equality holds if and only if diam($\overline{G}$)=2.

Proof For any vertex u in $\overline{G}$ there are $n-1-d_G(u)$ vertices which are at distance 1 and the remaining $d_G(u)$ vertices are at distance at least 2. Therefore,

$$\sigma_{\overline{G}}(u) \geq [n-1-d_G(u)] + 2d_G(u) = n-1+d_G(u).$$

Therefore,

$$\begin{aligned}
S_1^d(\overline{G}) &= \sum_{uv\notin E(\overline{G})} [\sigma_{\overline{G}}(u) + \sigma_{\overline{G}}(v)] d_{\overline{G}}(u,v) \\
&\geq \sum_{uv\notin E(\overline{G})} [n-1+d_G(u)+n-1+d_G(v)] d_{\overline{G}}(u,v) \\
&= \sum_{uv\notin E(\overline{G})} [2n-2+d_G(u)+d_G(v)] d_{\overline{G}}(u,v) \\
&= 2m(2n-2) + \sum_{uv\notin E(\overline{G})} [d_G(u)+d_G(v)] d_{\overline{G}}(u,v) \\
&= 4m(n-1) + \sum_{uv\in E(G)} [d_G(u)+d_G(v)]2 \\
&= 4m(n-1) + 2M_1(G).
\end{aligned}$$

And

$$\begin{aligned}
S_2^d(\overline{G}) &= \sum_{uv\notin E(\overline{G})} [\sigma_{\overline{G}}(u)\sigma_{\overline{G}}(v)] d_{\overline{G}}(u,v) \\
&\geq \sum_{uv\notin E(\overline{G})} [n-1+d_G(u)][n-1+d_G(v)] d_{\overline{G}}(u,v) \\
&= \sum_{uv\notin E(\overline{G})} \left[(n-1)^2 + (n-1)[d_G(u)+d_G(v)] + [d_G(u)d_G(v)]\right] d_{\overline{G}}(u,v) \\
&= 2m(n-1)^2 + \sum_{uv\in E(G)} (n-1)[d_G(u)+d_G(v)] d_{\overline{G}}(u,v)
\end{aligned}$$

$$+\sum_{uv\in E(G)}[d_G(u)d_G(v)]d_{\overline{G}}(u,v)$$
$$= 2m(n-1)^2 + 2(n-1)M_1(G) + 2M_2(G).$$ □

Corollary 2.3 *Let G be a graph with n vertices, m edges and diam ≥ 2 and let $\overline{G}$, the complement of G, be connected. Then,*

$$S_1^d(\overline{G}) \geq 2\left[4m(n-1) - \overline{M}_1(\overline{G})\right]$$

and

$$S_2^d(\overline{G}) \geq 2[4m(n-1)^2 - 2(n-1)\overline{M}_1(\overline{G}) + \overline{M}_2(\overline{G})],$$

with equality holds if and only if diam$(\overline{G}) = 2$.

Proof By definition, we have [16]

$$\overline{M}_1(\overline{G}) = 2m(n-1) - M_1(G) \tag{2.3}$$

and

$$\overline{M}_2(\overline{G}) = m(n-1)^2 - (n-1)M_1(G) + M_2(G). \tag{2.4}$$

Substituting (2.3) in (2.1) and (2.4) in (2.2) we get the required result. □

§3. Bounds for Status Coindex Distance Sum

Theorem 3.1 *Let G be a connected graph with n vertices, m edges and diam$(G) = D \geq 2$. Then,*

$$4(n-1)W(G) - 2S_1(G) \leq S_1^d(G) \leq 2D(n-1)W(G) - DS_1(G),$$

$$4(W(G))^2 - \sum_{u\in V(G)}(\sigma_G(u))^2 - 2S_2(G)$$
$$\leq S_2^d(G) \leq 2D(W(G))^2 - \frac{D}{2}\sum_{u\in V(G)}[(\sigma_G(u))^2 - DS_2(G)]$$

with equality holds if and only if $D = 2$.

Proof Let us first prove the lower bound. When $uv \notin E(G)$, the minimum distance between

u and v is 2. Therefore

$$\begin{aligned}
S_1^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)] d(u,v) \\
&\geq \sum_{uv \notin E(G)} [\sigma_G(u) + \sigma_G(v)] 2 \\
&= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u) + \sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u) + \sigma_G(v)] \right] 2 \\
&= \left[(n-1) \left(\sum_{u \in V(G)} \sigma_G(u) \right) - S_1(G) \right] 2 \\
&= [2(n-1)W(G) - S_1(G)] 2.
\end{aligned}$$

i.e.,

$$S_1^d(G) \geq 4(n-1)W(G) - 2S_1(G).$$

And

$$\begin{aligned}
S_2^d(G) &= \sum_{uv \notin E(G)} [\sigma_G(u) \sigma_G(v)] d(u,v) \\
&\geq \sum_{uv \notin E(G)} [\sigma_G(u) \sigma_G(v)] 2 \\
&= \left[\sum_{\{u,v\} \subseteq V(G)} [\sigma_G(u) \sigma_G(v)] - \sum_{uv \in E(G)} [\sigma_G(u) \sigma_G(v)] \right] 2 \\
&= \left[\frac{1}{2} \left[\left(\sum_{u \in V(G)} \sigma_G(u) \right)^2 - \sum_{u \in V(G)} \sigma_G(u)^2 \right] - S_2(G) \right] 2 \\
&= [2(W(G))^2 - \frac{1}{2} \sum_{u \in V(G)} (\sigma_G(u)^2) - S_2(G)] 2.
\end{aligned}$$

i.e.,

$$S_2^d(G) \geq 4(W(G))^2 - \sum_{u \in V(G)} (\sigma_G(u))^2 - 2S_2(G).$$

Now let us prove the upper bound. When $uv \notin E(G)$, the maximum distance between u

and v be $D(diameter)$. Then

$$\begin{aligned} S_1^d(G) &= \sum_{uv\notin E(G)} [\sigma_G(u)+\sigma_G(v)]d(u,v) \\ &\leq \sum_{uv\notin E(G)} [\sigma_G(u)+\sigma_G(v)]D \\ &= \left[\sum_{\{u,v\}\subseteq V(G)} [\sigma_G(u)+\sigma_G(v)] - \sum_{uv\in E(G)} [\sigma_G(u)+\sigma_G(v)]\right] D \\ &= \left[(n-1)\left(\sum_{u\in V(G)} \sigma_G(u)\right) - S_1(G)\right] D, \end{aligned}$$

i.e.,

$$S_1^d(G) \leq 2D(n-1)W(G) - DS_1(G).$$

And similarly

$$S_2^d(G) = \sum_{uv\notin E(G)} [\sigma_G(u)\sigma_G(v)]d(u,v).$$

We get that

$$S_2^d(G) \leq 2D(W(G))^2 - \frac{D}{2}\sum_{u\in V(G)} [(\sigma_G(u))^2 - DS_2(G)].$$

Thus the result follows and in both upper and lower bounds of $S_1^d(G)$ and $S_2^d(G)$, the equality holds for $D = 2$. □

§4. First Status Coindex Distance Sum of Line Graphs

Theorem 4.1([14]) *Let G be a graph with n-vertices and m-edges. Then,*

$$M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1).$$

Proposition 4.2([14]) *Let L be the line graph of the graph G. Then*

$$M_1(L(G)) = F - 4M_1 + 2M_2 + 4m$$

where, M_1, M_2, F are the first Zagreb index, second Zagreb index, and forgotten topological index of the parent graph G respectively.

Theorem 4.3([19]) *Let G be a connected graph with n vertices, m edges and $diam(G) = D \geq 2$. Then,*

$$4(n-1)(n(n-1)-2m) - D\overline{M_1}(G) \leq S_1^d(G) \leq (n-1)D^2(n(n-1)-2m) - 2(D-1)\overline{M_1}(G)$$

with equality holds for $Diam(G) = 2$.

F :

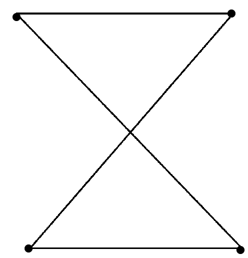

Figure 1

Theorem 4.4 *Let $L(G)$ be the line graph of the graph G with n-vertices, m-edges and $diam(G) = 2$. Then*

$$S_1^d(L(G)) = 4m[(m-1)^2 - 3(m-1) + 2] - M_1(G)[6(m-1) + 8] + 4M_2(G) + 2F.$$

Proof From the definition of line graphs [4], the number of vertices of $L(G)$ is $n_1 = m$ and the number of edges of $L(G)$ is [7] $m_1 = \frac{1}{2}\sum_{i=1}^{n} d_i^2 - m$. Since from [11], if $diam(G) \leq 2$ and G does not contain F (Figure 1.) as an induced subgraph of G and also G is not a star Graph S_n, then $diam(L(G)) = 2$. From [19],

$$S_1^d(G) = 4(n-1)[n(n-1) - 3m] + 2M_1(G)$$

Therefore, the status coindex distance sum of line graphs can be written as,

$$\begin{aligned} S_1^d(L(G)) &= 4(n_1-1)[n_1(n_1-1) - 3m_1] + 2M_1(L(G)) \\ &= 4(m-1)[m(m-1) - 3(\frac{1}{2}\sum_{i=1}^{n} d_i^2 - m)] + 2M_1(L(G)) \end{aligned}$$

From Proposition 4.2 and definition of Zagreb index [13]

$$M_1(G) = \sum_{i=1}^{n} d_i^2.$$

Hence,

$$S_1^d(L(G)) = 4m[(m-1)^2 - 3(m-1) + 2] - M_1(G)[6(m-1) + 8] + 4M_2(G) + 2F. \qquad \square$$

The following corollary directly follows from the Theorem 4.4.

Corollary 4.5 *Let G be a connected regular graph of degree r on n-vertices and m-edges and* *let* $diam(G) = 2$. *Then,*

$$S_1^d(L(G)) = 2m[4r^2 - 2r(3m-7) + 2(m-1)^2 - 6(m-1) + 4].$$

Proposition 4.6 *The first status coindex distance sum of line graph of complete bipartite graph* $K_{p,q}$

$$\begin{aligned} S_1^d(L(K_{p,q})) &= 4pq[(pq-1)^2 - 3(pq-1) + 2] \\ &\quad -pq(p+q)[6(pq-1)+8] + 4(pq)^2 + 2pq(p^2+q^2). \end{aligned}$$

Proof The graph $K_{p,q}$ has $n = p+q$ vertices and $m = pq$ edges. Also $diam(K_{p,q}) \leq 2$. The vertex set $V(K_{p,q})$ can be partitioned into two sets V_1 and V_2 such that for every edge uv of $K_{p,q}$, the vertex $u \in V_1$ and $v \in V_2$, where $|V_1| = p$ and $|V_2| = q$. Therefore $d(u) = q$ and $d(v) = p$ and hence,

$$M_1(K_{p,q}) = pq(p+q),\ M_2(K_{p,q}) = (pq)^2,\ F = pq(p^2+q^2).$$

Therefore by the Theorem 4.4 the result holds. □

Theorem 4.7 *Let* G *be a graph whose line graph* $L(G)$ *has* $diam(L(G)) > 3$, *then*

$$S_1^d(\overline{L(G)}) = 4(m-1)[m(m-1) - M_1(G) + 2m] - D[(M_1(G) - 2m)(n-1) - M_1(\overline{L(G)}).$$

Proof Let G be any graph with n-vertices and m-edges whose line graph $L(G)$ has $diam(L(G)) > 3$. Let $\overline{L(G)}$ be the complement of line graph.

We know from Theorem 4.3 that

$$4(n-1)(n(n-1) - 2m) - D\overline{M_1}(G) \leq S_1^d(G)$$

i.e.,

$$4(n-1)(n(n-1) - 2m) - D\left[2m(n-1) - M_1(G)\right] \leq S_1^d(G)$$

with equality holds for graphs of $diam = 2$. Since there exist a fact that for any graph G, if $diam(G) > 3$ then $diam(\overline{G}) \leq 2$ [27]. Since G is connected graph and $diam(L(G)) > 3$, then $\overline{L(G)}$ is connected and has diameter $D = 2$, then by Theorem 4.3,

$$\begin{aligned} S_1^d(\overline{L(G)}) &= 4(n_1-1)\left[n_1(n_1-1) - 2m_1\right] - D\left[2m_1(n_1-1) - M_1(\overline{L(G)})\right] \\ S_1^d(\overline{L(G)}) &= 4(m-1)[m(m-1) - M_1(G) + 2m] \\ &\quad -D[(M_1(G) - 2m)(n-1) - M_1(\overline{L(G)}). \end{aligned}$$

□

§5. Status Connectivity Coindices of Some Standard Graphs

A simple calculation enables us getting status connectivity coindice on a few standard graphs following.

Proposition 5.1 *For a complete bipartite graph $K_{s.t}$*

$$\overline{S}_1(K_{s,t}) = (2s+t-2)s(s-1) - (2t+s-2)t(t-1)$$

and

$$\overline{S}_2(K_{s,t}) = \frac{s(s-1)}{2}(2s+t-2)^2 + \frac{t(t-1)}{2}(2t+s-2)^2.$$

Proposition 5.2 *For a cycle C_n on $n \geq 4$ vertices*

$$\overline{S}_1(C_n) = \begin{cases} \frac{n^2}{4}[n(n-1)-2m], & if\ n\ is\ even; \\ \frac{n^2-1}{4}[n(n-1)-2m], & if\ n\ is\ odd. \end{cases}$$

and

$$\overline{S}_2(C_n) = \begin{cases} \frac{n^4}{32}(n(n-1)-2m), & if\ n\ is\ even; \\ \frac{(n^2-1)^2}{32}[n(n-1)-2m], & if\ n\ is\ odd. \end{cases}$$

Proposition 5.3 *For a wheel $W_{n+1}, n \geq 4$*

$$\overline{S}_1(W_{n+1}) = 3n(n-3) + 2n(n-3)^2$$

and

$$\overline{S}_2(W_{n+1}) = \frac{9n(n-3)}{2} + 6n(n-3)^2 + 2n(n-3)^3.$$

Proposition 5.4 *For a helm $H_n, n \geq 3$*

$$\overline{S}_1(H_n) = 2[12n^3 - 27n^2 + 18n]$$

and

$$\begin{aligned} \overline{S}_2(H_n) &= (21n^3 - 24n^2) + \frac{(n^2-n)}{2}(7n-8)^2 \\ &+ \frac{n^2-3n}{2}(5n-7)^2 + (7n^3 - 15n^2 + 8n)(5n-7). \end{aligned}$$

Proposition 5.5 *For a friendship graph F_n, $n \geq 2$*

$$\overline{S}_1(F_n) = 8n(2n-1)(n-1) \quad and \quad \overline{S}_2(F_n) = (4n-2)^2(2n^2-2n).$$

§6. Bounds for Status Connectivity Coindices

Theorem 6.1 *Let G be a connected graph with n vertices, m edges and $diam(G) = D \geq 2$. Then,*

$$2(n-1)(n(n-1)-2m) - \overline{M}_1(G) \leq \overline{S}_1(G) \leq D(n-1)(n(n-1)-2m) - (D-1)\overline{M}_1(G)$$

and

$$\begin{aligned}&2(n-1)^2(n(n-1)-2m) - 2(n-1)\overline{M}_1(G) + \overline{M}_2(G)\\ &\leq \overline{S}_2(G) \leq D^2(n-1)^2\left(\frac{n(n-1)}{2} - m\right) - D(D-1)(n-1)\overline{M}_1(G) + (D-1)^2\overline{M}_2(G)\end{aligned}$$

with equality holds if and only if $D = 2$.

Proof Let us first prove the lower bound. For any vertex u of G there are $d(u)$ which are at a distance 1 from the vertex u and the remaining $(n-1-d(u))$ vertices are at distance at least 2. Therefore

$$\sigma(u) \geq d(u) + 2(n-1-d(u)) = 2(n-1) - d(u) = 2(n-1) - d(u).$$

Therefore,

$$\begin{aligned}\overline{S}_1(G) &= \sum_{uv\notin E(G)} [\sigma(u)+\sigma(v)] \geq \sum_{uv\notin E(G)} [4n-4-(d(u)+d(v))]\\ &= \sum_{uv\notin E(G)} 4(n-1) - \sum_{uv\notin E(G)} d(u)+d(v)\\ &= 4(n-1)\left(\frac{n(n-1)}{2} - m\right) - \overline{M}_1(G)\\ &= 2(n-1)(n(n-1)-2m) - \overline{M}_1(G)\end{aligned}$$

and

$$\begin{aligned}\overline{S}_2(G) &= \sum_{uv\notin E(G)} [\sigma(u)\sigma(v)]\\ &\geq \sum_{uv\notin E(G)} (2n-2-d(u))(2n-2-d(u))\\ &= \sum_{uv\notin E(G)} [4(n-1)^2 - 2(n-1)(d(u)+d(v)) + d(u)d(v)]\\ &= \sum_{uv\notin E(G)} 4(n-1)^2 - (2n-2)\sum_{uv\notin E(G)} (d(u)+d(v)) + \sum_{uv\notin E(G)} d(u)d(v)\\ &= 4(n-1)^2\left(\frac{n(n-1)}{2} - m\right) - (2n-2)\overline{M}_1(G) + \overline{M}_2(G)\\ &= 2(n-1)^2(n(n-1)-2m) - 2(n-1)\overline{M}_1(G) + \overline{M}_2(G).\end{aligned}$$

Now we prove the upper bound. For any vertex u of G there are $d(u)$ which are at a distance 1 from the vertex u and the remaining $(n-1-d(u))$ vertices are at distance at most D. Hence,

$$\sigma(u) \leq d(u) + D(n-1-d(u)) = D(n-1) - (D-1)d(u).$$

Therefore

$$\begin{aligned}
\overline{S}_1(G) &= \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v)] \\
&\leq \sum_{uv \notin E(G)} [2D(n-1) - (D-1)(d(u) + d(v))] \\
&= 2D(n-1)\left(\frac{n(n-1)}{2} - m\right) - (D-1)\overline{M}_1(G) \\
&= D(n-1)(n(n-1) - 2m) - (D-1)\overline{M}_1(G)
\end{aligned}$$

and

$$\begin{aligned}
\overline{S}_2(G) &= \sum_{uv \notin E(G)} [\sigma(u)\sigma(v)] \\
&\leq \sum_{uv \notin E(G)} [D(n-1) - (D-1)d(u)][D(n-1) - (D-1)d(v)] \\
&= \sum_{uv \notin E(G)} [D^2(n-1)^2 - D(D-1)(n-1)(d(u) + d(v)) + (D-1)^2 d(u)d(v)] \\
&= D^2(n-1)^2\left(\frac{n(n-1)}{2} - m\right) - D(D-1)(n-1)\overline{M}_1(G) + (D-1)^2\overline{M}_2(G).
\end{aligned}$$

In both upper and lower bounds of $\overline{S}_1(G)$ and $\overline{S}_2(G)$, the equality holds for $D = 2$. □

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