International J.Math. Combin. Vol.3(2019), 54-60

On Radicals for Ternary Semirings

Swapnil P. Wani

University Institute of Chemical Technology
Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon - 425001, India

Kishor F. Pawar

Department of Mathematics, School of Mathematical Sciences
Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon - 425 001, India

E-mail: swapwani@gmail.com, kfpawar@nmu.ac.in

Abstract: The notion of radical theory of rings was introduced by Kurosh [1] in 1953. In the present paper, the concepts of radical theory of rings and semirings are generalized for ternary semirings. Later, the notions like semisimple class, upper radical and Hoehnke radical for class of ternary semirings are introduced. Also proved some consequences of semisimple and upper radicals.

Key Words: Ternary semiring, radical class, semisimple class, upper radical, Hoehnke radical.

AMS(2010): 16Y30.

§1. Introduction

The idea of a ternary algebraic system was first invented in 1924 by Prüfer in [5]. In 1971, Lister [8]investigated the notion of ternary ring and studied some properties of a ternary ring. The concept of semiring was first introduced in [6] by Vandiver in 1934. Later, the notion of a ternary semiring which generalizes the notion of ternary ring and semiring was introduced by Dutta and Kar [7] in 2003. Pawar and Deore in [2]-[4] generalizes concepts of radical classes for a class of semirings. The present paper extends the notions of radical theory of rings and semirings to a ternary semiring. The concept of radical class with few examples and results are introduced in Section 3. Section 4 introduces the notions of semisimple class, upper radical and their properties and relationship. In Section 5, the notion of Hoehnke radical for class of ternary semirings is introduced.

§2. Preliminary Definitions

Definition 2.1([7]) *A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an*

[1]Received April 3, 2019, Accepted August 28, 2019.

additive commutative semigroup satisfying the following conditions:

(i)(*Associative Law*) $(abc)de = a(bcd)e = ab(cde)$;

(ii)(*Right Distributive Law*) $(a + b)cd = acd + bcd$;

(iii)(*Lateral Distributive Law*) $a(b + c)d = abd + acd$;

(iv)(*Left Distributive Law*) $ab(c + d) = abc + abd$

for all $a, b, c, d, e \in T$.

Example 2.2([7]) Let Z_0^- be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication, Z_0^- forms a ternary semiring.

Definition 2.3([7]) *An additive subsemigroup* I *of a ternary semiring* S *is called ideal of* S *if* $SSI \subseteq I, SIS \subseteq I$ *and* $ISS \subseteq I$. *An ideal* I *of a ternary semiring* S *is called* k*-ideal (subtractive) if for* $a \in I, a + b \in I, b \in S$ *imply* $b \in I$. *We denote* $I \lhd S$, *a ternary semiring ideal in* S.

Definition 2.4([7]) *A ternary semiring* S *is said to be regular if for each element* a *in* S *there exists an element* x *in* S *such that* $a = axa$. *If the element* x *is unique and satisfies* $x = xax$, *then* S *is called an inverse ternary semiring.* x *is called the inverse of* a.

Definition 2.5([7]) *Let* S *be a ternary semiring and* M *be an ideal of* S. *Then* M *is called maximal (largest) ideal of* S *if* $M \neq S$ *and there does not exist any other ideal* I *of* S *such that* $M \subset I \subset S$.

§3. Radical Class

In this section, the radical class for ternary semiring is defined on the lines of Kurosh [1]. Also discussed some properties and theorems on radical classes for ternary semirings on the line of [2] and [4].

Definition 3.1 *A class* $\mathcal{R}$ *of ternary semirings is called radical class if*

(a) $\mathcal{R}$ *is homomorphically closed;*

(b) *Every ternary semiring* $S \in \mathbb{U}$, *where* $\mathbb{U}$ *is the universal class of ternary semirings, contains a largest* $\mathcal{R}$*-*k*-ideal,* $\mathcal{R}(S)$*;*

(c) *If* $S \in \mathbb{U}$, *then* $S/\mathcal{R}(S)$ *is* $\mathcal{R}$*-semisimple. i.e.* $\mathcal{R}(S/\mathcal{R}(S)) = 0$.

Proposition 3.2 *Assuming conditions* (a) *and* (b) *on a class* $\mathcal{R}$ *of ternary semirings, condition* (c) *is equivalent to*

(c') *If* I *is a* k*-ideal of the ternary semiring* S *and if both* $I, S/I \in \mathcal{R}$, *then* $S \in \mathcal{R}$.

Proof Let us consider that (c) holds and that both $I, S/I \in \mathcal{R}$. Then $I \subseteq \mathcal{R}(S)$ by condition (b) and $\phi : S/\mathcal{R}(S) \longmapsto (S/I)/(\mathcal{R}(S)/I)$ is isomorphic, which implies that $S/\mathcal{R}(S) \in \mathcal{R}$. But $0 = \mathcal{R}(S/\mathcal{R}(S)) = S/\mathcal{R}(S)$. Therefore, $S = \mathcal{R}(S)$ is in $\mathcal{R}$ and hence (c') hold.

Conversely, assume that condition (c') holds and that $\mathcal{R}(S/\mathcal{R}(S)) \neq 0$. Now, $\mathcal{R}(S/\mathcal{R}(S)) = K/\mathcal{R}(S)$ for some k-ideal K of S. Since both $\mathcal{R}(S)$ and $K/\mathcal{R}(S)$ are in $\mathcal{R}$, by (c') K is in $\mathcal{R}$. So, $K \subseteq \mathcal{R}(S)$ and $K/\mathcal{R}(S) = 0$, a contradiction. Thus (c) holds. □

The class $\mathcal{R}$ with the condition (c') is said to be closed under extensions.

Proposition 3.3 *Assuming conditions* (a) *and* (c') *on a class* $\mathcal{R}$ *of ternary semirings, condition* (b) *is equivalent to*

(b') *If* $I_1 \subset I_2 \subset \cdots \subset I_\lambda \subset \cdots$ *is an ascending chain of* k*-ideals of a ternary semiring* S *and if each* $I_\lambda \in \mathcal{R}$, *then* $\bigcup I_\lambda \in \mathcal{R}$.

Proof Consider that (b) holds and let $K = \bigcup I_\lambda$. Thus $K = \mathcal{R}(K)$ is in $\mathcal{R}$ and hence (b') holds.

Conversely, suppose that (b') holds. Then by applying the Zorn's lemma, we obtain a maximal (largest) $\mathcal{R}$-k-ideal K of S. If J is any $\mathcal{R}$-k-ideal of S, then $\phi : (K+J)/J \longmapsto K/(K \cap J)$ is isomorphic. Thus both J and $(K+J)/J$ are in $\mathcal{R}$ and by (c'), $K+J$ is in $\mathcal{R}$. Thus $\mathcal{R}(S) = K$ is in $\mathcal{R}$ and hence (b) holds. □

The class $\mathcal{R}$ with the condition (b') is said to has the inductive property.

Theorem 3.4 *A class* $\mathcal{R}$ *of ternary semirings is called radical class if*

(a) $\mathcal{R}$ *is homomorphically closed;*
(b') $\mathcal{R}$ *has the inductive property;*
(c') $\mathcal{R}$ *is closed under extensions.*

Theorem 3.5 *For any class* $\mathcal{R}$ *of ternary semirings, the following conditions are equivalent:*

(I) $\mathcal{R}$ *is radical class;*

(II) $(R1)$ *If* $S \in \mathcal{R}$, *then for every* $S \longmapsto T \neq 0$ *there is a* k*-ideal* I *in* T *such that* $0 \neq I \in \mathcal{R}$;

$(R2)$ *If* $S \in \mathbb{U}$ *and for every* $S \longmapsto T \neq 0$ *there is a* k*-ideal* I *in* T *such that* $0 \neq I \in \mathcal{R}$, *then* $S \in \mathcal{R}$;

(III) $\mathcal{R}$ *satisfies condition* $(R1)$, *has the inductive property and is closed under extensions.*

Proof $(I) \Longrightarrow (III)$: It is immediate from Theorem 3.4.

$(III) \Longrightarrow (II)$: Let S be a ternary semiring such that for every $S \longmapsto T \neq 0$ there is a k-ideal I in T such that $0 \neq I \in \mathcal{R}$ and that $S \notin \mathcal{R}$. By inductive property and applying Zorn's lemma, we obtain a maximal k-ideal $J \in S$ with respect to being in $\mathcal{R}$. Since $S \notin \mathcal{R}, S/J \neq 0$ holds. Then there exists an k-ideal I/J of S/J such that $0 \neq I/J \in \mathcal{R}$ which implies $I \in \mathcal{R}$. But this contradicts the maximality of J and thus we have $(R2)$ and hence (II).

$(II) \Longrightarrow (I)$: Its immediate from $(R2)$ that $\mathcal{R}$ is homomorphically closed. Let $I_1 \subset I_2 \subset \cdots \subset I_\lambda \subset \cdots$ is an ascending chain of k-ideals of a ternary semiring S such that each $I_\lambda \in \mathcal{R}$. Let $(\bigcup I_\lambda)/J$ be any factor ternary semiring of $\bigcup I_\lambda$. Then there exists an index λ such that $I_\lambda \nsubseteq J$ and thus $0 \neq (I_\lambda + J)/J$ is in $(\bigcup I_\lambda)/J$. Also $(I_\lambda + J)/J$ is isomorphic to $I_\lambda/(I_\lambda \cap J)$ which is in $\mathcal{R}$. Thus, by $(R2)$ we have $\bigcup I_\lambda \in \mathcal{R}$ and that $\mathcal{R}$ has the inductive property. Now, consider J and S/J both in $\mathcal{R}$. Let S/K be any non-zero factor ternary semiring of S. In this case when $J \subseteq K$, $0 \neq (S/K)$ is isomorphic to $(S/J)/(K/J)$ and this is in $\mathcal{R}$. In this case when $J \nsubseteq K$, $0 \neq (J+K)/K$ is in S/K and $(J+K)/K$ is isomorphic to $J/(J \cap K)$ and this is in $\mathcal{R}$. Thus, in both cases S/K has a non-zero k-ideal in $\mathcal{R}$ and by $(R2)$, S itself is in $\mathcal{R}$. Therefore $\mathcal{R}$ is closed under extensions and hence (I). □

Example 3.6 (1) *Nil Radical.* The class

$$\mathcal{N} = \{S \ | \ \forall\, a \in S \ \exists\, n > 1,\ n \text{ depending on } a, \text{ such that } a^n = 0\}$$

(i.e. the class of nil ternary semirings) is a radical class called the Nil-radical class.

(2) *Von-Neumann Radical.* A ternary semiring S is said to be Von-Neumann regular if for every $a \in S,\ a = aba,\ \forall b \in S$ or $a \in aSa$. The class

$$\mathcal{V} = \{S \ | \ S \text{ is Von-Neumann regular}\} = \{a \in S, a = aba,\ \forall\, b \in S\}$$

is a radical class.

§4. Semisimple Class and Upper Radical Class

In this section, the semisimple, hereditary and regular class for ternary semirings are defined on the lines of Kurosh [1]. Also discussed some properties and theorems on semisimple class for ternary semiring.

Definition 4.1 *A class $\mathcal{R}$ of ternary semirings is called hereditary if I is ideal of a ternary semiring S and $S \in \mathcal{R}$, then $I \in \mathcal{R}$.*

Definition 4.2 *A class $\mathcal{R}$ of ternary semirings is called regular if $S \in \mathcal{R}$ and I is non-zero ideal of a ternary semiring S, then there is a non-zero homomorphic image of I in $\mathcal{R}$.*

Remark 4.3 *In particular, every hereditary class is clearly regular.*

Definition 4.4 *A class $\mathcal{S}$ of ternary semirings is called semisimple class if*

$(S1)$ If $S \in \mathcal{S}$, then every non-zero ideal of S has a non-zero homomorphic image in $\mathcal{S}$
$(S2)$ If every non-zero ideal of S has a non-zero homomorphic image in $\mathcal{S}$, then $S \in \mathcal{S}$.

Proposition 4.5 *If $\mathcal{R}$ is a radical class of ternary semirings, then it admits a semisimple class $\mathbb{S}_{\mathcal{R}} = \{S \in \mathbb{U} : \mathcal{R}(S) = 0\}$.*

Proof Let $S \in \mathbb{S}_{\mathcal{R}}$ and I be any non-zero ideal of S such that I has no non-zero homomorphic image in $\mathbb{S}_{\mathcal{R}}$. As $\mathcal{R}$ is radical class, $\mathcal{R}(I/\mathcal{R}(I)) = 0$ and this implies $I/\mathcal{R}(I) \in \mathbb{S}_{\mathcal{R}}$. Thus $I/\mathcal{R}(I) = 0$ and $I = \mathcal{R}(I) \in \mathcal{R}$. Then $0 \neq I \subseteq \mathcal{R}(S)$, which is contradicting to $\mathcal{R}(S) = 0$ and $(S1)$ holds for $\mathbb{S}_{\mathcal{R}}$. Now, if $S \notin \mathbb{S}_{\mathcal{R}}$ then $\mathcal{R}(S) \neq 0$. Since $\mathcal{R}$ is homomorphically closed, no non-zero homomorphic image of $\mathcal{R}(S)$ is in $\mathbb{S}_{\mathcal{R}}$. Thus, a contrapositive form of $(S2)$ holds for $\mathbb{S}_{\mathcal{R}}$. □

The operator $\mathbb{S}$ is called the semisimple operator.

Theorem 4.6 *If $\mathcal{R}$ is a regular class of ternary semirings then the class*

$$\mathcal{U}_{\mathcal{R}} = \{S : \ S \text{ has no nonzero homomorphic image in } \mathcal{R}\}$$

is a radical class, $\mathcal{R} \cap \mathcal{U}_{\mathcal{R}} = 0$ and $\mathcal{U}_{\mathcal{R}}$ is the largest radical having zero intersection with $\mathcal{R}$.

Proof Let S has a non-zero homomorphic image T such that T has no non-zero ideal in $\mathcal{U}_{\mathcal{R}}$, then $S \notin \mathcal{U}_{\mathcal{R}}$. If such a T exists, then $T \notin \mathcal{U}_{\mathcal{R}}$ and T must have a non-zero homomorphic image V in $\mathcal{R}$ and which is also a non-zero homomorphic image of S in $\mathcal{R}$. Therefore $S \notin \mathcal{U}_{\mathcal{R}}$ and a contrapositive form of $(R1)$ holds for $\mathcal{U}_{\mathcal{R}}$.

Now, assume that $S \notin \mathcal{U}_{\mathcal{R}}$, then S has a non-zero homomorphic image T in $\mathcal{R}$. Since T is regular, every non-zero ideal of T has a non-zero homomorphic image in $\mathcal{R}$. Thus, a contrapositive form of $(R2)$ holds for $\mathcal{U}_{\mathcal{R}}$. Hence, by Theorem 3.5, $\mathcal{U}_{\mathcal{R}}$ is a radical class. □

The operator $\mathcal{U}$ is called the upper radical operator and $\mathcal{U}_{\mathcal{R}}$ is called the upper radical of the class $\mathcal{R}$.

Theorem 4.7 *For any semisimple class $\mathcal{S}$ and radical class $\mathcal{R}$ we have $\mathbb{S}\mathcal{U}_{\mathcal{S}} = \mathcal{S}$ and $\mathcal{U}\mathbb{S}_{\mathcal{R}} = \mathcal{R}$.*

Proof Let $S \in \mathcal{S}$ Then by using $(S1)$ and definition of upper radical we have $S \in \mathbb{S}\mathcal{U}_{\mathcal{S}}$. Also, by using $(S2)$ and definition of upper radical we have $\mathbb{S}\mathcal{U}_{\mathcal{S}} \subseteq \mathcal{S}$. Hence $\mathbb{S}\mathcal{U}_{\mathcal{S}} = \mathcal{S}$.

Similarly, using $(R1)$ and $(R2)$ we have $\mathcal{U}\mathbb{S}_{\mathcal{R}} = \mathcal{R}$. □

Theorem 4.8 *Every semisimple class $\mathcal{S}$ is closed under extensions.*

Proof Let I is a k-ideal of the ternary semiring S such that both $I, S/I \in \mathcal{S}$. Then $(\mathcal{U}_{\mathcal{S}}(S)+I)/I$ is isomorphic to $\mathcal{U}_{\mathcal{S}}(S)/(\mathcal{U}_{\mathcal{S}}(S) \cap I)$ and this is in $\mathcal{U}_{\mathcal{S}}$. Also $(\mathcal{U}_{\mathcal{S}}(S)+I)/I \triangleleft S/I \in \mathcal{S} = \mathbb{S}\mathcal{U}_{\mathcal{S}}$. Thus $(\mathcal{U}_{\mathcal{S}}(S)+I)/I$ must be 0 and so $\mathcal{U}_{\mathcal{S}}(S) \subseteq I$.

Now $\mathcal{U}_{\mathcal{S}}(S) \triangleleft S$, also $\mathcal{U}_{\mathcal{S}}(S) \triangleleft I$. Since $\mathcal{U}_{\mathcal{S}}(S) \in \mathcal{U}_{\mathcal{S}}$, we have $\mathcal{U}_{\mathcal{S}}(S) = \mathcal{U}_{\mathcal{S}}(I) = 0$. Therefore $S \in \mathbb{S}\mathcal{U}_{\mathcal{S}} = \mathcal{S}$ and hence the semisimple class $\mathcal{S}$ is closed under extensions. □

Theorem 4.9 *The classes $\mathcal{R}$ and $\mathcal{S}$ are corresponding radical and semisimple classes if and only if*

(*i*) *$S \in \mathcal{R}$ and $S \longrightarrow T \neq 0$ imply $T \notin \mathcal{S}$, that is, $\mathcal{R} \subseteq \mathcal{U}_{\mathcal{S}}$;*

(*ii*) *$S \in \mathcal{S}$ and a non-zero k-ideal T of S imply $T \notin \mathcal{R}$, that is, $\mathcal{S} \subseteq \mathbb{S}_{\mathcal{R}}$;*

(*iii*) *Every ternary semiring $S \in \mathbb{U}$ has an k-ideal I such that $I \in \mathcal{R}$ and $S/I \in \mathcal{S}$.*

Proof If $\mathcal{R}$ and $\mathcal{S}$ are corresponding radical and semisimple classes, then the if part is obvious (to get (*iii*) just take $T = \mathcal{R}(S)$). Conversely, suppose that the classes $\mathcal{R}$ and $\mathcal{S}$ satisfying these thrre conditions.

Now, let a ternary semiring $S \in \mathcal{U}_{\mathcal{S}}$. Then by (*iii*), S has an k-ideal $T \in \mathcal{R}$ such that $S/T \in \mathcal{S}$ and this implies that $S/T = 0$. Thus $S = T \in \mathcal{R}$ holds and proving $\mathcal{U}_{\mathcal{S}} \subseteq \mathcal{R}$. And by using (*i*) we have $\mathcal{R} = \mathcal{U}_{\mathcal{S}}$. Similarly, we have $\mathcal{S} = \mathbb{S}_{\mathcal{R}}$. Since, $\mathcal{S} = \mathbb{S}_{\mathcal{R}} = \mathbb{S}\mathcal{U}_{\mathcal{S}}$, also $\mathcal{S} \subseteq \mathbb{S}\mathcal{U}_{\mathcal{S}}$ holds and this is the regularity of the class $\mathcal{S}$. Hence $\mathcal{R} = \mathcal{U}_{\mathcal{S}}$ is a radical class and

$\mathcal{S} = \mathbb{S}\mathcal{U}_{\mathcal{S}} = \mathbb{S}_{\mathcal{R}}$ the corresponding semisimple class. □

Proposition 4.10 *A semisimple class $\mathcal{S}$ is hereditary if and only if the corresponding radical class $\mathcal{R} = \mathcal{U}_{\mathcal{S}}$ satisfies*

($S3$) $\mathcal{R}(I) \subseteq \mathcal{R}(S)$, *for every k-ideal I of S.*

Proof Let ($S3$) holds, then for any $S \in \mathcal{S}$ and any k-ideal I of S we have $\mathcal{R}(I) \subseteq \mathcal{R}(S) = 0$. Thus $I \in \mathcal{S}$ and hence $\mathcal{S}$ is hereditary.

Conversely, assume that a semisimple class $\mathcal{S}$ is hereditary. Then for any k-ideal I of S we have $(\mathcal{R}(I) + \mathcal{R}(S))/\mathcal{R}(S) \triangleleft (I + \mathcal{R}(S))/\mathcal{R}(S) \triangleleft S/\mathcal{R}(S) \in \mathcal{S}$. Since $\mathcal{S}$ is hereditary, $(I+\mathcal{R}(S))/\mathcal{R}(S) \in \mathcal{S}$ and $(\mathcal{R}(I)+\mathcal{R}(S))/\mathcal{R}(S) \in \mathcal{S}$. But this implies that $\mathcal{R}(I)/(\mathcal{R}(I)\cap\mathcal{R}(S)) \cong (\mathcal{R}(I) + \mathcal{R}(S))/\mathcal{R}(S) \in \mathcal{R} \cap \mathcal{S} = 0$. Hence $\mathcal{R}(I) \subseteq \mathcal{R}(S)$. □

§5. Hoehnke Radical

With an axiomatic point of view an assignment $\mathcal{R} : S \longrightarrow \mathcal{R}(S)$ designating a certain k-ideal $\mathcal{R}(S)$ to every ternary semiring S is called a Hoehnke radical if:

(i) $f(\mathcal{R}(S)) \subseteq \mathcal{R}(f(S))$, for every homomorphism $f : S \longrightarrow \mathcal{R}(S)$;

(ii) If $S \in \mathbb{U}$, then $S/\mathcal{R}(S)$ is $\mathcal{R}$-semisimple. i.e. $\mathcal{R}(S/\mathcal{R}(S)) = 0$.

A Hoehnke radical $\mathcal{R}$ may satisfy also the following conditions:

(iii) $\mathcal{R}$ is complete: if I is a k-ideal of S and $\mathcal{R}(I) = I$ then $I \subseteq \mathcal{R}(S)$;

(iv) $\mathcal{R}$ is idempotent: if $\mathcal{R}(\mathcal{R}(S)) = \mathcal{R}(S)$, for every ternary semiring S.

Proposition 5.1 *If $\mathcal{R}$ is a radical class then the assignment $\mathcal{R} : S \longrightarrow \mathcal{R}(S)$ is a complete, idempotent Hoehnke radical. Conversely, if $\mathcal{R}$ is a complete, idempotent Hoehnke radical, then there is a radical class $\wp$ such that $\mathcal{R}(S) = \wp(S)$ for every ternary semiring S. Moreover, $\wp = \{S/\mathcal{R}(S) = S\}$.*

Proof (i) and (ii) is obvious. Since $\mathcal{R}(S)$ is a largest $\mathcal{R}$-k-ideal of S. So, for any k-ideal I of S, $\mathcal{R}(I) = I$ implies that $I \subseteq \mathcal{R}(S)$. Also, for every ternary semiring S,

$$\mathcal{R}(\mathcal{R}(S)) = \mathcal{R}(S).$$

This proves (iii) and (iv).

Conversely, suppose that $\mathcal{R}$ is a complete, idempotent Hoehnke radical, and let define the class $\wp$ by $\wp = \{S/\mathcal{R}(S) = S\}$. If $S \in \wp$ and $f : S \longrightarrow T$ is a surjective homomorphism, then by (i),

$$T = f(S) = f(\mathcal{R}(S)) \subseteq \mathcal{R}(f(S)) = \mathcal{R}(T).$$

So $T \in \wp$. Thus (a) holds for $\wp$.

If I is any k-ideal of S and $\mathcal{R}(I) = I$ then $I \subseteq \wp(S)$ and by (iii), $I \subseteq \mathcal{R}(\wp(S))$, therefore $\wp(S) = \mathcal{R}(\wp(S))$ which implies that $\wp(S)$ is a largest $\wp$-k-ideal of S. Thus (b) holds for $\wp$.

Now, $\wp(S) = \mathcal{R}(\wp(S))$ and (iii) implies that $\wp(S) \subseteq \mathcal{R}(S)$. But by (iv), $\mathcal{R}(S) \subseteq \wp(S)$. Thus $\mathcal{R}(S) = \wp(S)$ for every ternary semiring S. Therefore

$$\wp(S/\wp(S)) = \mathcal{R}(S/\mathcal{R}(S)) = 0$$

and (c) holds for $\wp$. □

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