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On Prime Graph $PG_2(R)$ of a Ring

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Abstract: In the present paper we define a simple undirected graph $PG_2(R)$ with all the elements of a ring R as vertices, and two distinct vertices x, y are adjacent if and only if either $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in Z(R)$, the set of all zero divisors of R (including zero). We have proved that $PG_2(\mathbb{Z}_n)$ is Eulerian for any odd positive integer n. Also we discuss the Planarity and girth of $PG_2(R)$ and some cases which gives the degree of all vertices in $PG_2(R)$, over a ring $\mathbb{Z}_n$, for $n \leq 100$.

Key Words: Ring, prime graph of a ring $PG(R)$, degree, planarity, girth.

AMS(2010): 05C25, 05C90, 05C99.

§1. Introduction

The study of graph theory for a commutative ring began when Beck in [1] introduced the notion of zero divisor of the graph. The graph $\Gamma_2(R)$ defined by R. Sen Gupta et al. [2] as: let R be a ring with unity and let $G = (V, E)$ be an undirected graph in which $V = R$ - $\{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a zero divisor (including zero). Another graph structure associated to a ring called prime graph was introduced by Satyanarayana et al. [3]. Prime graph is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. Pawar and Joshi in [5] gave a simple formulation for finding the degrees of vertices of prime graph $PG(R)$ as well as it's complement $(PG(R))^c$. Also the number of triangles in $PG(R)$ and $(PG(R))^c$ have been calculated using simple combinatorial approach. We introduced the prime graph $PG_1(R)$ of a ring and discussed all the results related to degree of vertices, Eulerianity, planarity and girth in [6]. Here, we introduced a new type of graph called $PG_2(R)$ as a generalization of [2].

In second section of this paper we give definition and some examples of $PG_2(R)$. In next section we try to find the degree of vertices in $PG_2(R)$ by distributing the vertex set $V(G)$ into

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two sets viz. the set of all zero-divisors and the set of all units and discussed some more cases which gives the degree of all vertices in $PG_2(\mathbb{Z}_n)$, for $n \leq 100$. In last section, we discussed the eulerianity, planarity and girth of $PG_2(R)$.

We refer to [3]-[4] for basic terminology and definitions.

§2. The Prime Graph $PG_2(R)$ of a Ring

Definition 2.1 *The prime graph $PG_2(R)$ is a graph with all the elements of a ring R as vertices, and any two distinct vertices x, y are adjacent if and only if $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in Z(R)$, the set of all zero-divisors of R.*

Example 2.2 Consider $\mathbb{Z}_n$, the ring of integers modulo n.

(1) Let $R = \mathbb{Z}_2$. The vertex set $V(PG_2(\mathbb{Z}_2)) = \{0, 1\}$. Since $0R1 = 0$, the edge set $E(PG_2(\mathbb{Z}_2)) = \{01\}$ and the graph $PG_2(\mathbb{Z}_2)$ as shown in figure below.

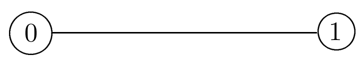

FIGURE 1. $PG_2(\mathbb{Z}_2)$

(2) Let $R = \mathbb{Z}_3$. The vertex set $V(PG_2(\mathbb{Z}_3)) = \{0, 1, 2\}$. Since $0R1 = 0, 0R2 = 0, 1+2 = 0$, the edge set $E(PG_2(\mathbb{Z}_3)) = \{01, 02, 12\}$ and the graph $PG_2(\mathbb{Z}_3)$ as shown in figure below-

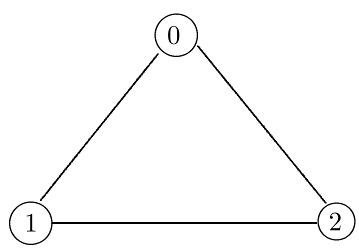

FIGURE 2. $PG_2(\mathbb{Z}_3)$

(3) Let $R = \mathbb{Z}_4$. The vertex set $V(PG_2(\mathbb{Z}_4)) = \{0, 1, 2, 3\}$, the edge set $E(PG_2(\mathbb{Z}_4)) = \{01, 02, 03, 13\}$ and the graph $PG_2(\mathbb{Z}_4)$ as shown in figure below-

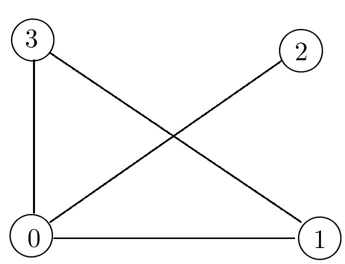

FIGURE 3. $PG_2(\mathbb{Z}_4)$

(4) Let $R = \mathbb{Z}_5$. The vertex set $V(PG_2(\mathbb{Z}_5)) = \{0, 1, 2, 3, 4\}$, the edge set $E(PG_2(\mathbb{Z}_5)) =$

$\{01, 02, 03, 04, 14, 23\}$ and the graph $PG_2(\mathbb{Z}_5)$ as shown in figure below.

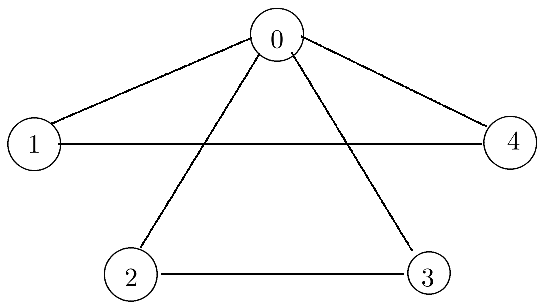

FIGURE 4. $PG_2(\mathbb{Z}_5)$

§3. Degree of Vertices in $PG_2(\mathbb{Z}_n)$

In this section, we find the degree of every vertex of $PG_2(\mathbb{Z}_n)$, for $n \leq 100$ by giving some illustrative examples.

Theorem 3.1 *$PG_2(\mathbb{Z}_n)$ is never complete graph unless $n = 2$ or 3.*

Proof From Figures 1 and 2 we conclude the theorem. □

Theorem 3.2 *$PG_2(\mathbb{Z}_{2^r})$, where $r \in \mathbb{N} - \{1\}$, has two components consisting of zero divisors and units of $(\mathbb{Z}_{2^r})$ respectively. The first is $K_{2^{r-1}}$ consists of all zero divisors and the other is $K_{2^{r-1}+1}$ consists of all the units and the element zero.*

Proof From Figure 3 we conclude the theorem. □

Theorem 3.3 *Let F be a finite field with $|F| = p^n, p \geq 3$ for some prime p and $n \in \mathbb{N}$, then $PG_2(F)$ is a union of $(p^n - 1)/2$ copies of K_3 in which the element zero is adjacent to all the vertices.*

Proof From Figure 4 we conclude the theorem. □

Example 3.4 Let $R = \mathbb{Z}_6$. The vertex set $V(PG_2(\mathbb{Z}_6)) = \{0, 1, 2, 3, 4, 5\}$, the edge set $E(PG_2(\mathbb{Z}_6)) = \{01, 02, 03, 04, 05, 12, 13, 15, 23, 24, 34, 35, 45\}$ and the graph $PG_2(\mathbb{Z}_6)$ as shown in figure below.

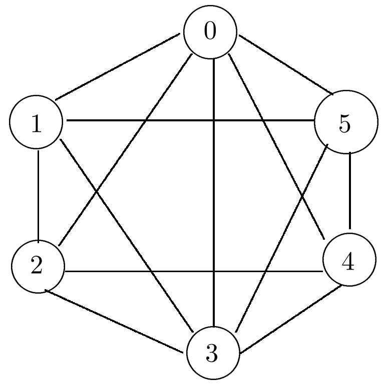

FIGURE 5. $PG_2(\mathbb{Z}_6)$

In $\mathbb{Z}_6$, zero-divisors $Z(\mathbb{Z}_6) = \{0, 2, 3, 4\}$, units $U(\mathbb{Z}_6) = \{1, 5\}$ and the value of $\phi(6) = 2$.

$$\begin{aligned} deg(0) &= n - 1 = 6 - 1 = 5 \\ deg(2) &= n - \phi(n) = 6 - 2 = 4 \\ deg(3) &= 2q - 1 = 2 \cdot 3 - 1 = 6 - 1 = 5 \\ deg(4) &= n - \phi(n) = 6 - 2 = 4 \end{aligned}$$

and as n is even,

$$\begin{aligned} deg(1) &= n - \phi(n) = 6 - 2 = 4 \\ deg(5) &= n - \phi(n) = 6 - 2 = 4. \end{aligned}$$

From Example 3.4 we conclude the following three results.

Theorem 3.5 *For any $n \in \mathbb{N}$, the degree of vertex zero in $PG_2(\mathbb{Z}_n)$ is $n - 1$.*

Theorem 3.6 *Let u be the unit element in a ring $\mathbb{Z}_n$, for any $n \in \mathbb{N}$, the degree of u in $PG_2(\mathbb{Z}_n)$ is*

$$\begin{aligned} deg(u) &= n - \phi(n), && \text{if } n \text{ is even} \\ &= n - \phi(n) + 1, && \text{if } n \text{ is odd.} \end{aligned}$$

Theorem 3.7 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, for any $n \in \mathbb{N}$ and $n = p \cdot q$, where p and q are distinct primes. Then the degree of z in $PG_2(\mathbb{Z}_n)$ is*

(a) If $p = 2$, then

$$\begin{aligned} deg(z) &= 2q - 1, && \text{if } z \text{ is multiple of } q \\ &= n - \phi(n), && \text{otherwise.} \end{aligned}$$

(*b*) *If* $p \neq 2$, *then*

$$
\begin{aligned}
deg(z) &= n - \phi(n) + (p-2), && \text{if } z \text{ is multiple of } p \\
&= 2q + (p-3), && \text{if } z \text{ is multiple of } q.
\end{aligned}
$$

Example 3.8 Let $R = \mathbb{Z}_9$. The vertex set $V(PG_2(\mathbb{Z}_9)) = \{0,1,2,3,4,5,6,7,8\}$, the edge set $E(PG_2(\mathbb{Z}_9)) = \{01,02,03,04,05,06,07,08,36,12,15,18,42,45,48,72,75,78\}$ and the graph $PG_2(\mathbb{Z}_9)$ as shown in figure below.

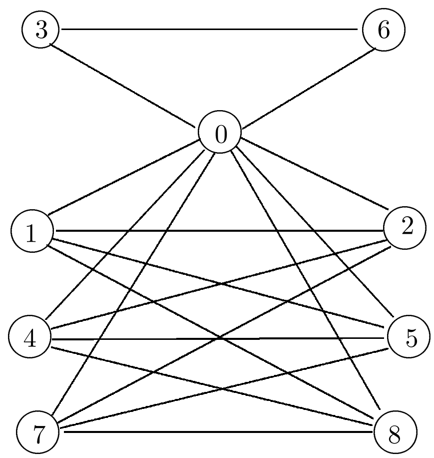

FIGURE 6. $PG_2(\mathbb{Z}_9)$

In $\mathbb{Z}_9$, zero-divisors $Z(\mathbb{Z}_9) = \{0,3,6\}$, units $U(\mathbb{Z}_9) = \{1,2,4,5,7,8\}$ and the value of $\phi(9) = 6$ and as n is odd,

$$
\begin{aligned}
deg(3,6) &= 9 - \phi(9) - 1 = 9 - 6 - 1 = 2 \\
deg(1,2,4,5,7,8) &= 9 - \phi(9) + 1 = 9 - 6 + 1 = 4.
\end{aligned}
$$

From Example 3.8 we conclude the following three results.

Theorem 3.9 *Let* $n = p^r$, *where* p *is an odd prime and* $r \in \mathbb{N} - \{1\}$ *then* $PG_2(\mathbb{Z}_n)$ *has* $(p+1)/2$ *components, one is* $K_{p^{r-1}}$ *consisting of the zero divisors and* $(p-1)/2$ *copies of* $K_{p^{r-1},p^{r-1}} \bigcup \{0\}$ *for the units and the element zero.*

Theorem 3.10 *Let* z *be a non-zero zero-divisor in a ring* $\mathbb{Z}_n$, *for any* $n \in \mathbb{N}$ *such that* $z^2 \equiv 0 \ (mod\ n)$. *Then the degree of* z *in* $PG_2(\mathbb{Z}_n)$ *is*

$$deg(z) = n - \phi(n) - 1.$$

Theorem 3.11 *Let* u *be the unit element and* z *be a non-zero zero-divisor in a ring* $\mathbb{Z}_{p^2}$, *for* *any prime p. Then from the Theorem 3.6 the degree of u is*

$$\begin{aligned} deg(u) &= n - \phi(n), && \text{if } n \text{ is even} \\ &= n - \phi(n) + 1, && \text{if } n \text{ is odd} \end{aligned}$$

and from the Theorem 3.10 the degree of z is

$$deg(z) = n - \phi(n) - 1.$$

Example 3.12 Let $R = \mathbb{Z}_{2^n p}$, for any $n \in \mathbb{N}$, where p is prime,

(a) If $p = 2$, then

(1) If $n = 1$, $R = \mathbb{Z}_4$, the non-zero zero-divisor is 2. Hence

$$deg(2) = 4 - \phi(4) - 1 = 4 - 2 - 1 = 1.$$

(2) If $n = 2$, $R = \mathbb{Z}_8$, the set of non-zero zero-divisors, $Z(\mathbb{Z}_8) - \{0\} = \{2, 4, 6\}$. So

$$deg(2, 4, 6) = 8 - \phi(8) - 1 = 8 - 4 - 1 = 3.$$

(3) If $n = 3$, $R = \mathbb{Z}_{16}$, the set of non-zero zero-divisors, $Z(\mathbb{Z}_{16}) - \{0\} = \{2, 4, 6, 8, 10, 12, 14\}$. Therefore

$$deg(2, 4, 6, 8, 10, 12, 14) = 16 - \phi(16) - 1 = 16 - 8 - 1 = 7.$$

Similarly, we find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{32}, \mathbb{Z}_{64}$ and so on. In general, we conclude that if $p = 2$, then

$$deg(z) = n - \phi(n) - 1.$$

(b) If $p \neq 2$, then

(1) If $n = 1$, $R = \mathbb{Z}_{2p}$ where $p = 3, 5, 7, \cdots$ then by Theorem 3.7

$$\begin{aligned} deg(z) &= n - \phi(n) + (p - 2), && \text{if } z \text{ is multiple of } p \\ &= 2q + (p - 3), && \text{if } z \text{ is multiple of } q. \end{aligned}$$

The results are same for $R = \mathbb{Z}_{10}, \mathbb{Z}_{14}$ and so on.

(2) If $n = 2$, $R = \mathbb{Z}_{4p}$, where $p = 3, 5, 7, \cdots$. Let $p = 3$, $R = \mathbb{Z}_{12}$, the set of non-zero zero-divisors, $Z(\mathbb{Z}_{12}) - \{0\} = \{2, 4, 6, 8, 10, 3, 9\}$ and $6^2 \equiv 0 (mod\ 12)$. Hence

$$\begin{aligned} & deg(6) = 12 - \phi(12) - 1 = 12 - 4 - 1 = 7 \\ & deg(3, 9) = 12 - 4 - 1 = 7, && \text{if } z \text{ is multiple of } p \end{aligned}$$

$$\begin{aligned}
deg(2) &= 12 - 4 - 1 = 7, && \text{if } z \text{ is multiple of } 2^1\\
deg(8) &= 12 - 4 - 1 = 7, && \text{if } z \text{ is multiple of } 2^1\\
deg(10) &= 12 - 4 - 1 = 7, && \text{if } z \text{ is multiple of } 2^1\\
deg(4) &= 12 - 4 - 1 + 2 = 9, && \text{if } z \text{ is multiple of } 2^2.
\end{aligned}$$

The results are same for $R = \mathbb{Z}_{20}, \mathbb{Z}_{28}$ and so on.

(3) If $n = 3$, $R = \mathbb{Z}_{8p}$, where $p = 3, 5, 7, \cdots$. Let $p = 3$, $R = \mathbb{Z}_{24}$, the set of non-zero zero-divisor, $Z(\mathbb{Z}_{24}) - \{0\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 3, 9, 15, 21\}$. Therefore

$$\begin{aligned}
deg(6, 18) &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2p\\
deg(12) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n)\\
deg(3, 9, 15, 21) &= n - \phi(n) + (p - 2), && \text{if } z \text{ is multiple of } p\\
deg(2, 4, 10, 14, 16, 20, 22) &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2^1\\
deg(8) &= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n.
\end{aligned}$$

The results are same for $R = \mathbb{Z}_{40}, \mathbb{Z}_{56}$ and so on. In general, we conclude that if $p \neq 2$, then

$$\begin{aligned}
deg(z) &= n - \phi(n) + (p - 2), && \text{if } z \text{ is multiple of } p\\
&= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n)\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2p\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2, 2^2, ., 2^{n-1}\\
&= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n.
\end{aligned}$$

From Example 3.12 we conclude the following theorem.

Theorem 3.13 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_{2^n p}$, for any $n \in \mathbb{N}$, where p is prime*

(*a*) *If $p = 2$, then*

$$deg(z) = n - \phi(n) - 1.$$

(*b*) *If $p \neq 2$, then*

$$\begin{aligned}
deg(z) &= n - \phi(n) + (p - 2), && \text{if } z \text{ is multiple of } p\\
&= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n)\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2p\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2, 2^2, \cdots, 2^{n-1}\\
&= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n.
\end{aligned}$$

Example 3.14 Let $R = \mathbb{Z}_{2^n p^2}$, for any $n \in \mathbb{N}$, where p is odd prime.

(a) If $n = 1$, then $R = \mathbb{Z}_{2p^2}$, where $p = 3, 5, 7, \cdots$.

(1) Let $p = 3$, $R = \mathbb{Z}_{18}$ and 6^2, $12^2 \equiv 0(mod\ 18)$. Hence

$$\begin{aligned}
deg(6,12) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0\ (mod\ n)\\
deg(3,15) &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p\\
deg(9) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2\\
deg(2,4,8,10,14,16) &= n - \phi(n), && \text{if } z \text{ is multiple of } 2^1.
\end{aligned}$$

(2) Let $p = 5$, $R = \mathbb{Z}_{50}$ and 10^2, 20^2, 30^2, $40^2 \equiv 0(mod\ 50)$. So

$$\begin{aligned}
deg(10,20,30,40) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0\ (mod\ n)\\
deg(5,15,35,45) &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p\\
deg(25) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2\\
deg(2,4,6,\cdots,48) &= n - \phi(n), && \text{if } z \text{ is multiple of } 2^1.
\end{aligned}$$

The results are same for $R = \mathbb{Z}_{98}, \mathbb{Z}_{242}$ and so on. In general, we conclude that if $n = 1$, then

$$\begin{aligned}
deg(z) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2\\
&= n - \phi(n) - 1, && \text{if } z^2 \equiv 0\ (mod\ n)\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p\\
&= n - \phi(n), && \text{if } z \text{ is multiple of } 2.
\end{aligned}$$

(b) If $n \neq 1$,

(1) If $n = 2$, $R = \mathbb{Z}_{4p^2}$, where $p = 3, 5, 7, \cdots$, then $R = \mathbb{Z}_{36}$, $\mathbb{Z}_{100}$ and so on. If $n = 3$, $R = \mathbb{Z}_{8p^2}$, where $p = 3, 5, 7, \cdots$, then $R = \mathbb{Z}_{72}$, $\mathbb{Z}_{200}$ and so on. Therefore, we conclude the result as

$$\begin{aligned}
deg(z) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2\\
&= n - \phi(n) - 1, && \text{if } z^2 \equiv 0\ (mod\ n)\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2, 2^2, \cdots, 2^{n-1}\\
&= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n.
\end{aligned}$$

From Example 3.14 we conclude the following theorem.

Theorem 3.15 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_{2^n p^2}$, for any $n \in \mathbb{N}$, where p is odd prime*

(a) *If $n = 1$, then*

$$\begin{aligned} deg(z) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2 \\ &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \\ &= n - \phi(n), && \text{if } z \text{ is multiple of } 2. \end{aligned}$$

(b) *If $n \neq 1$, then*

$$\begin{aligned} deg(z) &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2 \\ &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } 2, 2^2, \cdots, 2^{n-1} \\ &= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n. \end{aligned}$$

Example 3.16 Let $R = \mathbb{Z}_{2^n pq}$, for any $n \in \mathbb{N}$, where p and q are distinct odd primes and $p < q$. Then

(a) Let $n = 1$,

(1) $R = \mathbb{Z}_{2pq}$, where $p = 3$ and $q = 5, 7, 11, \cdots$. We find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{30}, \mathbb{Z}_{42}, \mathbb{Z}_{66}, \mathbb{Z}_{78}$ and so on.

(2) $R = \mathbb{Z}_{2pq}$, where $p = 5$ and $q = 7, 11, 13, \cdots$. We find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{70}, \mathbb{Z}_{110}$ and so on.

(b) Let $n \neq 1$.

(1) If $n = 2$, $R = \mathbb{Z}_{4pq}$, where $p = 3$ and $q = 5, 7, 11, \cdots$ then we find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{60}, \mathbb{Z}_{84}$ and so on.

(2) If $n = 3$, $R = \mathbb{Z}_{8pq}$, where $p = 3$ and $q = 5, 7, 11, \cdots$ then we find the degree of all non-zero zero-divisors in $R = \mathbb{Z}_{120}, \mathbb{Z}_{168}$ and so on.

From Example 3.16 and previous discussion we conclude results following.

Theorem 3.17 *Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_{2^n pq}$, for any $n \in \mathbb{N}$, where p and q are distinct odd primes and $p < q$.*

(a) *If $n = 1$, then*

$$\begin{aligned} deg(z) &= n - \phi(n), && \text{if } z \text{ is multiple of } 2 \\ &= n - \phi(n) + p - 2, && \text{if } z \text{ is multiple of } p \ or \ 2p \\ &= n - \phi(n) + q - 2, && \text{if } z \text{ is multiple of } q \ or \ 2q \\ &= 2pq - 1, && \text{if } z \text{ is multiple of } pq. \end{aligned}$$

(*b*) *If* $n \neq 1$, *then*

$$\begin{aligned} deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) + pq - (p+q), && \text{if } z \text{ is multiple of } pq \\ &= n - \phi(n) + p - 2, && \text{if } z \text{ is multiple of } p \\ &= n - \phi(n) + q - 2, && \text{if } z \text{ is multiple of } q \\ &= n - \phi(n) - 1 + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n \\ &= n - \phi(n) - 1 + 2^n, && \text{if } z \text{ is multiple of } 2^n p \\ &= n - \phi(n)/2 - 1, && \text{if } z \text{ is multiple of } 2^n q \\ &= n - \phi(n) - 1, && \text{otherwise.} \end{aligned}$$

We are also discussed some more cases in continuation to Theorem 3.5− Theorem 3.17 which calculates the degree of vertices in $PG_2(\mathbb{Z}_n)$, for $n \leq 100$.

Case 1. (*a*) Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = 3pq$ where $p = 3$, $q = 5, 7, 11, 13, \cdots$. Then

$$\begin{aligned} deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) + q - 2, && \text{if } z \text{ is multiple of } q \\ &= n - \phi(n) + 2p - 1, && \text{if } z \text{ is multiple of } 3p \\ &= n - \phi(n) - 1, && \text{otherwise.} \end{aligned}$$

(*b*) In this case when $p = q = 3$, then $deg(z) = n - \phi(n) - 1$.

Case 2. Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = 3p^2$, $p = 3, 5, 7, \cdots$. Then

$$\begin{aligned} deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \ and \ 3p \\ &= n - \phi(n) + 1, && \text{if } z \text{ is multiple of } 3 \\ &= n - \phi(n) - 1 + p(p-1), && \text{if } z \text{ is multiple of } p^2. \end{aligned}$$

Case 3. Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = 2p^3$, $p = 3, 5, 7, \cdots$, $p > 2$. Then

$$\begin{aligned} deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p \ and \ p^2 \\ &= n - \phi(n), && \text{if } z \text{ is multiple of } 2 \\ &= n - \phi(n) - p, && \text{if } z \text{ is multiple of } 2p \\ &= n - \phi(n) - p + 1, && \text{if } z \text{ is multiple of } 2p^2 \\ &= n - \phi(n) - 1 + 2p^2, && \text{if } z \text{ is multiple of } p^3. \end{aligned}$$

Case 4. Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = p^4$, $p = 2, 3, 5, 7, \cdots$. Then

$$deg(z) = n - \phi(n) - 1.$$

Case 5. Let z be a non-zero zero-divisor in a ring $\mathbb{Z}_n$, $n = 2p^2q$, $p = 3$, $q = 5, 7, 11, \cdots$. Then

$$\begin{aligned}
deg(z) &= n - \phi(n) - 1, && \text{if } z^2 \equiv 0 \ (mod\ n)\\
&= n - \phi(n) - 1, && \text{if } z \text{ is multiple of } p,\ 2p \textit{ and } pq\\
&= n - \phi(n) + 2p - 1, && \text{if } z \text{ is multiple of } 2p^2 \text{ and } p^2\\
&= n - \phi(n) + 2q + 1, && \text{if } z \text{ is multiple of } 2q\\
&= n - \phi(n) + q - 2, && \text{if } z \text{ is multiple of } q\\
&= n - 1, && \text{if } z \text{ is multiple of } p^2q\\
&= n - \phi(n), && \text{if } z \text{ is multiple of } 2.
\end{aligned}$$

§4. Eulerianity, Planarity and Girth of $PG_2(\mathbb{Z}_n)$

In this section, we proved that $PG_2(\mathbb{Z}_n)$ is Eulerian for any odd positive integer n and is planar if and only if $n = 4, 6$ or n is a prime number. Also, we found that the girth of $PG_2(\mathbb{Z}_n)$ is 3, for $n \neq 2$.

Theorem 4.1 *$PG_2(\mathbb{Z}_n)$ is Eulerian, when n is odd positive integer.*

Proof Let n be even, so from Theorem 3.5, we have that $deg(0) = n - 1$, which is an odd number, so not Eulerian. Again if n is odd, then by Theorems $3.6 - 3.11$ and from the above discussion, degree of every vertex in $PG_2(\mathbb{Z}_n)$ is an even number. Hence, $PG_2(\mathbb{Z}_n)$ is Eulerian, when n is odd positive integer. □

Theorem 4.2 *$PG_2(\mathbb{Z}_n)$ is planar if and only if $n = 4, 6$ or n is a prime.*

Proof We discuss different cases for planarity of $PG_2(\mathbb{Z}_n)$.

Case 1. For $n = 2$, $PG_2(\mathbb{Z}_2)$ is a complete graph K_2. Hence it is a planar graph.

Case 2. For $n = 3$, $PG_2(\mathbb{Z}_3)$ is complete graph K_3. Therefore it is a planar graph.

Case 3. If n is prime and $n > 3$, $PG_2(\mathbb{Z}_n)$ is a union of copies of K_3 in which again zero is a common vertex. So, the graph is planar when n is prime.

Case 4. If $n = 4$, $PG_2(\mathbb{Z}_4)$ has two components consisting of zero divisors and units of $\mathbb{Z}_2^2$. The first is K_2 and the other is K_3 in which zero is again a common vertex, hence planar.

Case 5. If $n = 6$, $PG_2(\mathbb{Z}_6)$ is union of eight copies of K_3 hence planar.

Case 6. If $n = 8$, the graph $PG_2(\mathbb{Z}_8)$ contains a subgraph K_5. So, it cannot be a planar graph.

Case 7. Let $n = 2^m$, $m > 2$ contains K_5 and hence cannot be planar.

Case 8. Let $p \geq 3$, $PG_2(\mathbb{Z}_p^m)$, where $m > 1$ contains $K_{3,3}$, hence it cannot be planar.

Case 9. Let n be even. If $n = 10$, then the subgraph induced by the vertices $\{0, 2, 4, 6, 8\}$ forms K_5 and for $n = 12$, the subgraph induced by the vertices $\{0, 2, 4, 6, 8\}$ forms again K_5. So, the subgraph of $PG_2(\mathbb{Z}_n)$ where n is even forms K_5 and hence the graph is not planar.

Case 10. Let n be odd. If $n = 15$ then the subgraph induced by $\{0, 3, 6, 9, 12\}$ forms K_5 and for $n = 21$ the subgraph induced by $\{0, 3, 6, 9, 12\}$ forms again K_5. So, the subgraph of $PG_2(\mathbb{Z}_n)$, where n is odd forms a subgraph K_5 and hence the graph is nonplanar. Hence the result. $\square$

Theorem 4.3 *The girth,* $gr(PG_2(\mathbb{Z}_n))$ *is* 3*, for* $n \geq 3$.

Proof We know that $PG_2(\mathbb{Z}_2)$ is a complete graph K_2, hence girth of $PG_2(\mathbb{Z}_2)$ is ∞. Now, let $n \geq 3$, then in $PG_2(\mathbb{Z}_n)$ always 3-cycle exist and hence $gr((PG_2(\mathbb{Z}_n)) = 3$, for $n \geq 3$. $\square$

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