International J.Math. Combin. Vol.3(2019), 23-29

Non-Holonomic Frame for
A Finsler Space with Some Special (α, β)-Metric

C.K.Yadav and M.K.Gupta

(Department of Pure & Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.), India)

E-mail: chiranjeev86@gmail.com, mkgiaps@gmail.com

Abstract: In this paper, we study the Finsler deformation of some special (α, β)-metric. We determine a nonholonomic Finsler frame for Finsler space with (α, β)-metric $F = \alpha e^{\frac{\beta}{\alpha}}$ and Randers change of Matsumoto metric.

Key Words: Finsler space, (α, β)-metric, Randers metric, matsumoto metric, exponential metric, non-holonomic Finsler frame.

AMS(2010): 53C22, 53C60, 53B40.

§1. Introduction

The concept of non-holonomic space which is more general than a Riemannian space and generalized the parallelism of Levi-Civita and geodesic curves in that space is introduced by G. Vranceanu [22]. A non-holonomic region as a space with a non-holonomic dynamical system was considered by Z. Horak [13] with another aspect. The non-holonomic space in a space of line elements with an affine connection was first conferred by T. Hosokawa [9]. The theory of non-holonomic system in Finsler space was introduced by Y. Kastsurada [15].

The concept of non-holonomic frame is a deformation arising from the consideration of a charged particle moving in external electromagnetic field in the studied of a unified formalism that uses a non-holonomic frame on space-time [11, 12]. Further, the gauge transformation is studied as a non-holonomic frame on the tangent bundle of a four-dimensional base manifold [3, 4]. The geometry which arises from these consideration gives a more unified approach to gravitation and gauge symmetries. In these papers, the common Finsler idea used by the physicist is the existence of non-holonomic frame on the vertical subbundle of a base manifold M. In [1, 2], P. L. Antonelli and I. Bucataru, determined such a non-holonomic frame for the two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces. Since Randers and Kropina spaces are the Finsler space with (α, β)-metric, is a member of the bigger class of Finsler space. It appears a natural question that how many Finsler spaces with (α, β) metrics have such a non-holonomic frame [7]? Yes, there is a number of Finsler space with (α, β) metrics. Some auther's which discusses the non-holonomic frame for (α, β)

[1]Received January 30, 2019, Accepted August 21, 2019.

metrics are [8, 20, 21]. In the present work, we determine the non-holonomic Finsler frame for the exponential (α, β) metric and Randers change of Matsumoto metric.

§2. Preliminaries

The physicists R. G. Beil in [3, 4] and R. R. Holland in [11, 12] are using non-holonomic Finsler frames to develop unified field theories. For a Finsler space with (α, β)-metric a non-holonomic frame is a product of two non-holonomic frames, each of these being determined by Finsler deformation.

Let U be an open set of TM and

$$V_i : u \in U \mapsto V_i(u) \in V_u TM, \quad i \in \{1, 2, \cdots, n\}$$

be a vertical frame over U. If $V_i(u) = V_i^j(u) \frac{\partial}{\partial y^j}|_u$ are the entries of a invertible matrix for all $u \in U$. Denote by $\tilde{V}_i^j(u)$ the inverse of this matrix. this means that:

$$V_j^i \tilde{V}_k^j = \delta_k^i, \quad \tilde{V}_j^i V_k^j = \delta_k^i.$$

We call V_j^i a non-holonomic Finsler frame.

An important class of Finsler space with (α, β)-metrics are given in [18]. The first Finsler space with (α, β)-metric was introduced in the forties by G. Randers and known as Randers space [19].

Definition 2.1 *A Finsler space $F^n = (M, F(x, y))$ is called with (α, β)-metric if there exists a two homogenius function L of two variables such that the Finsler metric $F : TM \to \Re$ is given by*

$$F^2(x, y) = L(\alpha(x, y), \beta(x, y))$$

where $\alpha(x, y) = \sqrt{a_{ij} y^i y^j}$, where a_{ij} is a Riemmannian metric on M and $\beta(x, y) = b_i(x) y^i$, is a 1-form on M.

Example 2.2 If $L(\alpha, \beta) = (\alpha + \beta)^2$, then the Finsler space with metric

$$F(x, y) = \sqrt{a_{ij} y^i y^j} + b_i(x) y^i$$

is called a Randers space.

Example 2.3 If $L(\alpha, \beta) = \frac{\alpha^4}{\beta^2}$, then the Finsler space with metric

$$F(x, y) = \frac{a_{ij} y^i y^j}{|b_i(x) y^i|}$$

is called a Kropina space.

The Randers space and Kropina space play an important role in Finsler geometry and are dual in sense of [10]. For a Finsler space with (α, β)-metric $F^2(x, y) = L(\alpha(x, y), \beta(x, y))$, the

Finsler invariants are [17]

$$\begin{cases} \rho = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}, \ \ \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}, \ \ \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}, \\ \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right), \end{cases} \tag{1}$$

For a Finsler space with (α, β)-metric, we have

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0. \tag{2}$$

With respect to above notation (i.e. Finsler invariants), the metric tensor g_{ij} of a Finsler space with (α, β)-metric is given by [18]

$$g_{ij} = \rho a_{ij}(x) + \rho_0 b_i(x) b_j(x) + \rho_{-1}(b_i(x) y_j + b_j(x) y_i) + \rho_{-2} y_i y_j. \tag{3}$$

We can be arranged the metric tensor g_{ij} of Finsler space into the form

$$\begin{aligned} g_{ij} &= \rho a_{ij}(x) + \tfrac{1}{\rho_{-2}}(\rho_{-1} b_i + \rho_{-2} y_i)(\rho_{-1} b_j + \rho_{-2} y_j) \\ &+ \tfrac{1}{\rho_{-2}}(\rho_0 \rho_{-2} - \rho_{-1}^2) b_i b_j. \end{aligned} \tag{4}$$

From the equation (4), we can see that the metric tensor g_{ij} is the result of the two deformations

$$\begin{cases} a_{ij} \mapsto h_{ij} = \rho a_{ij}(x) + \frac{1}{\rho_{-2}}(\rho_{-1} b_i + \rho_{-2} y_i)(\rho_{-1} b_j + \rho_{-2} y_j), \\ h_{ij} \mapsto g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0 \rho_{-2} - \rho_{-1}^2) b_i b_j. \end{cases} \tag{5}$$

The non-holonomic Finsler frame corresponding to the first deformation of equation (5) is according to Theorem 7.9.1 in reference [7], is given by

$$X_j^i = \sqrt{\rho}\, \delta_j^i - \frac{1}{A^2}\left(\sqrt{\rho} \pm \sqrt{\rho + \frac{A^2}{\rho_{-2}}}\right)(\rho_{-1} b^i + \rho_{-2} y^i)(\rho_{-1} b_j + \rho_{-2} y_j), \tag{6}$$

where $A^2 = a_{ij}(\rho_{-1} b^i + \rho_{-2} y^i)(\rho_{-1} b_j + \rho_{-2} y_j) = \rho_{-1}^2 b^2 + \beta \rho_{-1} \rho{-2}$. The metric temsor a_{ij} and h_{ij} are related by:

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{7}$$

Similarly, a non-holonomic frame Finsler frame corresponding to the second deformation of equation (5) is according to Theorem 7.9.1 in reference [7] given by

$$Y_j^i = \delta_j^i - \frac{1}{B^2}\left(1 \pm \sqrt{1 + \frac{\rho_{-2} B^2}{\rho_0 \rho_{-2} - \rho_{-1}^2}}\right) b^i b_j, \tag{8}$$

where $B^2 = h_{ij} b^i b_j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1} b^2 + \rho_{-2} \beta^2$. The metric temsor h_{ij} and g_{ij} are related by

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \tag{9}$$

From (7) and (9), we have $V_m^k = X_i^k Y_m^i$, with X_i^k and Y_m^i are given by (6) and (8), are the non-holonomic Finsler frame of the Finsler space with (α, β) metric.

§3. Non-Holonomic Frame for Finsler Space with $F = \alpha e^{\frac{\beta}{\alpha}}$ - Metric

We consider a (α, β)- metric given as

$$F = \alpha e^{\frac{\beta}{\alpha}}, \tag{10}$$

in a Finsler space. For the fundamental function $L = \alpha^2 e^{\frac{2\beta}{\alpha}}$, the Finsler invariants (1) are given by:

$$\begin{cases} \rho = \frac{(\alpha-\beta)}{\alpha} e^{\frac{2\beta}{\alpha}}, \quad \rho_0 = 2e^{\frac{2\beta}{\alpha}}, \quad \rho_{-1} = \frac{(\alpha-2\beta)}{\alpha^2} e^{\frac{2\beta}{\alpha}}, \\ \rho_{-2} = \frac{\beta(2\beta-\alpha)}{\alpha^4} e^{\frac{2\beta}{\alpha}}, \end{cases} \tag{11}$$

and

$$\begin{cases} A^2 = \frac{(\alpha-2\beta)(\alpha^3 b^2 - 2\alpha^2\beta b^2 + 2\beta^3 - \alpha^2\beta^2)}{\alpha^6} e^{\frac{4\beta}{\alpha}}, \\ B^2 = e^{\frac{2\beta}{\alpha}} \left[\frac{b^2(\alpha-\beta)}{\alpha} - \frac{(\alpha-2\beta)}{\beta} \left(b^2 - \frac{\beta}{\alpha^2} \right)^2 \right]. \end{cases} \tag{12}$$

The Finsler invariants satisfies the relation $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then by using equation (6) with respect to first deformation of (5), we get

$$\begin{aligned} X_j^i &= \sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \delta_j^i - \frac{\alpha^2}{(\alpha^3 b^2 - 2\alpha^2\beta b^2 + 2\beta^3 - \alpha^2\beta^2)} \\ &\quad \times \left[\sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \pm \sqrt{\frac{e^{\frac{2\beta}{\alpha}}(\alpha^2\beta - \alpha\beta^2 - \alpha^3 b^2 + 2\alpha^2\beta b^2 - 2\beta^3 + \alpha^2\beta^2)}{\alpha^2\beta}} \right] \\ &\quad \times \left(b^i - \frac{\beta}{\alpha^2} y^i \right) \left(b_j - \frac{\beta}{\alpha^2} y_j \right), \end{aligned} \tag{13}$$

and also using equation (6) with respect to second deformation of (5), we get

$$\begin{aligned} Y_j^i = \delta_j^i &- \frac{\alpha^4\beta}{e^{\frac{2\beta}{\alpha}} \{(\alpha^4\beta - \alpha^3\beta^2)b^2 - (\alpha-2\beta)(\alpha^2 b^2 - \beta)^2\}} \\ &\times \left[1 \pm \sqrt{1 + \frac{\alpha^4\beta - \alpha^3\beta^2 - (\alpha-2\beta)(\alpha^2 b^2 - \beta)^2}{\alpha^4}} \right] b^i b_j. \end{aligned} \tag{14}$$

By using X_k^i and Y_j^k, we obtain the non-holonomic Finsler frame as follows,

$$\begin{aligned} V_j^i &= X_k^i Y_j^k = \sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \delta_j^i - [1 \pm D] E b^i b_j \delta_k^i \sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \\ &\quad - \frac{\alpha^2 C[1 - E(1 \pm D) b^i b_j]}{(\alpha^3 b^2 - 2\alpha^2\beta b^2 + 2\beta^3 - \alpha^2\beta^2)} \left(b^i - \frac{\beta}{\alpha^2} y^i \right) \left(b_j - \frac{\beta}{\alpha^2} y_j \right), \end{aligned} \tag{15}$$

where

$$C = \sqrt{\frac{(\alpha-\beta)e^{\frac{2\beta}{\alpha}}}{\alpha}} \pm \sqrt{\frac{e^{\frac{2\beta}{\alpha}}(\alpha^2\beta - \alpha\beta^2 - \alpha^3 b^2 + 2\alpha^2\beta b^2 - 2\beta^3 + \alpha^2\beta^2)}{\alpha^2\beta}},$$

$$
\begin{aligned}
D &= 1 \pm \sqrt{1+\frac{\alpha^4\beta-\alpha^3\beta^2-(\alpha-2\beta)(\alpha^2b^2-\beta)^2}{\alpha^4}},\\
E &= \frac{\alpha^4\beta}{e^{\frac{2\beta}{\alpha}}\{(\alpha^4\beta-\alpha^3\beta^2)b^2-(\alpha-2\beta)(\alpha^2b^2-\beta)^2\}}.
\end{aligned}
$$

Theorem 3.1 *Consider a Finsler space $F^n=(M,F)$ with $L=(\alpha e^{\frac{\alpha}{\beta}})^2$, for which the condition (2) is true. Then $V^i_j=X^i_kY^k_j$ is a non-holonomic Finsler frame given in equation (15), where X^i_k and Y^k_j are given by (13) and (14) respectively.*

§4. Non-Holonomic Frame for Finsler Space with Randers Change of Matsumoto Metric

We consider a Randers change of Matsumoto metric as given by

$$
F=\frac{\alpha^2}{\alpha+\beta}+\beta, \tag{16}
$$

in a Finsler space. For the fundamental function $L=\left(\frac{\alpha^2}{\alpha+\beta}+\beta\right)^2$, the Finsler invariants (1) are given by

$$
\begin{cases}
\rho=\frac{(\alpha^3-\alpha^2\beta-3\alpha\beta^2+2\beta^3)}{(\alpha-\beta)^3},\\[2mm]
\rho_0=\frac{6\alpha^4+\beta^4-6\alpha^3\beta+6\alpha^2\beta^2-4\alpha\beta^3}{(\alpha-\beta)^4},\\[2mm]
\rho_{-1}=\frac{(2\alpha^3-8\alpha^2\beta+3\alpha\beta^2)}{(\alpha-\beta)^4},\\[2mm]
\rho_{-2}=\frac{8\alpha^2\beta^2-2\alpha^3\beta-3\alpha\beta^3)}{\alpha^2(\alpha-\beta)^4},
\end{cases} \tag{17}
$$

and

$$
\begin{cases}
A^2=\frac{b^2S(\alpha,\beta)-T(\alpha,\beta)}{(\alpha-\beta)^8},\\
B^2=\frac{\{b^2(2\alpha^4-8\alpha^3\beta+3\alpha^2\beta^2)+(8\alpha\beta^3-2\alpha^2\beta^2-3\beta^4)\}^2}{\alpha^2(\alpha-\beta)^8}\\
\qquad+\frac{b^2U(\alpha,\beta)}{(\alpha+\beta)^3},
\end{cases} \tag{18}
$$

where

$$
\begin{aligned}
&S(\alpha,\beta)=4\alpha^6-32\alpha^5\beta+76\alpha^4\beta^2-48\alpha^3\beta^3+9\alpha^2\beta^4,\\
&T(\alpha,\beta)=4\alpha^4\beta^2--32\alpha^3\beta^3+76\alpha^2\beta^4-48\alpha\beta^5+9\beta^6,\\
&U(\alpha,\beta)=(\alpha^3-2\alpha^2\beta-\alpha\beta^2+2\alpha\beta-4\beta^2-2\beta^3).
\end{aligned}
$$

The Finsler invariants satisfies the relation $\rho_{-1}\beta+\rho_{-2}\alpha^2=0$. Then by using equation (6) with respect to first deformation of (5), we get

$$
\begin{aligned}
X^i_j=&\sqrt{\frac{(\alpha^3-\alpha^2\beta-3\alpha\beta^2+2\beta^3)}{(\alpha-\beta)^3}}\delta^i_j-\frac{(2\alpha^8-8\alpha^5\beta+3\alpha^4\beta)^2}{b^2S(\alpha,\beta)-T(\alpha,\beta)}\\
&\times\left[\sqrt{\frac{(\alpha^3-\alpha^2\beta-3\alpha\beta^2+2\beta^3)}{(\alpha-\beta)^3}}\pm\sqrt{\frac{(\alpha^3-\alpha^2\beta-3\alpha\beta^2+2\beta^3)}{(\alpha-\beta)^3}+\frac{\alpha(b^2S(\alpha,\beta)-T(\alpha,\beta))}{\beta(\alpha-\beta)^4(8\alpha\beta-2\alpha^2-3\beta^2)}}\right]\\
&\times\left(b^i-\frac{\beta}{\alpha^2}y^i\right)\left(b_j-\frac{\beta}{\alpha^2}y_j\right),
\end{aligned} \tag{19}
$$

and also using equation (6) with respect to second deformation of (5), we get

$$Y^i_j = \delta^i_j \; - \frac{\alpha^2(\alpha-\beta)^8}{V} \times \left[1 \pm \sqrt{1 + \frac{V\beta}{\alpha^2 W}} \right] b^i b_j. \qquad (20)$$

Where

$$V = \alpha^2(\alpha-\beta)^5 b^2 U(\alpha,\beta) + \{b^2(2\alpha^4 - 8\alpha^3\beta + 3\alpha^2\beta^2) + (8\alpha\beta^3 - 2\alpha^2\beta^2 - 3\beta^4)\}^2,$$

$$W = (2\alpha^6 - 8\alpha^5\beta + 6\alpha^4\beta + 3\alpha^4\beta^2 - 6\alpha^3\beta 2 + 6\alpha^2\beta^4 - 4\alpha\beta^5 + \beta^5).$$

By using X^i_k and Y^k_j, we obtain the non-holonomic Finsler frame as follows.

Theorem 4.1 *Consider a Finsler space* $F^n = (M,F)$ *with* $L = (\frac{\alpha^2}{\alpha+\beta} + \beta)^2$ *for which the condition (*2*) is true. Then* $V^i_j = X^i_k Y^k_j$ *is a non-holonomic Finsler frame, where* X^i_k *and* Y^k_j *are given by (*19*) and (*20*) respectively.*

References

[1] P. L. Antonelli and I. Bucataru, On Holland's frame for Randers space and its applications in physics, Steps in differential geometry, *Procedings of the Colloquium on Differential Geometry*, 25-30, July, 2000, Debrecen, Hungary.

[2] P. L. Antonelli and I. Bucataru, Finsler connections in an holonomic geometry of a Kropina space, *Nonlinear Stud.*, 8(1)(2001) 171-184.

[3] R. G. Beil, Finsler gauge transformations and general relativity, *Intern. J. Theor. Phys.*, 31,6(1992) 1025-1044.

[4] R. G. Beil, Finsler and Kaluza-Klein Gauge Theories, *Intern. J. Theor. Phys.*, 32(1993) 1021-1031.

[5] I. Bucataru, Nonholonomic frame in Finsler geometry, *Balkan Journal of Geometry and and Its Applications*, 7,1(2002) 13-27.

[6] I. Bucataru, Nonholonomic frame in Finsler spaces with (α,β) metrics , *Procrdings of the Conference on Finsler and Lagrange Geometries,* Iasi, August 26-31,2001, Kluwer Academic Publishers (2003) 69-78.

[7] I. Bucataru and R. Miron, *Finsler-Lagrange Geometry: Applications to Dynamical Systems*, Bucuresti: Editura Academiei, (2007).

[8] K. Chandru, S. K. Narasimhamurthy and M. Ramesha, A study of nonholonomic Finsler frame on gauge transformations, *Bull. Trans. Univ. Brasov*, 9(1)(2016) 29-38.

[9] T. Hosokawa, Ueber nicht-holonome Uebertragung in allgemeiner Manninfaltigkeit T., *Jour. Fac. Sci. Hokkaido Imp. Univ.,*, I, 2(1934) 1-11.

[10] D. Hrimiuc and H. Shimada, On the L-duality between Lagrange and Hamilton manifolds, *Nonlinear World*, 3(1996) 613-641.

[11] P. R. Holland, Electromagnetism, particles and anholonomy, *Phys. Lett.*, 91A(1982) 275-278.

[12] P. R. Holland, Philippids, Anholonomic deformations in the other: a significance for the

electrodynamics potentials, in the volume *Quantum Implications*, B. J. Hiley and F. D. Peat (eds), Routledge and Kegan Paul, London and New York (1987), 295-311.

[13] Z. Horak, Sur une generalisation de la notion de variete, *Publ. Fac. Sc. Univ. Masaryk, Brno*, 86(1927) 1-20.

[14] R. S. Ingarden, Differential geometry and physics, *Tensor N. S.*, 30(1976) 201-209.

[15] Y. Kasturada, On the theory of nonholonomic system in Finsler Spaces, *Tohuku Math. J.*, (2)3, 2(1951) 140-148.

[16] V. K. Kropina, On projective two-dimensional Finsler spaces with a special metrics, *Trudy Sem. Vektor. Tenzor. Anal.*, 11(1961) 271-292.

[17] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Ostu, Japan (1996).

[18] M. Matsumoto, Theory of Finsler spaces with (α, β) metrics, *Rep. Math. Phys.* 31(1991) 43-83.

[19] G. Randers, On an asymmetric metric in the four-spaces of general relativity, *Phys. Rev.*, 59(1941),195-199.

[20] Gauri Shanker and Sruthy Asha Baby, Nonholonomic frames for a Finsler space with general (α, β)- metric, *Inter. Jour. Pure and Applied Mathematics*, 104(4)(2016) 1013-1023.

[21] Brijesh Kumar Tripathi, Nonholonomic Frames for Finsler Space with Deformed Special (α, β) Metric, *International J. Math. Combin.*, Vol.1, (2018), 61-67.

[22] G. Vranceanu, Sur les espaces non holonomes, Sur le calcul differential absolu pour les varietes non holonomes, *C. R.*, 183(1926) 852-1083.